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Cohomological Interpretation of Some Classic Formulas Satisfied by Hypergeometric Functions

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Introduction

The purpose of the present work is to give a cohomological interpretation of linear and quadratic relations satisfied by the hypergeometric function

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} \lambda^n $$

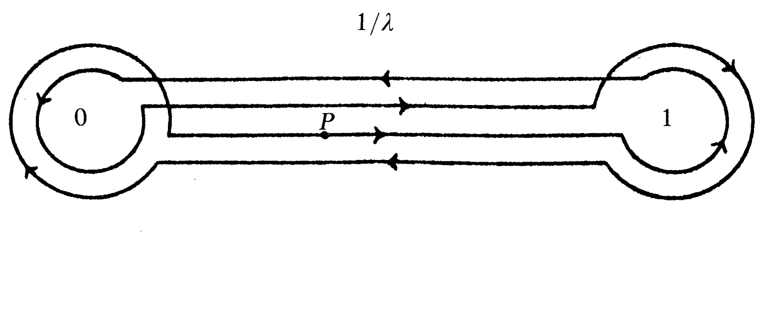
where $(d)_n = d(d+1)\cdots(d+n-1)$ and c is not a negative integer. We also study the cohomology of the generalized hypergeometric functions ${}_3F_2$ and, as an application, we explain the formula of symmetry

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] = {}_2F_1\left[\begin{matrix} b, a \\ c \end{matrix}; \lambda\right], $$

which has non-trivial meaning in cohomology. We study the classical integral representation ([2] §2.1.3)

$$ (0.1) \qquad \begin{aligned} {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] &= k(b, c) \int_{\gamma} x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a} dx \\ [k(b, c)]^{-1} &= (1-e^{2\pi i b})(1-e^{2\pi i(c-b)}) \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \end{aligned} $$

which holds for $c, c-b \neq 1, 2, \ldots,$ and $\lambda \in \mathbb{C} \setminus [1, \infty)$. In formula (0.1) integration is done over the Pochhammer loop γ where $P \in]0, 1[$ is the starting and ending point of γ. The branch of the integrand



$$f(x) = x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a}$$

is chosen in such a way that $\arg x$ and $\arg(1-x)$ are continuous in x and are reduced to zero at the starting point of γ, while $\arg(1-\lambda x)$ is continuous in λ, x and it is reduced to zero at $\lambda = 0$ and the starting point of γ. It is easily seen that the integrand function $f(x)$ takes the same value at the starting and ending points of γ, i.e. corresponds to a closed loop on the Riemann surface of $f(x)$ (for details see [2] § 1.6). When a, b and c are rational, then for each λ the Riemann surface of $f(x)$ is the algebraic curve defined by

$$y^N - x^{N-B}(1-x)^{N+B-C}(1-\lambda x)^A = 0$$

where $y^{-1} = f(x)$ while A, B, C, N are nonzero integer numbers such that $a = A/N$, $b = B/N$, $c = C/N$ and g.c.d. $(A, B, C, N) = 1$. If A, B, C and N are the previous integer numbers then

$$\mathscr{C} = \{(x, y, \lambda) \in \mathbb{C}^2 \times S \mid y^N = x^{N-B}(1-x)^{N+B-C}(1-\lambda x)^A,\ y \neq 0\}$$

is a family of curves parametrized by

$$\pi: \mathscr{C} \longrightarrow S$$

where S is the space of the parameter λ. Then π is a smooth affine morphism and for each $\lambda \in S$ the fiber $\pi^{-1}(\lambda)$ is an irreducible non-singular affine curve which is an N-cyclic covering of $\mathbb{C} \setminus \{0, 1, 1/\lambda\}$. Therefore (0.1) may be written in the form

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] = k(b, c) \int_{\hat{\gamma}} \frac{dx}{y}$$

where $\hat{\gamma}$ is a suitable covering of γ on $\mathscr{C}$ and now ${}_2F_1$ is the multivalued function, period of the S-relative algebraic form dx/y. There exists an integral representation for ${}_3F_2$ involving a double integral, in which case families of irreducible non-singular affine surfaces come into the game. Consider, in general, a family X of non-singular affine varietes with an affine parameter space and let

$$\pi: X \longrightarrow S \tag{0.2}$$

be a smooth affine morphism. In order to formulate our problem, we recall some notions. Let $\Omega^{\bullet}_{X/S}$ the complex of sheaves of germs of the S-relative holomorphic algebraic forms on X, that is $\omega \in \Omega^{\bullet}_{X/S}$ is a differential form which depends holomorphically upon λ and for each λ, $\omega \in \Omega^{\bullet}_X$, $X_\lambda = \pi^{-}(\lambda)$. Then we study $H^q(\Gamma(X, \Omega^{\bullet}_{X/S}))$, $q \in N$, the cohomology of the complex of the global section on X, which has the structure of $\mathcal{O}_S$-differential module. It means that every derivation D of $\mathcal{O}_S$ may be uniquely extended to a derivation $\tilde{D}$ of $H^q(\Gamma(X, \Omega^{\bullet}_{X/S}))$. To define such an extention $\tilde{D}$ we follow [4] and [6]. Since X and S are affine varieties and π is an affine morphism, then

$$H^q(\Gamma(X, \Omega^{\bullet}_{X/S})) \simeq \Gamma(S, H^q_{DR}(X/S))$$

where the relative de-Rham cohomology $H^q_{DR}(X/S) = \mathbb{R}^q\pi_*(\Omega^{\bullet}_{X/S})$, $\mathbb{R}^q\pi_*$ the q-th hyperderivated functor of π_* and $\tilde{D} = \nabla_{X/S}(D)$ where $\nabla_{X/S}$ denotes the Gauss-Manin connection (see [5]). If $\pi: X \to S$ is a morphism of type (0.2) then following Pham ([9], annexe A) we define $\mathscr{H}_q(X/S)$, the q-dimensional ($q \in \mathbb{N}$) S-relative homology of X, to be the sheaf over the topological space S associated to the pre-sheaf defined for every open set U of S, by

$$U \longrightarrow \mathscr{H}_{q+r}(X, \pi^{-1}(S \setminus U)),$$

$r = \dim_{\mathbb{R}} S$, $\mathscr{H}_{\bullet}(X, A)$ being the usual relative homology of the pair (X, A), $A \subseteq X$. Because of the previous definition it follows that $\mathscr{H}_q(X/S)_\lambda$ the fiber of $\mathscr{H}_q(X/S)$ over $\lambda \in S$, is $H_q(\pi^{-1}(\lambda))$, the homology of the fiber X over λ (see [9]). For given two families

$$X \longrightarrow S; \quad Y \longrightarrow S'$$

of type (0.2), forms $\omega \in H^q(\Gamma(X, \Omega^{\bullet}_{X/S}))$, $\omega' \in H^q(\Gamma(Y, \Omega^{\bullet}_{Y/S'}))$ and cycles $\mathfrak{s} \in \mathscr{H}_q(X/S)$, $\mathfrak{s}' \in \mathscr{H}_q(Y/S')$, put

$$F(\lambda) = \int_{\mathfrak{s}} \omega, \quad F'(\lambda') = \int_{\mathfrak{s}'} \omega'.$$

If there are morphisms φ and τ

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow & \divideontimes & \downarrow \\ S & \xrightarrow{\tau} & S' \end{array} \tag{0.3}$$

such that $\varphi^*(\omega') = \omega$, $\varphi_{\#}\mathfrak{s} = \mathfrak{s}'$ and $\lambda' = \tau(\lambda)$, then by changing variables

$$\int_{\mathfrak{s}'} \omega' = \int_{\varphi_{\#}\mathfrak{s}} \omega' = \int_{\mathfrak{s}} \varphi^*\omega' = \int_{\mathfrak{s}} \omega$$

we have the equality $F(\lambda) = F'(\lambda')$. Then diagram (0.3) is called a *cohomological interpretation* of $F(\lambda) = F'(\lambda')$. Note that $\mathscr{H}_q(X/S)$ has not necessarily global section and so $\mathfrak{s}$ and $\mathfrak{s}'$ must be considered local sections over $D(\lambda_o, \varepsilon)$ and $D(\lambda'_o, \varepsilon')$ respectively ($\lambda'_o = \tau(\lambda_o)$).

Therefore $F(\lambda) = F'(\lambda')$ has to be interpreted as equality between two function elements $(\int_{\mathfrak{s}}\omega, D(\lambda_o, \varepsilon))$ and $(\int_{\mathfrak{s}'} w', D(\lambda_o', \varepsilon'))$ of the multivalued functions $F(\lambda)$ and $F'(\lambda')$ respectively.

Let D and D' be derivations on S and S' respectively such that for every $h \in \mathcal{O}(S')$

$$\tau^* D'(h) = D(\tau^* h)$$

then for every $\eta \in H^q(\Gamma(Y, \Omega^{\bullet}_{X/S'}))$

$$\varphi^* \nabla_{Y/S'}(D')\eta = \nabla_{X|S}(D)\varphi^*\eta$$

Since previous ω' satisfies in $H^q(\Gamma(Y, \Omega^{\bullet}_{X/S'}))$ the equation

$$\sum_{j=0}^{m} a_j(\lambda') \nabla_{Y/S'}(D')^j \omega' = 0 \tag{0.4}$$

with $a_j(\lambda') \in \mathcal{O}(S')$, $j = 0, \ldots, m$ then applying φ^* we obtain in $H^q(\Gamma(x, \Omega^{\bullet}_{X/S}))$ the relation

$$\sum_{j=0}^{m} a_j(\tau(\lambda)) \nabla_{X/S}(D)^j \omega = 0 \tag{0.5}$$

$a_j(\tau(\lambda)) \in \mathcal{O}(S)$, $j = 0, \ldots, m$.

This means that ω satisfies the same equation of ω' after substitution $\lambda' = \tau(\lambda)$. The same result holds for $F(\lambda)$ and $F'(\lambda')$. In fact in this work we consider q-cycles $\mathfrak{s}$ and $\mathfrak{s}'$ such that

$$D[F(\lambda)] = \int_{\mathfrak{s}} \nabla_{X/S}(D)\omega, \quad D'[F'(\lambda')] = \int_{\mathfrak{s}'} \nabla_{Y/S'}(D')\omega'$$

and so integrating (0.4) and (0.5) over $\mathfrak{s}$ and $\mathfrak{s}'$ respectively we have the differential equations

$$\sum_{j=0}^{m} a_j(\lambda')(D')^j[F'(\lambda')] = 0$$

and

$$\sum_{j=0}^{m} a_j(\tau(\lambda)) D^j[F(\lambda)] = 0.$$

In this work we give some explicit diagram of type (0.3) and their implications. In Chapter I we explain Kummer list of the 24 solutions of the hypergeometric equation

$$\lambda(1-\lambda)(d/d\lambda)^2 u + [c - (a+b+1)\lambda](d/d\lambda)u - ab\,u = 0 \tag{0.6}$$

which has, among the others, $u = {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right]$ as solution. In this case we use diagrams (0.3) where the varieties X and Y are curves, the morphisms φ and τ are isomorphisms and the cohomology classes ω and ω' both satisfy the hypergeometric equation (0.6). Linear transformations for ${}_2F_1$ are so explained since they are equalities between some couples of functions among those of Kummer list.

In Chapter II we study the relative de-Rham cohomology of a family of affine surfaces coming from the integral representation of ${}_3F_2$.

We show in Chapter III that reduction formulas of ${}_3F_2$ (that is, when ${}_3F_2$ coincides with ${}_2F_1$) follow from diagrams of type (0.3) in which φ is birational and τ is the identity map.

The cases of quadratic relations between hypergeometric functions essentially involves diagrams (0.3) where φ and τ are morphisms of degree two as shown in Chapter IV.

NOTATIONS

$i = \sqrt{-1}$,
$\varepsilon(\alpha) = e^{2\pi i\alpha}$,
$(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, $n \in \mathbb{Z}$,
$I = [0, 1]$,
$D(x, r)$; closed complex disk of radius r and center x,
$D(U, r) = \bigcup_{u \in U} D(u, r)$, $U \subset \mathbb{C}$,
$k(a, b) = \Gamma(b)[(1-\varepsilon(a))(1-\varepsilon(b-a))\Gamma(a)\Gamma(b-a)]^{-1}$,
$\mathbb{B}(a, b)$; beta function of arguments a and b.

Chapter I. On Kummer list of 24 solutions of the hypergeometric equation

In this chapter we consider the family of curves

$$\mathscr{C} \longrightarrow S = \mathbb{C} \setminus \{0, 1\}$$
$$\mathscr{C} = \{(x, y; \lambda) \in \mathbb{C}^2 \times S \mid y^N = x^{N-B}(1-x)^{N+B-C}(1-\lambda x)^A,\ y \neq 0\}$$

where A, B, C, N are nonzero integer numbers, g.c.d.$(A, B, C, N) = 1$ and $A, B, C, B-C, A-C \not\equiv 0 \bmod N$. We shall give some properties of such a family. Let $\theta = \varepsilon(1/N)$. Then there exists an S-automorphism $\Theta: \mathscr{C} \to \mathscr{C}$ given by $(x, y) \to (x, \theta y)$. The corresponding morphism Θ^* acts on the complex $\Omega^\bullet = \Gamma(\mathscr{C}, \Omega^\bullet_{\mathscr{C}/S})$ and so $\Omega^\bullet$ splits into a direct sum of subcomplexes $\Omega_j^\bullet$ on which Θ^* acts via multiplication with θ^{-j}. If

$$\mathcal{L} = \mathcal{O}(S)[x, (x(1-x)(1-\lambda x))^{-1}]$$

then $\Omega_j^{\bullet}$ is the complex

$$0 \longrightarrow \frac{1}{y^j}\mathcal{L} \longrightarrow \frac{1}{y^j}\mathcal{L}\,dx \longrightarrow 0$$

for $j = 0,\dots, N-1$ and so $\Omega^{\bullet} = \bigoplus_{j=0}^{N-1} \Omega_j^{\bullet}$.

Since cohomology commutes with direct sum then Θ^* gives the following decomposition

$$H^q(\Omega^{\bullet}) = \bigoplus_{j=0}^{N-1} H^q(\Omega_j^{\bullet}), \qquad q = 0, 1.$$

Let $D = \nabla_{\mathscr{C}/S}(d/d\lambda)$, then it is defined on $\Omega^{\bullet}$ by

$$D(x) = 0, \quad D(y) = -\frac{xyA}{N(1-\lambda x)}$$

(the second formula comes from the first and the equation of $\mathscr{C}$) and then extended to $\Omega^{\bullet}$ by

$$D(f\,dg) = [D(f)]\,dg + f\,dD(g).$$

Passing to the quotient D is so defined on $H^q(\Omega^{\bullet})$. Note that D commutes with θ^*, so $H^1(\Omega^{\bullet})$ consists of N differential modules $H^1(\Omega_j^{\bullet}), j = 0,\dots, N-1$.

Theorem 1.1. *If $j = 1,\dots, N-1$ then $H^1(\Omega_j^{\bullet})$ is a free $\mathcal{O}(S)$-module of rank 2. For $j = 0$, $H^1(\Omega_o^{\bullet})$ is a free $\mathcal{O}(S)$-module of rank 3.*

Proof. For $j = 1,\dots, N-1$ The theorem follows by direct calculations using reduction formulas coming from the equation of $\mathscr{C}$. In the case $j = 0$ we observe that $\Omega_o^{\bullet}$ is the complex of the regular forms on $\mathbb{C}\setminus\{0, 1, 1/\lambda\}$ and so the result immediately follows.

See [7] and [1] for similar results.

Proposition 1.2. *The relative cohomology class $\omega = [dx/y] \in H^1(\Omega_1^{\bullet})$ satisfies the hypergeometric equation*

$$\lambda(1-\lambda)D^2\omega + [c - (a+b+1, \lambda]D\omega - ab\omega = 0 \tag{1.1}$$

$a = A/N$, $b = B/N$, $c = C/N$ and $D = \nabla_{\mathscr{C}/S}(d/d\lambda)$.

Proof. From the definition of D it follows that

$$D(dx/y) = \frac{ax\,dx}{(1-\lambda x)\,y}$$

and then the proposition follows by easy calculations.

If $\mathfrak{s}$ is a suitable covering of the usual Pochhammer type loop of base points 0 and 1 (see γ of the Introduction) then $\mathfrak{s} \in \mathscr{H}_1(\mathscr{C}/S)$ and

$$ {}_2F_1\begin{bmatrix} a,\, b \\ c \end{bmatrix}; \lambda \Bigg] = k(b,\, c)\int_{\mathfrak{s}} \omega, $$

$$ (d/d\lambda)\, {}_2F_1\begin{bmatrix} a,\, b \\ c \end{bmatrix}; \lambda \Bigg] = k(b,\, c)\int_{\mathfrak{s}} D\omega, $$

where, as usual, $a = A/N$, $b = B/N$ $c = C/N$, and $D = \nabla_{\mathscr{C}/S}(d/d\lambda)$. Therefore the hypergeometric equation (0.6) is nothing but equation (1.1) integrated over the S-relative cycle $\mathfrak{s}$.

Kummer gave a list of 24 solutions of (0.6), see list 1, where each of these solutions is expressed in terms of hypergeometric functions. From the integral representations of these 24 solutions come out families of curves

$$\mathscr{C}_j \longrightarrow S, \quad j = 1,\ldots,24$$

where

$$\begin{aligned} \mathscr{C}_j &= \mathscr{C}(h;\, \alpha,\, \beta,\, \gamma;\, \lambda') \\ &= \{(x',\, y';\, \lambda') \in \mathbb{C}^2 \times S \mid y'^N = hx'^{N-\beta}(1-x')^{N+\beta-\gamma}(1-\lambda' x')^{\alpha},\ y' \neq 0\} \end{aligned}$$

with $h \in \mathscr{D}(S)$ and α, β, γ integer numbers as shown in list 2. Furthermore g.c.d, $(\alpha, \beta, \gamma, N) = 1$ and $\alpha, \beta, \gamma, \beta-\gamma, \alpha-\gamma \not\equiv 0 \bmod N$, so the results previously stated for $\mathscr{C}$ still hold for families $\mathscr{C}_j$, $j = 1,\ldots,24$.

Theorem 1.3. *Let $\mathscr{C}_j$, $j = 1,\ldots,24$ defined as in list 2. Then the following diagram commutes*

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\varphi_j} & \mathscr{C}_j \\ \downarrow & & \downarrow \\ S & \xrightarrow{\tau_j} & S \end{array}$$

where φ_j, $\tau_j(\tau_j(\lambda) = \lambda')$ are defined as in list 2.

Proof. Direct computation.

Corollary 1.4. *The relative cohomology classes $\omega_j = [dx'/y'] \in H^1(\Gamma(\mathscr{C}_j, \Omega^{\bullet}_{\mathscr{C}_j/S}))$, $j = 1,\ldots,24$ satisfy the hypergeometric equation* (1.1).

List 1

1) $u_1 = F[a, b; c; \lambda]$
2) $\quad = (1-\lambda)^{c-a-b} F[c-a, c-b; c; \lambda]$
3) $\quad = (1-\lambda)^{-a} F[a, c-b; c; \lambda/(\lambda-1)]$
4) $\quad = (1-\lambda)^{-b} F[c-a, b; c; \lambda/(\lambda-1)]$
5) $u_2 = F[a, b; a+b+1-c; 1-\lambda]$
6) $\quad = \lambda^{1-c} F[b+1-c, a+1-c; a+b+1-c; 1-\lambda]$
7) $\quad = \lambda^{-a} F[a, a+1-c; a+b+1-c; 1-1/\lambda]$
8) $\quad = \lambda^{-b} F[b+1-c, b; a+b+1-c; 1-1/\lambda]$
9) $u_3 = (-\lambda)^{-a} F[a, a+1-c; a+1-b; 1/\lambda]$
10) $\quad = (-\lambda)^{b-c}(1-\lambda)^{c-a-b} F[1-b, c-b; a+1-b; 1/\lambda]$
11) $\quad = (1-\lambda)^{-a} F[a, c-b; a+1-b; 1/(1-\lambda)]$
12) $\quad = (-\lambda)^{1-c}(1-\lambda)^{c-a-1} F[1-b, a+1-c; a+1-b; 1/(1-\lambda)]$
13) $u_4 = (-\lambda)^{-b} F[b+1-c, b; b+1-a; 1/\lambda]$
14) $\quad = (-\lambda)^{a-c}(1-\lambda)^{c-a-b} F[c-a, 1-a; b+1-a; 1/\lambda]$
15) $\quad = (-\lambda)^{1-c}(1-\lambda)^{c-b-1} F[b+1-c, 1-a; b+1-a; 1/(1-\lambda)]$
16) $\quad = (1-\lambda)^{-b} F[c-a, b; b+1-a; 1/(1-\lambda)]$
17) $u_5 = \lambda^{1-c} F[b+1-c, a+1-c; 2-c; \lambda]$
18) $\quad = \lambda^{1-c}(1-\lambda)^{c-a-b} F[1-b, 1-a; 2-c; \lambda]$
19) $\quad = \lambda^{1-c}(1-\lambda)^{c-b-1} F[b+1-c, 1-a; 2-c; \lambda/(\lambda-1)]$
20) $\quad = \lambda^{1-c}(1-\lambda)^{c-a-1} F[1-b, a+1-c; 2-c; \lambda/(\lambda-1)]$
21) $u_6 = (1-\lambda)^{c-a-b} F[c-a, c-b; c+1-a-b; 1-\lambda]$
22) $\quad = \lambda^{1-c}(1-\lambda)^{c-a-b} F[1-b, 1-a; c+1-a-b; 1-\lambda]$
23) $\quad = \lambda^{a-c}(1-\lambda)^{c-a-b} F[c-a, 1-a; c+1-a-b; 1-1/\lambda]$
24) $\quad = \lambda^{b-c}(1-\lambda)^{c-a-b} F[1-b, c-b; c+1-a-b; 1-1/\lambda]$

$$F[a, b; c; \lambda] := {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right].$$

List 2

j	h	α	β	γ	$\lambda' = \tau_j(\lambda)$	x'
1	1	A	B	C	λ	x
2	$(-1)^N(1-\lambda)^{A+B-C}$	$C-A$	$C-B$			$\frac{1-x}{1-\lambda x}$
3	$(-1)^N(1-\lambda)^A$	A	$C-B$		$\frac{\lambda}{\lambda-1}$	$1-x$
4	$(1-\lambda)^B$	$C-A$	B			$\frac{(1-\lambda)x}{1-\lambda x}$
5	$(-1)^B$	A	B	$A+B+N-C$	$1-\lambda$	$\frac{x}{x-1}$
6	$(-1)^{N-B}\lambda^{C-N}$	$B+N-C$	$A+N-C$			$\frac{1}{1-\lambda x}$

7	$(-1)^{N-B}\lambda^{A}$	A	$A+N-C$		$1-\dfrac{1}{\lambda}$	$\dfrac{1}{1-x}$
8	$(-\lambda)^{B}$	$B+N-C$	B			$\dfrac{\lambda x}{1-\lambda x}$
9	$(-1)^{B-C}(-\lambda)^{A}$	A	$A+N-C$	$A+N-B$	$1/\lambda$	$1/x$
10	$(-1)^{N}\lambda^{C-B}(1-\lambda)^{A+B-C}$	$N-B$	$C-B$			$\dfrac{\lambda(1-x)}{1-\lambda x}$
11	$(-1)^{N+B-C}(1-\lambda)^{A}$	A	$C-B$		$\dfrac{1}{1-\lambda}$	$\dfrac{x-1}{x}$
12	$(-1)^{N-B}\lambda^{C-N}(1-\lambda)^{N+A-C}$	$N-B$	$A+N-C$			$\dfrac{1-\lambda}{1-\lambda x}$
13	λ^{B}	$B+N-C$	B	$B+N-A$	$1/\lambda$	λx
14	$(-1)^{N}\lambda^{C-A}(\lambda-1)^{A+B-C}$	$C-A$	$N-A$			$\dfrac{1-\lambda x}{1-x}$
15	$(-1)^{N}\lambda^{C-N}(\lambda-1)^{N+B-C}$	$B+N-C$	$N-A$		$\dfrac{1}{1-\lambda}$	$1-\lambda x$
16	$(\lambda-1)^{B}$	$C-A$	B			$\dfrac{(1-\lambda)x}{x-1}$
17	$(-1)^{A+B-C}\lambda^{C-N}$	$B+N-C$	$A+N-C$	$2N-C$	λ	$1/\lambda x$
18	$(-1)^{N}\lambda^{C-N}(\lambda-1)^{A+B-C}$	$N-B$	$N-A$			$\dfrac{1-\lambda x}{\lambda(1-x)}$
19	$(-1)^{N-A}\lambda^{C-N}(\lambda-1)^{N+B-C}$	$B+N-C$	$N-A$		$\dfrac{\lambda}{\lambda-1}$	$1-x/\lambda$
20	$(-1)^{N-B}\lambda^{C-N}(\lambda-1)^{N+A-C}$	$N-B$	$A+N-C$			$\dfrac{1-\lambda}{\lambda(x-1)}$
21	$(-1)^{N-A}(\lambda-1)^{A+B-C}$	$C-A$	$C-B$	$C+N-A-B$	$1-\lambda$	$\dfrac{x-1}{(1-\lambda)x}$
22	$(-1)^{A}\lambda^{C-N}(\lambda-1)^{A+B-C}$	$N-B$	$N-A$			$\dfrac{1-\lambda x}{1-\lambda}$
23	$(-1)^{A}\lambda^{C-A}(\lambda-1)^{A+B-C}$	$C-A$	$N-A$		$1-\dfrac{1}{\lambda}$	$\dfrac{1-\lambda x}{(1-\lambda)x}$
24	$(-1)^{A}\lambda^{C-B}(\lambda-1)^{A+B-C}$	$N-B$	$C-B$			$\dfrac{\lambda(1-x)}{\lambda-1}$

N. B. To make computation easier h is defined in terms of λ rather than λ'. The morphisms $\varphi_j\colon \mathscr{C}\to\mathscr{C}_j$ are defined by the previous $x'=x'(x)$ and $y'=y(d/dx)x'$.

Proof. By definition of the morphism φ_j we have that $(\varphi_j^{-1})^* \omega = \omega_j$ for $j = i, \ldots, 24$. Therefore applying $(\varphi_j^{-1})^*$ to equation (1.1) Corollary follows.

It is interesting to note that Kummer list arises in correspondence to the 24 Moebius transformations which map the set $\{0, 1, \infty\}$ is $\{0, 1, 1/\lambda, \infty\}$. If $\mathfrak{s}_j$ is the S-relative cycle on $\mathscr{C}_j$ covering the loop of Pochhammer type with base points 0 and 1 then $(\varphi_j^{-1})_\# \mathfrak{s}_j$ is a S-relative cycle on $\mathscr{C}$ which covers a loop of Pochhammer type with (two) base points in $\{0, 1, 1/\lambda, \infty\}$. Hence every period of ω integrated over any S-relative cycle of such a type is a solution of (0.6) as pointed out in [2], §2.1.3.

Furthermore for $j = 1, 2, 3, 4$, morphisms φ_j are stable on $\{0, 1\}$ and then $(\varphi_j^{-1})_\# \mathfrak{s}_j = \mathfrak{s}$. This implies the equalities in list 1 which are the linear transformations of ${}_2F_1$.

It is, perhaps, surprising that the curve of equation

$$v^N = u^{N-A}(1-u)^{N+A-C}(1-\lambda u)^B$$

coming from $\mathscr{C}$ by transposition of A and B does not appear on list 2 even though by integral representation

$${}_2F_1\left[\begin{matrix} b, a \\ c \end{matrix}; \lambda\right] = k(a, c) \int_{\mathfrak{s}} du/v$$

and

$${}_2F_1\left[\begin{matrix} b, a \\ c \end{matrix}; \lambda\right] = {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right].$$

In Chapter III we shall see that a cohomological interpretation of the symmetric relation involves cohomology of surfaces.

Chapter II. The cohomology of the generalized hypergeometric function ${}_3F_2$

The generalized hypergeometric function ${}_3F_2$ is defined as the series

$${}_3F_2\left[\begin{matrix} a, b_1, b_2 \\ c_1, c_2 \end{matrix}; \lambda\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b_1)_n (b_2)_n}{n!\,(c_1)_n (c_2)_n} \lambda^n$$

where c_1, c_2 are non-negative integer numbers and $|\lambda| < 1$. If δ denotes $\lambda(d/d\lambda)$ then ${}_3F_2$ satisfies the third order linear differential equation

$$(2.1) \qquad [\delta(\delta + c_1 - 1)(\delta + c_2 - 1) - \lambda(\delta + a)(\delta + b_1)(\delta + b_2)]\, {}_3F_2 = 0.$$

Equation (2.1) is of Fuchsian type with regular singularities $\lambda = 0$, 1 and ∞.

To state the cohomological situation we consider the parameters a, b_1, b_2, c_1, c_2 as rational numbers and the integer numbers A, B_1, B_2, C_1, C_2

and N are chosen such that

$$a = A/N,\ b_1 = B_1/N,\ b_2 = B_2/N,\ c_1 = C_1/N,\ c_2 = C_2/N$$
$$\text{g.c.d.}\,(A, B_1, B_2, C_1, C_2, N) = 1,$$

and $A, B_1, B_2, C_1, C_2, B_1 - C_1, B_2 - C_2 \not\equiv 0 \mod N$.

We define

$$P(x, y, z; \lambda) = z^N - x^{N-B_1}(1-x)^{N+B_1-C_1}y^{N-B_2}(1-y)^{N+B_2-C_2}(1-\lambda xy)^A.$$

Then the family $\mathscr{S}$ of affine surfaces

$$\mathscr{S} = \{(x, y, z; \lambda) \in \mathbb{C}^3 \times S \mid P(x, y, z; \lambda) = 0,\ \ z \neq 0\}$$

is parametrized by

$$\pi: \mathscr{S} \longrightarrow S = \mathbb{C} \setminus \{0, 1\}$$

where S is the space of the parameter λ.

The family $\mathscr{S}$ has some properties analogous to those of $\mathscr{C}$ in Chapter I. Let $\theta = \varepsilon(1/N)$. There exists an automorphism Θ on $\mathscr{S}$ given by $(x, y, z) \to (x, y, \theta z)$. The corresponding morphism Θ^* acts on the complex $\Omega^\bullet = \Gamma(\mathscr{S}, \Omega^\bullet_{\mathscr{S}/S})$ and so $\Omega^\bullet$ splits into a direct sum of subcomplexes $\Omega^\bullet_j$ on which Θ^* acts via multiplication by θ^{-j}.

More precisely, if

$$\mathscr{L} = \mathcal{O}(S)[x, y, (x(1-x)y(1-y)(1-\lambda xy))^{-1}]$$

then $\Omega^\bullet_j$ is

$$0 \longrightarrow \frac{1}{z^j}\mathscr{L} \longrightarrow \left(\frac{1}{z^j}\mathscr{L}\right)dx \oplus \left(\frac{1}{z^j}\mathscr{L}\right)dy \longrightarrow \left(\frac{1}{z^j}\mathscr{L}\right)dx \wedge dy \longrightarrow 0$$

for $j = 0, \ldots, N - I$.

By functoriality Θ^* gives the following decomposition in cohomology

$$H^q(\Omega^\bullet) = \bigoplus_{j=0}^{N-1} H^q(\Omega^\bullet_j), \qquad q = 0, 1, 2.$$

Let $D = \nabla_{\mathscr{S}/S}(d/d\lambda)$ then D is defined on Ω^o by

$$D(x) = D(y) = 0, \quad D(z) = -\frac{Axy}{N(1-\lambda xy)z}$$

and extended to $\Omega^\bullet$ in the appropriate way. Passing to the quotient D is defined on $H^q(\Omega^\bullet)$, $q = 0, 1, 2$. Furthermore $H^2(\Omega^\bullet)$ consists of N differential modules $H^2(\Omega^\bullet_j)$, $j = 0, \ldots, N-1$, since D commutes with Θ^*.

From now on $\omega = [dx \wedge dy/z] \in H^2(\Omega_1^{\bullet})$.

Proposition 2.1. *There exists* $\mathfrak{s} \in \Gamma(S \setminus (1, +\infty), \mathscr{H}_2(\mathscr{S}/S))$ *such that*

(2.2) $$ {}_3F_2\left[\begin{matrix} a, b_1, b_2 \\ c_1, c_2 \end{matrix}; \lambda\right] = k(b_1, c_1)k(b_2, c_2)\int_{\mathfrak{s}} \omega, $$

(2.3) $$ (d/d\lambda)\,{}_3F_2\left[\begin{matrix} a, b_1, b_2 \\ c_1, c_2 \end{matrix}; \lambda\right] = k(b_1, c)k(b_2, c_2)\int_{\mathfrak{s}} D\omega. $$

Proof. It is easily seen that for every $\lambda_o \in S \setminus (1, +\infty)$ there exist r, r', r'' positive nonzero real numbers such that

$$ D(I, r') \cap \{\lambda^{-1} | \lambda \in D(\lambda_o, r)\} = \emptyset, $$
$$ D(I, r'') \cap \{(\lambda t)^{-1} | \lambda \in D(\lambda_o, r), t \in D(I, r')\} = \emptyset. $$

Let $\lambda_o \in (0, 1)$ and γ a parametrization of a loop of Pochhammer type consisting of intervals $[\rho, 1 - \rho]$, circles $|t| = \rho$ and $|1 - t| = \rho$ with $0 < \rho < \min(r', r'')$ and real starting point $\gamma(0)$. Let $\mathfrak{s}$ the covering map

$$ \begin{array}{ccc} & \overset{\mathfrak{s}}{\nearrow} & \mathscr{S} \\ & & \downarrow p \\ I^4 & \longrightarrow & \mathbb{C}^2 \times S \end{array} $$

$$ (t_1, t_2, s_1, s_2) \longrightarrow (\gamma(t_1), \gamma(t_2), \lambda_o + rs_1\varepsilon(s_2)) $$

$P(x, y, z, \lambda) = (x, y, \lambda)$ and with $\mathfrak{s}(0, 0, 0, 0) = (\gamma(0), \gamma(0), z_o, \lambda_o) \in \mathbb{R}^4_{>o}$. If $U = \operatorname{int}(D(\lambda_o, r))$ then $\mathfrak{s} \in \mathscr{H}_4(\mathscr{S}, \pi^{-1}(S \setminus U))$ and so $\mathfrak{s} \in \Gamma(U, \mathscr{H}_2(\mathscr{S}/S))$. Furthermore $\mathfrak{s}$ may be extended to a section on $\{\lambda \in S | \lambda xy \neq 1, x, y \in D(I, \rho)\}$ and so to $S \setminus (1, +\infty)$ since ρ is arbitrary.

Let

$$ g(x, y; \lambda) = x^{b_1 - 1}(1 - x)^{c_1 - b_1 - 1}\, y^{b_2 - 1}(1 - y)^{c_2 - b_2 - 1}(1 - \lambda xy)^{-a} $$

then $g(x, y; \lambda)\, dx \wedge dy$ is the branch of $dx \wedge dy/z$ such that $g(\gamma(0), \gamma(0); \lambda_0) \in \mathbb{R}_{>0}$.

Then we shall have that for $\lambda \in U$

(2.4) $$ k(b_1, c_1)k(b_2, c_2)\int_{\mathfrak{s}} g(x, y; \lambda)\, dx \wedge dy $$

coincides with the series expansion of ${}_3F_2$ previously given. In fact, because of (0, 1) integral (2.4) becomes

$$k(b_1, c_1) \int_\gamma x^{b_1 - 1}(1 - x)^{c_1 - b_1 - 1} {}_2F_1 \left[\begin{matrix} a, b_2 \\ c_2 \end{matrix} ; \lambda x \right] dx$$

and then statement follows expanding in series ${}_2F_1$ and integrating term by term using the integral representation of the classical beta-function

$$\mathbb{B}(\alpha, \beta) = [(1 - \varepsilon(\alpha))(1 - \varepsilon(\beta))]^{-1} \int_\gamma t^{\alpha - 1}(1 - t)^{\beta - 1}\, dt, \tag{2.5}$$

γ the Pochhammer loop defined in the Introduction (see [2], § 1.6). For general $\lambda \in S \setminus (1, +\infty)$ integral (2.4) is the analytic continuation of ${}_3F_2$. Finally equality (2.3) comes by derivation with respect to λ of equality (2.2).

The differential modules $H^2(\Omega_j^\bullet)$ will be the main object of study of this chapter.

Theorem 2.2. *If j is an integer $1 \leq j < N$ and* g.c.d. $(j, N) = 1$ *then $H^2(\Omega_j^\bullet)$ is a free $\mathcal{O}(S)$-module of rank* 3.

Proof. The theorem follows by an elementary but somewhat tedious recursion based on reduction formulas coming from the equation of $\mathcal{S}$.

Observation 2.3. By Theorem 2.2 $H^2(\Omega_1^\bullet)$ has rank 3 over $\mathcal{O}(S)$. Using reduction formulas we can prove that a base of $H^2(\Omega_1^\bullet)$ over $\mathcal{O}(S)$ is given by three cohomology classes among

$$[dx \wedge dy/z], \; [x\, dx \wedge dy/z], \; [y\, dx \wedge dy/z] \quad \text{and} \quad [xy\, dx \wedge dy/z]$$

which, for simplicity, we call ω, $x\omega$, $y\omega$, $xy\omega$ respectively. Since

$$\begin{aligned} &d[(1 - x)(1 - y)(y\, dx + x\, dy)/z] \\ &\qquad = [b_1 - b_2 + (b_2 - c_1)x - (b_1 - c_2)y + (c_1 - c_2)xy]\, dx \wedge dy/z \end{aligned}$$

then there exists the following relation in cohomology

$$(b_1 - b_2)\omega + (b_2 - c_1)x\omega - (b_1 - c_2)y\omega + (c_1 - c_2)xy\omega = 0. \tag{2.6}$$

Therefore the choice of the base of $H^2(\Omega_1^\bullet)$ depends upon the values of b_1, b_2, c_1, c_2. Note that (2.6) is not trivial since its coefficients are not all zeros at the same time. Otherwise $b_1 = c_1$ and $b_2 = c_2$ in contradiction with our hypothesis.

Theorem 2.4. *Let $(\omega, y\omega, xy\omega)$ be a base over $\mathcal{O}(S)$ of $H^2(\Omega_1^\bullet)$. Then the connection matrix of $H^2(\Omega_1^\bullet)$ is*

$$(2.7)\qquad (1-\lambda)\nabla_{\mathscr{S}/S}(d/d\lambda)\begin{bmatrix}\omega\\ y\omega\\ xy\omega\end{bmatrix}=\begin{bmatrix} b_2 & b_1-c_2 & a-c_1\\ \dfrac{b_2}{\lambda} & b_1-\dfrac{c_2}{\lambda} & a-c_1\\ \dfrac{b_2}{\lambda} & \dfrac{b_1-c_2}{\lambda} & a-\dfrac{c_1}{\lambda}\end{bmatrix}\begin{bmatrix}\omega\\ y\omega\\ xy\omega\end{bmatrix}$$

Proof. First we write some formulas that we shall use in the demonstration ($D=\nabla_{\mathscr{S}/S}(d/d\lambda)$);

$$(2.8)\qquad D(x^{\alpha-1}y^{\beta-1}\,dx\wedge dy/z)=\frac{a\lambda x^{\alpha}y^{\beta}\,dx\wedge dy}{(1-\lambda xy)z}$$

$$(2.9)\qquad d[x^{\alpha}y^{\beta}(1-x)\,dy/z-x^{\alpha-1}y^{\beta}(1-y)\,dx/z]$$
$$=x^{\alpha-1}y^{\beta-1}[(b_2-1+\beta)+(b_1-c_2+\alpha-\beta)y+(a-c_1+1-\alpha)xy]\,dx\wedge dy/z$$
$$-(1-\lambda)x^{\alpha}y^{\beta}\frac{a\lambda}{I-\lambda xy}dx\wedge dy/z$$

$$(2.10)\qquad d[x^{\alpha}y^{\beta-1}(1-x)\,dy/z-x^{\alpha}y^{\beta}(1-y)\,dy/z]$$
$$=[(b_1-1+\alpha)+(b_2-c_1+\beta-\alpha)x+(a-c_2+1-\beta)xy]x^{\alpha-1}y^{\beta-1}\,dx\wedge dy/z$$
$$-(1-\lambda)x^{\alpha}y^{\beta}\frac{a\lambda}{1-\lambda xy}dx\wedge dy/z$$

with $\alpha,\beta\in\mathbb{Z}$ and $\alpha,\beta>0$.

Using (2.9) and (2.8) with $\alpha=\beta=1$

$$(2.11)\qquad (1-\lambda)D\omega=b_2\omega+(b_1-c_2)y\omega+(a-c_1)xy\omega,$$

from (2.10) and (2.8) with $\alpha=1,\beta=2$

$$(2.12)\qquad (1-\lambda)Dy\omega=b_1y\omega+(b_2-c_1+1)xy\omega+(a-c_2-1)xy^2\omega$$

and from (2, 9) and (2.8) with $\alpha=\beta=2$

$$(2.13)\qquad (1-\lambda)Dxy\omega=(b_2+1)xy\omega+(b_1-c_2)xy^2\omega+(a-c_1-1)x^2y^2\omega$$

where, as usual, $xy^2\omega=[xy^2\,dx\wedge dy/z]$ and $x^2y^2\omega=[x^2y^2\,dx\wedge dy/z]$.

From

$$d[-y(1-y)(1-\lambda xy)\,dx/z]$$
$$=[b_2-c_2y+\lambda(a-b_2-1)xy+\lambda(c_2-a+1)xy^2]\,dx\wedge dy/z$$

we obtain

$$(2.14)\qquad \lambda(a-c_2-1)xy^2\omega=b_2\omega-c_2y\omega+\lambda(a-b_2-1)xy\omega$$

In the same way, by

$$d[xy(1-x)(1-\lambda xy)\,dy/z]$$
$$= [b_1 y - c_1 xy + \lambda(a - b_1 - 1)xy^2 + \lambda(c_1 - a + 1)x^2y^2]\,dx \wedge dy/z$$

we have that

$$\lambda(a - c_1 - 1)x^2y^2\omega + \lambda(b_1 - c_2)xy^2\omega$$
$$= b_1 y\omega - c_1 xy\omega + \lambda(a - c_2 - 1)xy^2\omega$$

and then

$$(2.15)\qquad = b_2\omega + (b_1 - c_2)y\omega - [c_1 - \lambda(a - b_2 - 1)]xy\omega$$

because of (2.14).

Therefore the connection matrix (2.7) comes from (2.11), from (2.12) and (2.14) by elimination of $xy^2\omega$ and from (2.13) and (2.15) by elimination of $xy^2\omega$, $x^2y^2\omega$.

Observation 2.5. If $(\omega, x\omega, xy\omega)$ is a base of $H^2(\Omega_1^\bullet)$ then the computation of the connection matrix is the same. Such a matrix is that of Theorem 2.4 after the changing x, y, b_1, b_2, c_1, c_2 into y, x, b_2, b_1, c_2, c_1 respectively. The other cases are similar.

Proposition 2.6. *The cohomology class* ω *satisfies the third order generalized hypergeometric equation*

$$(2.16)\qquad [\nabla_{\mathscr{S}/S}(\delta)(\nabla_{\mathscr{S}/S}(\delta) + c_1 - 1)(\nabla_{\mathscr{S}/S}(\delta) + c_2 - 1)$$
$$- \lambda(\nabla_{\mathscr{S}/S}(\delta) + a)(\nabla_{\mathscr{S}/S}(\delta) + b_1)(\nabla_{\mathscr{S}/S}(\delta) + b_2)]\,\omega = 0$$

where $\delta = \lambda(d/d\lambda)$.

Proof. To make notation easier $D = \nabla_{\mathscr{S}/S}(d/d\lambda)$. Applying D to (2.11) and using the connection matrix (2.7)

$$(2.17)\qquad \lambda(1-\lambda)D^2\omega + [c_1 - \lambda(a + b_1 + b_2 + 1 - c_2)]D\omega - b_2(a + b_1 - c_2)\omega$$
$$= (b_1 - c_2)(a - c_2)y\omega$$

and finally applying D to (2.17) it follows the equation

$$\lambda^2(1-\lambda)D^3\omega + [c_1 + c_2 - 1 - \lambda(a + b_1 + b_2 + 3)]\lambda D^2\omega$$
$$+ [c_1c_2 - \lambda(a + b_1 + b_2 + ab_1 + ab_2 + b_1b_2 + 1)]D\omega - ab_1b_2\omega = 0$$

which is equivalent to (2.16).

Note that relations (2.7) still work even if $(\omega, y\omega, xy\omega)$ is not a base of $H^2(\Omega_1^\bullet)$ and then (2.16) always holds.

In analogy with the hypergeometric function we have that the differential equation (2.1) satisfied by ${}_3F_2$ is obtained by integration of (2.16) over the S-relative 2-cycle defined in Proposition 2.1.

Chapter III. Cohomological interpretation of degeneration formulas of ${}_3F_2$

§ 0. Introduction

In Chapter II we proved that for every value of the rational parameters a, b_1, b_2, c_1, c_2 the cohomological space $H^2(\Omega_1^\bullet)$ is a free $\mathcal{O}(S)$-module of rank 3 (Th. 2.2). We are now interested in the submodule M of $H^2(\Omega_1^\bullet)$ spanned by $\omega = [dx \wedge dy/z]$ and its derivatives $D^n\omega$, $n = 1, 2$ $(D = \nabla_{\mathcal{S}/S}(d/d\lambda))$. By Theorem 2.4 we have the following relations

$$(3.0.1) \quad \lambda(1-\lambda)D^2\omega + [c_1 - \lambda(a + b_1 + b_2 + 1 + c_2)]\,D\omega - b_2(a + b_1 - c_2)\omega = (b_1 - c_2)(a - c_2)y\omega$$

$$(3.0.2) \quad (1-\lambda)D\omega = b_2\omega + (b_1 - c_2)y\omega + (a - c_1)xy\omega$$

and their symmetric changing x, y, b_1, b_2, c_1, c_2 into y, x, b_2, b_1, c_2, c_1 respectively.

Therefore it is evident that $\operatorname{rank}_{\mathcal{O}(S)}M$ depends upon the parameters a, b_1, b_2, c_1 and c_2.

Theorem 3.0.1. *Let a, b_1, b_2, c_1, c_2 noninteger rational numbers such that $b_1 \neq c_1$ and $b_2 \neq c_2$.*

If $a \neq c_1$, $a \neq c_2$, $b_1 \neq c_2$ and $b_2 \neq c_1$ then $M = H^2(\Omega_1^\bullet)$ and so $\operatorname{rank}_{\mathcal{O}(S)}M = 3$. *Otherwise* $\operatorname{rank}_{\mathcal{O}(S)}M < 3$.

Proof. Suppose $a \neq c_1$, $a \neq c_2$, $b_1 \neq c_2$ and $b_2 \neq c_1$. Then $(\omega, y\omega, xy\omega)$ is a base of $H^2(\Omega_1^\bullet)$ (see (2.6)). By the previous (3.0.1) and (3.0.2) we have that

$$[\omega, D\omega, D^2\omega]^t = H\,[\omega, y\omega, xy\omega]^t$$

where

$$H = \left[\begin{array}{cc} 1 & 0 \\ \dfrac{b_2}{1-\lambda} & \dfrac{b_1 - c_2}{1-\lambda} \\ \dfrac{b_2[a+b_1-c_1-c_2+\lambda(1-b_2)]}{\lambda(1-\lambda)^2} & \dfrac{(b_1-c_2)[a-c_1-c_2+\lambda(1+b_1+b_2)]}{\lambda(1-\lambda)^2} \end{array}\right.$$

$$\begin{matrix} 0 \\ \dfrac{a-c_1}{1-\lambda} \\ \dfrac{(c_1-a)[c_1-\lambda(a+b_1+b_2+1-c_2)]}{\lambda(1-\lambda)^2} \end{matrix}\Bigg]$$

Since

$$\det H = \frac{(a-c_1)(a-c_2)(b_1-c_2)}{\lambda(\lambda-1)}$$

then $(\det H)^{-1} \in \mathcal{O}(S)$ and so $M = H^2(\Omega_1^\bullet)$. Otherwise we have that two parameters coincide. For example let $b_1 = c_2$. From (3.0.1) the class $D^2\omega$ may be written as a linear combination of ω and $D\omega$ with coefficients in $\mathcal{O}(S)$ and then $\text{rank}_{\mathcal{O}(S)} M < 3$.

The other cases are similar.

It is perhaps obvious to note that the reductions of $\text{rank}_{\mathcal{O}(S)} M$ is connected to the reductions of the generalized hypergeometric function ${}_3F_2\left[\begin{matrix} a, b_1, b_2 \\ c_1, c_2 \end{matrix}; \lambda\right]$ into a ${}_2F_1$. Cases $b_1 = c_1$ and $b_2 = c_2$ do not appear in the cohomological context since the integral representation of ${}_3F_2$ does not hold any more.

There are different types of situations.

First we consider that among the equalities $a = c_1$, $a = c_2$, $b_1 = c_2$, $b_1 = c_1$ only one occurs.

If $b_1 = c_2$ or $a = c_1$ then $\text{rank}_{\mathcal{O}(S)} M = 2$ and the relation among $D^2\omega$, $D\omega$ and ω is stated by (3.0.1) which is an hypergeometric equation. The cases $b_1 = c_1$ and $a = c_2$ are analogous to the cases $b_1 = c_2$ and $a = c_1$ respectively. The following § 3.1 is about cases $b_1 = c_2$ or $b_2 = c_1$, while § 3.2 concerns the cases $a = c_1$ or $a = c_2$.

When two equalities occur there are more possibilities. If $b_1 = c_2$ and $a = c_1$ then $\text{rank}_{\mathcal{O}(S)} M = 1$ because of (3.0.2) which becomes of binomial type.

Case $b_2 = c_1$ and $a = c_2$ is analogous. These two cases may be investigated using the results of § 3.1 and § 3.2.

When $b_1 = c_2$ and $b_2 = c_1$ then $\text{rank}_{\mathcal{O}(S)} M = 1$ and the relation between $D\omega$ and ω is given by (2.6) and (3.0.2).

The other cases $a = c_1 = c_2$, $a = c_1 = b_2$, and $a = c_2 = b_1$ are connected to the symmetry of ${}_2F_1\left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; \lambda\right]$ with respect to the transposition α and β (see § 3.2) and in these cases $\text{rank}_{\mathcal{O}(S)} M = 2$.

Finally we observe that the consistent cases in which three equalities occur were previously considered.

§ 1. Reduction cases $b_1 = c_2$ and $b_2 = c_1$

In this paragraph we shall state the connection between the curve $\mathscr{C}$ of Chapter I and the surface $\mathscr{S}$ of Chapter II when $B_1 = C_2$. At the same time we shall discuss in cohomological way the formula

$$ {}_3F_2\left[\begin{matrix} a, f, b \\ c, f \end{matrix}; \lambda\right] = {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] \tag{3.1.1}$$

which corresponds to a degeneration into ${}_2F_1$ of ${}_3F_2$ (case $b_1 = c_2$).

For this purpose we consider A, B, C, F and N as integer numbers such that g.c.d. $(A, B, C, F, N) = 1$ and $A, B, F, B-C, F-C, B-F \not\equiv 0 \mod N$ and define

$$P(x_1, x_2, x_3; \lambda) = x_3^N - x_1^{N-F}(1-x_1)^{N+F-C}x_2^{N-B}(1-x_2)^{N+B-F}(1-\lambda x_1 x_2)^A,$$
$$Q(y_1, y_2, y_3; \lambda) = y_3^N - y_1^{N+B-F}(1-y_1)^{N+F-C}y_2^{N-B}(1-y_2)^{N+B-C}(1-\lambda y_2)^A.$$

Therefore

$$X = \{(x_1, x_2, x_3; \lambda) \in \mathbb{C}^3 \times S \mid P(x_1, x_2, x_3; \lambda) = 0,\ x_3 \neq 0\},$$
$$Y = \{(y_1, y_2, y_3; \lambda) \in \mathbb{C}^3 \times S \mid Q(y_1, y_2, y_3; \lambda) = 0,\ y_3 \neq 0\}$$

are families of affine surfaces of the usual type (0.2)

$$X \longrightarrow S, \quad Y \longrightarrow S$$

with space of parameters $S = \mathbb{C} \setminus \{0, 1\}$.

Let $a = A/N$, $b = B/N$, $c = C/N$ and $f = F/N$.

Note that X coincides with $\mathscr{S}$ previously defined with $B_1 = C_2$ and so its cohomology was studied in Chapter II.

Therefore by Proposition 2.1

$${}_3F_2\left[\begin{matrix} a, f, b \\ c, f \end{matrix}; \lambda\right] = k(f, c)k(b, f)\int_{\mathfrak{s}} dx_1 \wedge dx_2/x_3,$$

$\mathfrak{s}$ a suitable 2-cycle.

In the same way, because of equality (2.5)

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] = k(b, c)k(f-b, c-b)\int_{\mathfrak{s}'} dy_1 \wedge dy_2/y_3$$

where the definition of $\mathfrak{s}'$ is analogous to that of the $\mathfrak{s}$ of Proposition 2.1.

Since

$$(1 - \varepsilon(c - b))k(b, c)k(f - b, c - b) = (1 - \varepsilon(f))k(f, c)k(b, f)$$

then (3.1.1) assumes the cohomological form

$$(1 - \varepsilon(c - b)) \int \omega_x = (1 - \varepsilon(f)) \int \omega_y \tag{3.1.2}$$

where $\omega_x = [dx_1 \wedge dx_2/x_3]$, $\omega_y = [dy_1 \wedge dy_2/y_3]$ and integration is done on appropriate 2-cycles.

Furthermore ω_x satisfies the second order equation (3.0.1) with $b_1 = c_2 = f$, $b_2 = b$ and $c_1 = c$ which is the hypergeometric equation (1.3).

Now we shall study the cohomology of family Y and relation (3.1.2) will be explained later.

Let

$$\mathscr{C}' = \{(u_1, y_1) \in \mathbb{C}^2 \mid u_1^N = y_1^{N+B-F}(1 - y_1)^{N+F-C},\ u_1 \neq 0\},$$
$$\mathscr{C}'' = \{(u_2, y_2; \lambda) \in \mathbb{C}^2 \times S \mid u_2^N = y_2^{N-B}(1 - y_2)^{N+B-C}(1 - \lambda y_2)^A,\ u_2 \neq 0\}.$$

If the integer numbers A, B, C, F and N satisfy the extra conditions g.c.d. $(A, B, C, N) = 1$ and g.c.d. $(B - F, F - C, N) = 1$ then $\mathscr{C}'$ and $\mathscr{C}''$ are families of curves of type (0.2). We denote by $\mathscr{C}' \times \mathscr{C}'' \to S$ the family of surfaces (of type (0.2) again) consisting for each λ of the affine product of $\mathscr{C}'$ and $\mathscr{C}''$.

Therefore there exists the commutative diagram

$$\begin{array}{ccc} \mathscr{C}' \times \mathscr{C}'' & \xrightarrow{\psi} & \mathscr{C}' \times \mathscr{C}'' \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & Y & \end{array}$$

where ψ and τ are the S-morphisms

$$\psi : (u_1, y_1, u_2, y_2) \longrightarrow (\theta u_1, y_1, \theta^{-1} u_2, y_2),$$
$$\tau : (u_1, y_1, u_2, y_2) \longrightarrow (y_1, y_2, u_1, u_2)$$

and $\theta = \varepsilon(1/N)$.

The automorphism ψ permits us to determine the structure of the complex $\Gamma(Y, \Omega^{\bullet}_{Y/S})$ and its cohomology.

Proposition 3.1.1. *Let* $\langle\psi\rangle$ *the group of order N generated by* ψ. *Then* $Y = (\mathscr{C}' \times \mathscr{C}'')/\langle\psi\rangle$, *i.e.*
Y is the topological quotient of $\mathscr{C}' \times \mathscr{C}''$ *with respect to* $\langle\psi\rangle$,
$\Gamma(Y, \Omega^{\bullet}_{Y/S}) \cong \Gamma(\mathscr{C}' \times \mathscr{C}'', \Omega^{\bullet}_{\mathscr{C}' \times \mathscr{C}''/S})^{\langle\psi^*\rangle}$, *the subcomplex of the* $\langle\psi^*\rangle$ *invariant forms.*

Proof. Direct verifications.

For the properties of the quotient varieties see [8].

Let

$$\text{(3.1.3)}\qquad \begin{aligned} &\Gamma(Y, \Omega^{\bullet}_{Y/S}) = \bigoplus_{j=0}^{N-1} Y_j^{\bullet} \\ &\Gamma(\mathscr{C}', \Omega^{\bullet}_{\mathscr{C}'}) = \bigoplus_{j=0}^{N-1} \mathscr{C}_j'^{\bullet},\ \Gamma(\mathscr{C}'', \mathscr{C}^{\bullet}_{\mathscr{C}''/S}) = \bigoplus_{j=0}^{N-1} \mathscr{C}_j''^{\bullet} \end{aligned}$$

the usual decompositions with respect to the automorphisms of $Y, \mathscr{C}', \mathscr{C}''$ induced by multiplication by θ.

From the Kunnet formula of the product $\mathscr{C}' \times \mathscr{C}''$ we have for $q = 0, 1, 2$

$$\begin{aligned} &H^q(\Gamma(\mathscr{C}' \times \mathscr{C}'', \Omega^{\bullet}_{\mathscr{C}' \times \mathscr{C}''/S})) \\ &\cong \bigoplus_{n=0}^{q} [H^n(\Gamma(\mathscr{C}', \Omega^{\bullet}_{\mathscr{C}'})) \otimes H^{q-n}(\Gamma(\mathscr{C}'', \Omega^{\bullet}_{\mathscr{C}''/S}))] \end{aligned}$$

and then by Proposition 3.1.1 and (3.1.3)

$$\text{(3.1.4)}\qquad \begin{aligned} H^q(\bigoplus_{j=0}^{N-1} Y_j^{\bullet}) &\cong [\bigoplus_{n=0}^{q} H^n(\bigoplus_{j=0}^{N-1} \mathscr{C}_j'^{\bullet}) \otimes H^{q-n}(\bigoplus_{j=0}^{N-1} \mathscr{C}_j''^{\bullet})]^{\langle\psi^*\rangle} \\ &\cong \bigoplus_{n=0}^{q} \bigoplus_{j=0}^{N-1} H^n(\mathscr{C}_j'^{\bullet}) \otimes H^{q-n}(\mathscr{C}_j''^{\bullet}) \end{aligned}$$

Therefore by (3.1.4) we have for $j = 0, \ldots, N-1$ and $q = 0, 1, 2$

$$\text{(3.1.5)}\qquad H^q(Y_j^{\bullet}) \cong \bigoplus_{n=0}^{q} H^n(\mathscr{C}_j'^{\bullet}) \otimes H^{q-n}(\mathscr{C}_j''^{\bullet})$$

Furthermore if $\eta \in H^q(\Gamma(Y, \Omega^{\bullet}_{Y/S}))$ and $\eta = \eta' \otimes \eta''$ for some $\eta' \in H^n(\Gamma(\mathscr{C}', \Omega^{\bullet}_{\mathscr{C}'}))$ and $\eta'' \in H^{q-n}(\Gamma(\mathscr{C}'', \Omega^{\bullet}_{\mathscr{C}''/S}))$ then

$$\text{(3.1.6)}\qquad \nabla_{Y/S}(d/d\lambda)\eta = \eta' \otimes \nabla_{\mathscr{C}''/S}(d/d\lambda)\eta''.$$

Proposition 3.1.2. *For* $j = 0, \ldots, N-1$

$$H^2(Y_j^{\bullet}) \cong H^1(\mathscr{C}_j'^{\bullet}) \otimes H^1(\mathscr{C}_j''^{\bullet})$$

as differential modules. In particular ω_y *satisfies equation* (1.1).

Proof. It follows from (3.1.5) and (3.1.6).

Proposition 3.1.3. *The families X and Y previously defined are birationally equivalent by mean of*

$$\varphi : X \longrightarrow Y$$

defind by

$$y_1 = \frac{x_1(1-x_2)}{1-x_1x_2}, \quad y_2 = x_1x_2, \quad y_3 = \frac{x_1x_3}{1-x_1x_2}.$$

Proof. Morphism φ has inverse φ^{-1}:

$$x_1 = y_1 + y_2 - y_1y_2, x_2 = \frac{y_2}{y_1+y_2-y_1y_2}, x_3 = \frac{(1-y_2)y_3}{y_1+y_2-y_1y_2}.$$

It is easily shown that by restriction

$$\varphi: X\setminus\{x_1x_2=1\} \longrightarrow Y\setminus\{y_1+y_2=y_1y_2\}$$

is an isomorphism and then Proposition is proved.

If $X' = X\setminus\{x_1x_2=1\}$ and $Y' = Y\setminus\{y_1+y_2=y_1y_2\}$ then $X' \to S$ and $Y' \to S$ are surfaces of type (0.2).

Let $\xi_x: X' \subsetneq X$ and $\xi_y: Y' \subsetneq Y$ the inclusion maps of X' and Y' in X and Y respectively.

Theorem 3.1.4. *Let* $q = 0, 1, 2.$
Then

$$\varphi^*: H^q(\Gamma(Y', \Omega^{\bullet}_{Y'/S})) \longrightarrow H^q(\Gamma(X', \Omega^{\bullet}_{X'/S}))$$

is an isomorphism and $\varphi^*\nabla_{Y'/S} = \nabla_{X'/S}\varphi^*$.
For $V = X$ *and* Y

$$\xi_V^*: H^q(\Gamma(V, \Omega^{\bullet}_{V/S})) \longrightarrow H^q(\Gamma(V', \Omega^{\bullet}_{V'/S}))$$

is a monomorphism.

Proof. The statement is an easy consequence of Prop. 3.1.3 and explicit verifications.

Corollary 3.1.5. *The cohomology classes* ω_x *and* ω_y *satisfy the same equation of second order, namely the hypergeometric equation* (1.1).

Proof. It comes because $\varphi^*(dy_1 \wedge dy_2/y_3) = dx_1 \wedge dx_2/x_3$.

Observation 3.1.6. Prop. 3.1.3 does imply that $\varphi^*\xi_y^* H^2(Y_1^{\bullet})$ is isomorphic to the $\mathcal{O}(S)$-module spanned by $\xi_x^*\nabla_{X/S}(d/d\lambda)^n\omega_x$, $n = 0, 1, 2$.

Finally formula (3.1.1) comes by integration of the equality $\varphi^*(dy_1 \wedge dy_2/y_3) = dx_1 \wedge dx_2/x_3$ over I^2 using the usual Euler's integral representations of ${}_2F_1$ (over I), of ${}_3F_2$ (over I^2) and beta function (over I) (see [2], formula (1) of pag. 9 and (10) of pag. 59).

To prove formula (3.1.2) we need a $\sum\in\mathscr{H}_2(X'/S)$ such that $\xi_{X^\#}\sum = (1-\varepsilon(c-b))\mathfrak{s}$ where $\mathfrak{s}$ is the usual 2-cycle defined in Prop. 2.1. The construction of such a $\sum$ is complicated, so we don't give it here.

The reduction case $b_2 = c_1$ is treated in the same way by transposition of indexes 1 and 2.

§ 2. Reduction cases $a = c_1$ and $a = c_2$

In analogy with the previous paragraph let A, B, C, F and N integer numbers such that g.c.d. $(A, B, C, F, N) = 1$ and $A, B, F, B-C, A-F \not\equiv 0 \mod N$.

Let

$$P(x_1, x_2, x_3; \lambda) = x_3^N - x_1^{N-B}(1-x_1)^{N+B-C}x_2^{N-A}(1-x_2)^{N+A-F}(1-\lambda x_1 x_2)^F$$

$$Q(z_1, z_2, z_3; \lambda) = z_3^N - z_1^{N-B}(1-z_1)^{N+B-C}(1-\lambda z_1)^A z_2^{N-A}(1-z_2)^{N+A-F}.$$

Hence

$$X = \{(x_1, x_2, x_3; \lambda)\in\mathbb{C}^3\times S \mid P(x_1, x_2, x_3; \lambda) = 0,\ x_3 \neq 0\},$$

$$Z = \{(z_1, z_2, z_3; \lambda)\in\mathbb{C}^3\times S \mid Q(z_1, z_2, z_3; \lambda) = 0,\ z_3 \neq 0\}$$

are families of affine surfaces of type (0.2) over $S = \mathbb{C}\setminus\{0, 1\}$.

The family X is then the $\mathscr{S}$ of Chapther II with $A = C_2$ and so

$${}_3F_2\left[\begin{matrix} f, b, a \\ c, f \end{matrix}; \lambda\right] = k(b, c)k(a, f)\int_{\mathfrak{s}} dx_1 \wedge dx_2/x_3,$$

$\mathfrak{s}$ the usual 2-cycle.

Furthermore using equality (2.5) we have

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] = k(b, c)k(a, f)\int_{\mathfrak{s}'} dz_1 \wedge dz_2/z_3,$$

$\mathfrak{s}'$ a 2-cycle defined as $\mathfrak{s}$.

The treatment of this reduction case is completely the same of that of case $b_1 = c_2$.

The main result we state here is the following.

Theorem 3.2.1. *The families X and Z are birationally equivalent because of*

$$\varphi: X \longrightarrow Z$$

defined by

$$z_1 = x_1,\ z_2 = \frac{x_2(1-\lambda x_1)}{1-\lambda x_1 x_2},\ z_3 = \frac{x_3(1-\lambda x_1)}{(1-\lambda x_1 x_2)^2}.$$

Proof. The inverse map φ^{-1} is defined by

$$x_1 = z_1,\ x_2 = \frac{z_2}{1 - \lambda z_1(1 - z_2)},\ x_3 = \frac{z_3(1 - \lambda z_1)}{[1 - \lambda z_1(1 - z_2)]^2}$$

and the restriction

$$\varphi\colon X \setminus \{1 - \lambda x_1 = 0\} \longrightarrow Z \setminus \{1 - \lambda z_1(I - z_2) = 0\}$$

is an isomorphism.

As in the previous case we have that $\varphi^*(dz_1 \wedge dz_2/z_3) = dx_1 \wedge dx_2/x_3$ and so both $\omega_x = [dx_1 \wedge dx_2/x_3]$ and $\omega_z = [dz_1 \wedge dz_2/z_3]$ satisfy the equation (1.3).

Let $X' = X \setminus \{1 - \lambda x_1 = 0\}$. It is easily seen that the definition of $\mathfrak{s}$ in Prop. 2.1 still holds after restriction of X to X'. So by integration of ω_x over such a restricted $\mathfrak{s}(\mathfrak{s} \in \mathscr{H}_2(X'/S))$

$$\int_{\mathfrak{s}} \omega_x = \int_{\varphi^{\#}\mathfrak{s}'} \omega_x = \int_{\mathfrak{s}'} \varphi^* \omega_x = \int_{\mathfrak{s}'} \omega_z$$

which corresponds to the reduction case $a = c_2$.

In the same way case $a = c_1$ is treated.

The symmetric formula for the hypergeometric function can be cohomologically interpretated using reduction cases of ${}_3F_2$.

In fact we can consider family $\mathscr{S}$ of Chapter II with parameters A, B_1, B_2, C_1, C_2 equal to A, A, B, C, A respectively.

Hence both degeneration cases $b_1 = c_2$ and $a = c_2$ appear. Therefore the families of surfaces

$$\tilde{Y} = \{y_3^N - y_1^{N+B-A}(1 - y_1)^{N+A-C} y_2^{N-B}(1 - y_2)^{N+B-C}(1 - \lambda y_2)^A,\ y_3 \neq 0\},$$
$$\tilde{Z} = \{z_3^N - z_1^{N-A}(1 - z_1)^{N+A-C}(1 - \lambda z_1)^B z_2^{N-B}(1 - z_2)^{N+B-A},\ z_3 \neq 0\}$$

are birationally equivalent by means of the morphism $\tilde{\varphi}\colon \tilde{Y} \to \tilde{Z}$ coming by composition of morphisms of Prop. 3.1.3 and Th. 3.2.1

As a first consequence symmetric formula comes out from $\tilde{\varphi}^*(dz_1 \wedge dz_2/z_3) = dy_1 \wedge dy_2/y_3$ by integration over a suitable 2-cycle.

By Prop. 3.1.1 families $\tilde{Y}$ and $\tilde{Z}$ are quotient varieties (with respect to a finite group of automorphisms) of the affine product of a curve which does not depend upon λ and the hypergeometric curve $\mathscr{C}$ and $\mathscr{C}'$ respectively ($\mathscr{C}$ is defined in Chapter I and $\mathscr{C}'$ comes from $\mathscr{C}$ after transposition of A and B). Cohomology of $\mathscr{C}$ and $\mathscr{C}'$ are then related though $\tilde{\varphi}^*$.

As pointed out in [2], pag. 78, symmetric formula can be also cohomologically explained setting A, B_1, B_2, C_1 and C_2 equal to C, B, A, C, C respectively, i.e. using reduction cases $a = c_1$ and $a = c_2$.

Chapter IV. Quadratic transformations for hypergeometric function ${}_2F_1$

In [3] Goursat gave a complete list of the transformations of polynomial type for the hypergeometric functions. Those of quadratic type are all derived from

$$(4.1)\qquad {}_2F_1\left[\begin{matrix} a/2, (a+1)/2 \\ b+1/2 \end{matrix}; \left(\frac{\lambda}{\lambda-2}\right)^2\right] = (1-\lambda/2)^a {}_2F_1\left[\begin{matrix} a, b \\ 2b \end{matrix}; \lambda\right]$$

$$(4.2)\qquad {}_2F_1\left[\begin{matrix} a, \quad b \\ (a+b+1)/2 \end{matrix}; \lambda\right] = {}_2F_1\left[\begin{matrix} a/2, b/2 \\ (a+b+1)/2 \end{matrix}; 4\lambda(1-\lambda)\right]$$

by means of linear transformations.

The terminology "quadratic" stems from the fact that the variable λ undergoes a quadratic transformation. Rewriting (4.1) using Euler integral representation of ${}_2F_1$

$$\frac{\Gamma(b+1/2)}{\Gamma((a+1)/2)\Gamma(b-a/2)} \int_I t^{(a-1)/2}(1-t)^{b-a/2-1}(1-\lambda' t)^{-a/2}\,dt$$
$$= (1-\lambda/2)^a \frac{\Gamma(2b)}{[\Gamma(b)]^2} \int_I [t(1-t)]^{b-1}(1-\lambda t)^{-a}\,dt$$

where $\lambda' = \lambda^2/(\lambda-2)^2$.

The quotient of the two beta-factors in the previous equality is

$$\frac{\Gamma(2b)\Gamma((a+1)/2)\Gamma(b-a/2)}{[\Gamma(b)]^2\Gamma(b+1/2)} = 2^{2b-1}\frac{\Gamma((a+1)/2)\Gamma(b-a/2)}{\Gamma(b)\Gamma(1/2)}$$
$$= 2^{2b-1}\frac{\mathbb{B}(-a/2, (a+1)/2)}{\mathbb{B}(-a/2, b)}$$

because of multiplication formula

$$\Gamma(b)\Gamma(b+1/2) = 2^{1-2b}\Gamma(2b)\Gamma(1/2).$$

Then using the beta integral

$$(4.3)\qquad \int_{I^2} t_1^{-a/2-1}(1-t_1)^{b-1} t_2^{(a-1)/2}(1-t_2)^{b-a/2-1}(1-\lambda' t_2)^{-a/2}\,dt_1\,dt_2$$
$$= 2^{2b-a-1}(2-\lambda)^a \int_{I^2} x_1^{-a/2-1}(1-x_1)^{(a-1)/2}[x_2(1-x_2)]^{b-1}(1-\lambda x_2)^{-a}\,dx_1\,dx_2$$

Note that the results of Chapter 2.1 may be applied to the first integral of (4.3), explicitly by changing variables

$$t_1 = \frac{y_1(1-y_2)}{(1-y_1y_2)}, \qquad t_2 = y_1y_2$$

and equality (4.3) may be rewritten

$$(4.4) \qquad \int_{I^2} y_1^{-1/2}(1-y_1)^{b-1}y_2^{(a-1)/2}(1-y_2)^{-a/2-1}(1-\lambda' y_1y_2)^{-a/2}\,dy_1\,dy_2$$

$$= 2^{2b-a-1}(2-\lambda)^a \int_{I^2} x_1^{-a/2-1}(1-x_1)^{(a-1)/2}[x_2(1-x_2)]^{b-1}(1-\lambda x_2)^{-a}\,dx_1\,dx_2$$

So we have to consider two families of surfaces

$$X \longrightarrow S = \mathbb{C} \setminus \{0, 1, 2\}$$

$$X = \{(x_1, x_2, x_3; \lambda) \in \mathbb{C}^3 \times S \mid P(x_1, x_2, x_3; \lambda) = 0,\ x_3 \neq 0\}$$

$$P = x_3^{2N}(2-\lambda)^{2A}4^{2B-A-N} - x_1^{2N+A}(1-x_1)^{N-A}[x_2(1-x_2)]^{2N-2B}(1-\lambda x_2)^{2A}$$

and

$$Y \longrightarrow S' = \mathbb{C} \setminus \{0, 1\}$$

$$Y\{(y_1, y_2, y_3; \lambda') \in \mathbb{C}^3 \times S' \mid Q(y_1, y_2, y_3; \lambda') = 0,\ \ y_3 \neq 0\}$$

$$Q = y_3^{2N} - y_1^N(1-y_1)^{2N-2B}y_2^{N-A}(1-y_2)^{2N+A}(1-\lambda' y_1y_2)^A$$

where $a = A/N$ and $b = B/N$ for integer numbers A, B and N such that g.c.d. $(A, B, N) = 1$, g.c.d. $(A, 2, N) = 1$ and $A, B \not\equiv 0 \mod N$.

We now introduce a third family of surfaces Z which permits us to establish a relation between X and Y. Let

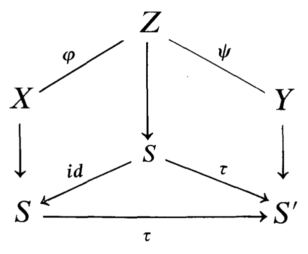
$$Z \longrightarrow S$$

$$Z = \{(z_1, z_2, z_3; \lambda) \in \mathbb{C}^3 \times S \mid R(z_1, z_2, z_3; \lambda) = 0,\ z_3 \neq 0\}$$

$$R = z_3^{2N} - [z_1(z_1-1)]^{N-A}(1-z_2^2)^{2N-2B}\left[1 - z_1\left(1 - \left(\frac{\lambda}{\lambda-2}\right)^2 z_2^2\right)\right]^A.$$

Then the following result holds.

Theorem 4.1. *There exists the commutative diagram*



where φ and ψ are rational morphisms defined by

$$\varphi: x_1 = \frac{1-z_1(1-\eta^2 z_2^2)}{[1-z_1(1-\eta z_2)]^2},\ x_2=(1-z_2)/2,\ x_3 = \frac{(1-\eta z_1)[1-z_1(1-\eta^2 z_2^2)]z_3}{2[1-z_1(1-\eta z_2)]^3}$$

$$\psi: y_1 = z_2^2,\ y_2 = \frac{z_1}{z_1-1},\ y_3 = \frac{z_1 z_3}{(1-z_1)^2},$$

$\eta = \lambda/(\lambda-2)$, *and* $\tau(\lambda) = \lambda'$.

Furthermore there are the birational S-automorphims

$$\alpha, \beta: Z \longrightarrow Z$$

$$\alpha(z_1, z_2, z_3) = \left(\frac{1-z_1}{1-z_1(1-\eta^2 z_2^2)}, z_2, \frac{z_2 z_3}{1-z_1(1-\eta^2 z_2^2)}\right),$$

$$\beta(z_1, z_2, z_3) = (z_1, -z_2, -z_3)$$

with the following properties

$$\alpha^2 = \text{Id},\ \beta^2 = \text{Id},\ \varphi\alpha = \varphi,\ \varphi\beta = \psi.$$

Proof. Direct computations.

First we shall show how to derive quadratic transformations.

The birational S-automorphisms α and β generate a group G. This group is isomorphic to the diedral group D_8 of eight elements.

More precisely

$$G = \{1, \xi, \xi^2, \xi^3, \alpha, \alpha\xi, \alpha\xi^2, \alpha\xi^3\}$$

where

$$\xi = \alpha\beta,\ \xi(z_1, z_2, z_3) = \left(\frac{1-z_1}{1-z_1(1-\eta^2 z_2^2)}, -z_2, \frac{\eta z_2 z_3}{1-z_1(1-\eta^2 z_2^2)}\right)$$

$$\xi^2(z_1, z_2, z_3) = (z_1, z_2, -z_3),$$

$$\xi^3(z_1, z_2, z_3) = \left(\frac{1-z_1}{1-z_1(1-\eta^2 z_2^2)}, -z_2, \frac{-\eta z_2 z_3}{1-z_1(1-\eta^2 z_2^2)}\right),$$

$$\alpha\xi = \beta,$$

$$\alpha\xi^2(z_1, z_2, z_3) = \left(\frac{1-z_1}{1-z_1(1-\eta^2 z_2^2)}, z_2, \frac{-\eta z_2 z_3}{1-z_1(1-\eta^2 z_2^2)}\right)$$

$$\alpha\xi^3(z_1, z_2, z_3) = (z_1, -z_2, z_3).$$

Let

$$\omega_x = \varphi^*(dx_1 \wedge dx_2/x_3) = \left(\frac{\eta z_2}{1 - z_1(1-\eta^2 z_2^2)} - 1\right)(dz_1 \wedge dz_2/z_3)$$

$$\omega_y = \psi^*(dy_1 \wedge dy_2/y_3) = 2\,dz_1 \wedge dz_2/z_3.$$

Then

$$2\omega_x + \omega_y - \omega' = 0 \tag{4.5}$$

with

$$\omega' = \frac{2\eta z_2}{1 - z_1(1-\eta^2 z_2^2)}\, dz_1 \wedge dz_2/z_3$$

and ω_x, ω_y, ω' satisfy

$$\alpha^*\omega_x = \omega_x,\ (\alpha\xi)^*\omega_y = \omega_y,\ (\alpha\xi^3)^*\omega' = \omega'.$$

Proposition 4.2. *The quadratic relation* (4.1) *comes from* (4.5) *by integration over a suitable* $\mathfrak{s} \in H_2(Z'|S)$ *where*

$$Z' = Z \setminus \{z_2(1 - z_1(1-\eta z_2))(1 - z_1(1+\eta z_2))(1-\eta^2 z_2^2) = 0\}.$$

Proof. Definition of $\mathfrak{s}$.

Consider $\lambda_o \in (-\infty, 0)$ i.e. $0 < \eta_o < 1$, $\eta_o = \lambda_o/(\lambda_o - 2)$. Let γ_1 a parametrization of a Pochhammer loop of type $(0+, (-\infty)+, 0-, (-\infty)-)$ (it means loops around 0, $-\infty$ of positive or negative directions according to signs $+$ or $-$) consisting of intervals $[-(r')^{-1}, -r']$, circles $|t| = r'$, $|t^{-1}| = r'$ with $0 < r' < (1-\eta)^{-1}$ and $\gamma_1(0) = -1$.

Let γ_2 a parametrization of a loop $(1+, (-1)+, 1-, (-1)-)$ consisting of intervals $[-1+r'', -r'']$, $[r'', 1-r'']$, circles $|1+t| = r''$, $|1-t| = r''$ and semicircles $|t| = r''$, $\operatorname{Im} t \geq 0$ with $0 < r'' < 1/2$, $\gamma_2(0) = 1/2$.

It is easily seen that there exist r' and r'' such that the following map is defined

$$I^2 \longrightarrow V$$

$$(t_1, t_2) \longrightarrow (\gamma_1(t_1), \gamma_2(t_2))$$

$$V = \{(z_1, z_2) \in \mathbb{C}^2 | z_1 z_2 (1-z_1)(1-z_2^2)(1-\eta^2 z_2^2)(1 - z_1(1-\eta^2 z_2^2)) \cdot (1 - z_1(1-\eta z_2))(1 - z_1(1+\eta z_2)) \neq 0\}.$$

Then $\mathfrak{s}$ is the covering map

$$\begin{array}{ccc} & & Z' \\ & \overset{\mathfrak{s}}{\nearrow} & \downarrow p \\ I^4 & \longrightarrow & V \times S \end{array}$$

$$(t_1, t_2, s_1, s_2) \longrightarrow (\gamma_1(t_1), \gamma_2(t_2), \lambda_o + rs_1\varepsilon(s_2))$$

where $P(z_1, z_2, z_3; \lambda) = (z_1, z_2; \lambda)$, $r > 0$ suitably chosen and $\mathfrak{s}(0, 0, 0, 0) = (-1, 1/2, \underline{z}_3, \lambda_o)$ with $\underline{z}_3 > 0$. Therefore $\mathfrak{s} \in \mathscr{H}_4(Z', \pi^{-1}(S \setminus U))$ where $U = \operatorname{int}(D(\lambda_o, r))$ i.e. $\mathfrak{s} \in \mathscr{H}_2(Z'S)$.

Note that $(\alpha\varepsilon^3)_\# \mathfrak{s} = -\mathfrak{s}$.

Integrating (4.5) over $\mathfrak{s}$

$$2\int_{\mathfrak{s}} \omega_x + \int_{\mathfrak{s}} \omega_y - \int_{\mathfrak{s}} \omega' = 0$$

and then

$$2\int_{\mathfrak{s}} \omega_x + \int_{\mathfrak{s}} \omega_y = 0$$

since $(\alpha\xi^3)^* \omega' = \omega'$ and $(\alpha\xi^3)_\# \mathfrak{s} = -\mathfrak{s}$.

Then

$$2\int_{\mathfrak{s}_x} dx_1 \wedge dx_2/x_3 + \int_{\mathfrak{s}_y} dy_1 \wedge dy_2/y_3 = 0$$

where $\varphi_\# \mathfrak{s} = \mathfrak{s}_x$, $\mathfrak{s}_x$ a covering of

$$x_1 \in (1+, 0+, 1-, 0-)$$
$$x_2 \in (0+, 1+, 0-, 1-)$$

and $\varphi_\# \mathfrak{s} = -\mathfrak{s}_y$, $\mathfrak{s}_y$ a covering of

$$y_1 \in (1+, 0+, 1+, 0-, 1-, 0+, 1-, 0-)$$
$$y_2 \in (0+, 1+, 0-, 1-)$$

(the sign -1 comes from the twisting y_1 with y_2). Therefore the previous equality may be rewritten in terms of integrals over I^2

$$\int_{\mathfrak{s}_x} dx_1 \wedge dx_2/x_3 = -(1 - \varepsilon(b))^2 \varepsilon(a/2)(1 - \varepsilon(-a)) \int_{I^2} dx_1\, dx_2/x_3$$

$$\int_{\mathfrak{s}_y} dy_1 \wedge dy_2/y_3 = 2(1 - \varepsilon(b))^2 \varepsilon(a/2)(1 - \varepsilon(-a)) \int_{I^2} dy_1\, dy_2/y_3$$

and so (4.4) follows.

Proposition 4.3. *The cohomology classes* $[dx_1 \wedge dx_2/x_3]$ *and* $[dy_1 \wedge dy_2/y_3]$ *satisfy the same second order equation (which is the differential equation of the two functions in equality* (4.1)).

Proof. We recall some equalities:

$$\varphi^*(dy_1 \wedge dy_2/y_3) = 2\,dz_1 \wedge dz_2/z_3,$$
$$\varphi^*(dx_1 \wedge dx_2/x_3) = -(1+\alpha^*)\,dz_1 \wedge dz_2/z_3.$$

By §3.1 the cohomology class $[dy_1 \wedge dy_2/y_3]$ satisfies the same equation of the function on the left side of (4.1) and because of the previous equalities so $[dz_1 \wedge dz_2/z_3]$ does and therefore also $[dx_1 \wedge dx_2/x_3]$.

In the other way $[dx_1 \wedge dx_2/x_3]$ must satisfies the same equation of the function of the right side of (4.1) and then the two equations (after substitution $\lambda' = \lambda'(\lambda)$) must coincide because of irreducibility.

Theorem 4.4. *Let*

$$X' = X \setminus \{x_2 = 1/2\},$$
$$Z' = Z \setminus \{z_2(1-\eta z_2)(1 - z_1(1-\eta z_2)) = 0\}.$$

Then $X' \cong Z'/\langle\alpha\rangle$ *and* $Y \cong Z/\langle\beta\rangle$.

Proof. We shall prove $X' \cong Z'/\langle\alpha\rangle$ only since the proof of the other isomorphism is analogous. A straightforward calculation using the definitions of φ, X' and Z' establishes the surjectivity of $\varphi: Z' \to X'$.

As a first consequence of that

$$\varphi^*: \Omega^\bullet_{X'} \longrightarrow \Omega^\bullet_{Z'}$$

$(\Omega^\bullet_{X'} := \Gamma(X', \Omega^\bullet_{X'/S})$ and $\Omega^\bullet_{Z'} := \Gamma(z', \Omega^\bullet_{Z'/S}))$ is an injective morphism of complexes and

$$\varphi^*(\Omega^q_{X'}) \subseteq (\Omega^q_{Z'})^{\langle\alpha^*\rangle}, \quad q = 0, 1, 2.$$

We shall prove that for $q = 0, 1, 2$ equality holds in the previous relation, i.e. $(\Omega^\bullet_{X'}) \cong (\Omega^\bullet_{Z'})^{\langle\alpha^*\rangle}$ and so Theorem will be proved.

Case $q = 0$. Let $K(X')$, $K(Z')$ be the function fields of X' and Z' respectively. In view of the relation $\varphi\alpha = \varphi$

$$\varphi^* K(X') = \alpha^* \varphi^* K(X')$$

whence

$$\varphi^* K(X') \subseteq K(Z')^{\langle\alpha^*\rangle}.$$

Since $K(Z')$ is an algebraic extension of degree two over $\varphi^* K(X')$ then $\varphi^* K(X') = K(Z')^{\langle\alpha^*\rangle}$ because $K(Z')^{\langle\alpha^*\rangle} \neq K(Z')$.

By what we have just proved

$$\mathcal{O}(Z') \supset \mathcal{O}(Z')^{\langle\alpha^*\rangle} = K(Z')^{\langle\alpha^*\rangle} \cap \mathcal{O}(Z')$$
$$= \varphi^* K(X') \cap \mathcal{O}(Z') \supseteq \varphi^* \mathcal{O}(X') \cap \mathcal{O}(Z') = \varphi^* \mathcal{O}(X').$$

Since X' is smooth affine variety it follows that $\mathcal{O}(X')$ is integrally closed in $K(X')$. Since φ^* is injective $\varphi^*\mathcal{O}(X')$ is integrally closed in $\varphi^* K(X')$ and so too in $\mathcal{O}(Z')^{\langle\alpha^*\rangle} \supseteq \varphi^*\mathcal{O}(X')$.

Next $\mathcal{O}(Z')$ is finitely generated as a $\varphi^*\mathcal{O}(X')$-module, which will imply that $\mathcal{O}(Z')^{\langle\alpha^*\rangle}$ is also finitely generated as $\varphi^*\mathcal{O}(X')$-module and thus that $\mathcal{O}(Z')^{\langle\alpha^*\rangle} = \varphi^*\mathcal{O}(X')$.

Let $u = [1 - z_1(1 - \eta z_2)]$ then

$$\mathcal{O}(Z') = \mathcal{O}(S)[z_1, z_2, z_3, z_2^{-1}, z_3^{-1}, (1 - \eta z_2)^{-1}, u^{-1}]/(R).$$

From the equalities

$$z_2^{\pm 1} = \varphi^*(1 - 2x_2)^{\pm 1},$$
$$(z_3/u)^{\pm 1} = \varphi^*\left(x_3 \frac{2 - \lambda}{x_1(1 - \lambda x_2)}\right)^{\pm 1},$$
$$(1 - \eta z_2)^{-1} = \varphi^*[(1 - \lambda/2)(1 - \lambda x_2)^{-1}]$$

it follows that

$$\varphi^*\mathcal{O}(X') \supseteq \mathcal{O}(S)[z_2, z_3/u, z_2^{-1}, u/z_3, (1 - \eta z_2)^{-1}]/(R).$$

Let

$$r = 2\frac{\lambda - 1 - \lambda x_2}{(\lambda - 2)x_1}, \quad s = \frac{\lambda(1 - 2x_2)}{(\lambda - 2)x_2}$$

then $r, s, s^{-1} \in \mathcal{O}(X')$.

Furthermore

$$u^2 - \varphi^*(r)u + \varphi^*(s) = 0,$$
$$(u^{-1})^2 - \varphi^*(rs^{-1})u^{-1} + \varphi^*(s^{-1}) = 0$$

and so u, u^{-1} and $z_1 = (1 - u)/(1 - \eta z_2)$ are integral over $\varphi^*\mathcal{O}(X')$. Then case $q = 0$ is proved.

Cases $q = 1, 2$ easily follow from case $q = 0$.

Finally we shall derive quadratic relation (4.2). Our idea is to integrate (4.5) over a domain with the same properties of the previous $\mathfrak{s}$ in Prop. 4.2. The difference is that in this case we shall not integrate over a 2-cycle. Let $\tilde{Z} = \{(z_1, z_2, z_3; \lambda) \in \mathbb{C}^3 \times S \mid R(z_1, z_2, z_3; \lambda) = 0\}$. Consider $\lambda_o \in (-\infty, 0)$ and so $0 < \eta_o < 1$, $\eta_o = \lambda_o/\lambda_o - 2)$. Let γ_1', γ_2' parametrizations of $[0, (1 - \eta z_2)^{-1}]$ and $(-\infty, -1)$ respectively and L the covering map

$$\begin{array}{ccc} & & \tilde{Z} \\ & \overset{L}{\nearrow} & \downarrow p \\ I^4 & \longrightarrow & \mathbb{C}^2 \times S \end{array}$$

$$(t_1, t_2, s_1, s_2) \longrightarrow (\gamma_1'(t_1), \gamma_2'(t_2), \lambda_o + rs_1\varepsilon(s_2))$$

where $P(z_1, z_2, z_3; \lambda) = (z_1, z_2; \lambda)$, $r > 0$ suitably chosen, and $L(1/2, 1/2, 0, 0) = (\gamma_1'(1/2), \gamma_2'(1/2), \underline{z}_3, \lambda_o)$ with $\underline{z}_3 > 0$.

Let $\mathfrak{s}' = (1 - (\alpha\xi^3)_\#)(1 - \xi_\#^2)(1 + \alpha_\#)L$, where $(1 + \zeta_\#)L$ denotes the addition in the group of simplicial maps and so $\int_{(1+\zeta_\#)L} = \int_L + \int_{\zeta_\# L}$.

If ω_x, ω_y and ω' are defined as in (4.5) then

$$\int_{\mathfrak{s}'} \omega' = 0,$$

since $(\alpha\xi^3)^*\omega' = \omega'$, $(\alpha\xi^3)_\#\mathfrak{s}' = -\mathfrak{s}'$,

$$\int_{\mathfrak{s}'} \omega_y = 2\int_{(1-\xi_\#^2)(1+\alpha_\#)L} \omega_y = 4\int_{(1+\alpha_\#)L} \omega_y = 4\int_{(1-(\alpha\xi^2)_\#)L} \omega_y$$

because $(\alpha\xi^3)^*\omega_y = -\omega_y$, $(\xi^2)^*\omega_y = -\omega_y$, and

$$\int_{\mathfrak{s}'} \omega_x = \int_{(1-\xi_\#^2)(1+\alpha_\#)L} \omega_x = 2\int_{(1+\alpha_\#)L} \omega_x = 4\int_L \omega_x$$

since $(\alpha\xi^3)_\#(1 - \xi_\#^2)(1 + \alpha_\#)L = (1 - \alpha_\#)(1 - \xi_\#^2)\xi_\# L$ and $\alpha^*\omega_x = \omega_x$, $(\xi^2)^*\omega_x = -\omega_x$.

Hence by integration of (4.5) over $\mathfrak{s}'$

$$2\int_L \omega_x + \int_{(I-(\alpha\xi^2)_\#)L} \omega_y = 0.$$

That is

$$2\int_{\varphi_\# L} dx_1 \wedge dx_2/x_3 + \int_{\psi_\#(1-(\alpha\xi^2)_\#)L} dy_1 \wedge dy_2/y_3 = 0.$$

It is easily seen that $\varphi_\# L = L'$, L' a covering of $(x_1, x_2) \in [1, +\infty) \times [1, +\infty)$ while $\psi_\#(1 - (\alpha\xi^2)_\#)L = -L''$, L'' a covering of $(y_1, y_2) \in [1, +\infty) \times (-\infty, 0]$ (since $(1 - \alpha\xi^2)_\# L$ covers $(z_1, z_2) \in [0, 1] \times [-1, -\infty)$).

Therefore we have that

$$2\int_{x_1=1}^{+\infty}\int_{x_2=1}^{+\infty} dx_1\, dx_2/x_3 = \int_{y_1=1}^{+\infty}\int_{y_2=-\infty}^{0} dy_1\, dy_2/y_3$$

and so by substitution

$$x_1 = 1/u_1,\ x_2 = 1/u_2,$$
$$y_1 = (1 - v_1 + v_1 v_2)/(v_1 v_2),\ y_2 = v_1(v_2 - 1)/(1 - v_1 + v_1 v_2)$$

we have

$$4^b\left(\frac{2-\lambda}{2\lambda}\right)^a \int_{I^2} u_1^{-1/2}(1-u_1)^{(a-1)/2} u_2^{a-2b}(1-u_2)^{b-1}(1-u_2/\lambda)^{-a} du_1\, du_2$$
$$= (\lambda')^{-a/2}\int_{I^2} v_1^{a/2-b}(1-v_1)^{b-1} v_1^{a/2-b-1/2}(1-v_2)^{(a-1)/2}$$
$$\times\left(1 - v_2\frac{\lambda'-1}{\lambda'}\right)^{-a/2} dv_1\, dv_2.$$

By means of Euler beta and hypergeometric integrals

$$2^{2b-a}\frac{\Gamma(1/2)\Gamma((a+1)/2)\Gamma(a+1-2b)\Gamma(b)}{\Gamma(a/2+1)\Gamma(a+1-b)}\ {}_2F_1\left[\begin{matrix} a,\ a+1-2b \\ a+1-b\end{matrix};\lambda^{-1}\right]$$
$$= \frac{\Gamma(a/2-b+1)\Gamma(b)\Gamma((a+1)/2-b)\Gamma((a+1)/2)}{\Gamma(a/2+1)\Gamma(a+1-b)}$$
$$\times\ {}_2F_1\left[\begin{matrix} a/2,\ (a+1)/2-b \\ a+1-b\end{matrix};\frac{4(\lambda-1)}{\lambda^2}\right].$$

We can drop the two constant factors since they are equal by Gauss multiplication formula of gamma-function. After redefinition of parameters and variable as follows

$$\lambda = t^{-1},\ a = a',\ b = (a'-b'+1)/2$$

we obtain (4.2).

Observation 4.5. In § 3.1 the cohomology of the family Y is given in term of the cohomology of the usual hypergeometric curve. In the same way the cohomology of family X may be given.

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