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Existence of Nonoscillatory Solutions of nth Order Neutral Delay Differential Equations

By

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1. Introduction

In this paper, we consider the following NDDE (neutral delay differential equation)

$$(x(t) + cx(t-\tau))^{(n)} + f(t, x(t-\sigma(t))) = 0, \qquad t \geqslant t_0 \tag{1}$$

where c is a real number, $\tau > 0$, $n \geqslant 2$, $\sigma(t) \in C([t_0, +\infty), R^+)$, $t - \delta(t) \to +\infty (t \to +\infty)$, $f(t, x) \in C([t_0, +\infty) \times R, R)$.

Several papers[1-5] have discussed the existence of the nonoscillatory solutions of nth order DDE (delay differential equation) and obtained some interesting results. Recently, Grove, Kulenovic and Ladas[6] has given some sufficient conditions for the first order NDDE to have nonoscillatory solutions. But there are few paper dealing with the existence of nonoscillatory solutions of nth order NDDE. In this paper, the author gives some sufficient conditions for (1) to have nonoscillatory solutions.

Definition. A solution of (1) is called nonoscillatory if it is eventually positive or eventually negative.

In this paper, we shall use the following Kranoselskii fixed point theorem:

Theorem A (Kranoselskii). *Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Let U, $S: X \to \Omega$ satisfy the following conditions:*

(i) *$Ux + Sy \in X$ for any $x, y \in X$;*
(ii) *U is a contraction mapping;*
(iii) *S is completely continuous.*

Then $U + S$ has a fixed point in X.

2. Main results

Theorem 1. *Suppose that $|c| \neq 1$, $xf(t, x) > 0$ $(x \neq 0)$ and $|f(t, x)| \leqslant$*

$|f(t, y)|$ for $|x| \leqslant |y|$, $xy > 0$. If

$$\int^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} |f(s, K)|\, ds\, ds_1 \cdots\cdots ds_{n-1} < +\infty \tag{2}$$

for some $K \neq 0$, then (1) *has a bounded nonoscillatory solution.*

Proof. Without loss of generality, we assume that $K > 0$ (a similar argument holds for $K < 0$).

(a) Case $|c| < 1$. Take T enough large such that $t - \tau > t_0$, $t - \sigma(t) > t_0$ for $t \geqslant T$ and

$$\int_T^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, K)\, ds\, ds_1 \cdots\cdots ds_{n-1} < \frac{1}{4}(1 - |c|)K. \tag{3}$$

Let $T^* = \inf\{t - \tau,\ t - \sigma(t) \colon t \geqslant T\}$. We introduce the Banach space

$$C_B[T^*, +\infty) = \{x \colon x \in C([T^*, +\infty), R),\ \sup_{t \geqslant T^*} |x(t)| < +\infty\}.$$

$$\|x\| = \sup_{t \geqslant T^*} |x(t)|.$$

Set $X = \left\{x \in C_B[T^*, +\infty) \colon \dfrac{K}{2} \leqslant x(t) \leqslant K\right\}$.

Then X is a bounded convex and closed subset of $C_B[T^*, +\infty)$. Define two operators U and $S\colon X \to C_B[T^*, +\infty)$ by

$$(Ux)(t) = \begin{cases} \dfrac{3(1+c)K}{4} - cx(T - \tau), & T^* \leqslant t < T \\[2ex] \dfrac{3(1+c)K}{4} - cx(t - \tau), & t \geqslant T \end{cases}$$

and

$$(Sx)(t) = \begin{cases} 0, & T^* \leqslant t < T \\ \displaystyle\int_T^t \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}, & t \geqslant T \end{cases}$$

for n even,

$$(Sx)(t) = \begin{cases} \displaystyle\int_T^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}, & T^* \leqslant t < T \\ \displaystyle\int_t^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}, & t \geqslant T \end{cases}$$

for n odd.

We are going to demonstrate that U and S satisfy the conditions of Theorem A. Without loss of generality, we assume that n is even (the case that n is odd can be treated similarly).

(i) $Ux + Sy \in X$ for any $x, y \in X$.

Since $x, y \in X$, we have

$$0 < K/2 \leqslant x(t) \leqslant K,\ 0 < K/2 \leqslant y(t) \leqslant K. \tag{4}$$

By the condition that $|f(t, x)| \leqslant |f(t, y)|$ for $|x| \leqslant |y|$, $xy > 0$, we have

$$f(t, x(t - \sigma(t))) \leqslant f(t, K), \qquad \text{for } t \geqslant T; \tag{5}$$

$$f(t, y(t - \sigma(t))) \leqslant f(t, K), \qquad \text{for } t \geqslant T. \tag{6}$$

For $t \geqslant T$, $0 \leqslant c < 1$, by (3), (4), (6), we have

$$\begin{aligned}(Ux)(t) + (Sy)(t) &= \frac{3(1+c)K}{4} - cx(t-\tau) \\ &+ \int_T^t \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, y(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1} \\ &\geqslant \frac{3(1+c)K}{4} - cx(t-\tau) \geqslant \frac{3(1+c)K}{4} - c \cdot K = \left(\frac{3}{4} - \frac{c}{4}\right)K \geqslant K/2,\end{aligned}$$

$$\begin{aligned}(Ux)(t) + (Sy)(t) &= \frac{3(1+c)K}{4} - cx(t-\tau) \\ &+ \int_T^t \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, y(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1} \\ &\leqslant \frac{3(1+c)K}{4} - c \cdot \frac{K}{2} + \int_T^t \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, K)\, ds\, ds_1 \cdots\cdots ds_{n-1} \\ &\leqslant \frac{3(1+c)K}{4} - c \cdot \frac{K}{2} + \frac{(1-c)K}{4} = K.\end{aligned}$$

Similarly, for $t \geqslant T$, $-1 < c < 0$, we have

$$(Ux)(t) + (Sy)(t) \geqslant \frac{3(1+c)K}{4} - c \cdot \frac{K}{2} = \left(\frac{3}{4} + \frac{c}{4}\right)K \geqslant \frac{K}{2},$$

$$(Ux)(t) + (Sy)(t) \leqslant \frac{3(1+c)K}{4} - c \cdot K + \frac{(1+c)K}{4} = K.$$

For $T^* \leqslant t < T$, $|c| < 1$, it is easy to see that

$$\frac{K}{2} \leqslant (Ux)(t) + (Sy)(t) = (Ux)(T) + (Sy)(T) \leqslant K.$$

Then $Ux + Sy \in X$.

(ii) U is a contraction mapping.

Let $x, y \in X$. Then for $T^* \leqslant t < T$,

$$|(Ux)(t) - (Uy)(t)| = |c|\,|x(T-\tau) - y(T-\tau)| \leqslant |c| \sup_{t \geqslant T^*} |x(t) - y(t)|$$

and for $t \geqslant T$,

$$|(Ux)(t) - (Uy)(t)| = |c|\,|x(t-\tau) - y(t-\tau)| \leqslant |c| \sup_{t \geqslant T^*} |x(t) - y(t)|.$$

Thus $\| Ux - Uy \| \leqslant |c|\, \| x - y \|$.

Since $|c| < 1$, U is a contraction mapping.

(iii) S is completely continuous.

Let $x_k \in X$, $\| x_k - x \| \to 0$ $(k \to +\infty)$. Since X is closed, we have $x \in X$. Then

$$|(Sx_k)(t) - (Sx)(t)| \begin{cases} = 0 & \text{for } T^* \leqslant t < T, \\ \leqslant \displaystyle\int_T^t \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} |F_k(s)|\, ds\, ds_1 \cdots\cdots ds_{n-1} & \text{for } t \geqslant T, \end{cases}$$

where $F_k(s) = f(s, x_k(s - \sigma(s))) - f(s, x(s - \sigma(s)))$. Noting that (5) holds, we have

$$|F_k(s)| = |f(s, x_k(s - \sigma(s))) - f(s, x(s - \sigma(s)))| \leqslant 2f(s, K).$$

Using the Lebesgue dominated convergence theorem, we get

$$\lim_{k \to +\infty} \| Sx_k - Sx \| = 0.$$

Then S is continuous. Next, we prove that SX is relatively compact. It suffices to show that the family of functions $\{Sx : x \in X\}$ is uniformly bounded and equicontinuous on $[T^*, +\infty)$. It is easy to see that $\{Sx : x \in X\}$ is uniformly bounded. We need only to show the equicontinuity. For any $\varepsilon > 0$, take $T' \geqslant T$ enough large such that, for $t \geqslant T'$,

$$\int_{T'}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, K)\, ds\, ds_1 \cdots\cdots ds_{n-1} < \varepsilon.$$

Then, for $t_2 > t_1 \geqslant T'$, we have

$$|(Sx)(t_2) - (Sx)(t_1)| \leqslant \int_{t_1}^{t_2} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, K)\, ds\, ds_1 \cdots\cdots ds_{n-1} < \varepsilon$$

and for $T^* \leqslant t_1 < t_2 \leqslant T' + 1$, we have

$$|(Sx)(t_2) - (Sx)(t_1)| = \begin{cases} \int_{t_1}^{t_2} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}, & T \leqslant t_1 < t_2 \\ \int_{T}^{t_2} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}, & t_1 < T < t_2 \\ 0, & t_1 < t_2 \leqslant T \end{cases}$$

or

$$|(Sx)(t_2) - (Sx)(t_1)| \leqslant \begin{cases} \int_{t_1}^{t_2} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, K)\, ds\, ds_1 \cdots\cdots ds_{n-1}, & T \leqslant t_1 \leqslant t_2 \\ \int_{T}^{t_2} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, K)\, ds\, ds_1 \cdots\cdots ds_{n-1}, & t_1 < T < t_2 \\ 0, & T^* \leqslant t_1 < t_2 \leqslant T. \end{cases}$$

Since $t_1, t_2 \leqslant T' + 1$, there is $\delta > 0$ such that $|t_2 - t_1| < \delta$ imply

$$|(Sx)(t_2) - (Sx)(t_1)| < \varepsilon.$$

Therefore $\{Sx: x \in X\}$ is equicontinuous on $[T^*, +\infty)$ and SX is relatively compact. Since S is continuous and SX is relatively compact, S is completely continuous. By Kranoselskii fixed point theorem (Theorem A), we have a fixed point $x^* \in X$ of $U + S$. That means there is $x^* \in X$ such that

$$(U + S)x^* = x^*.$$

Then, we have

$$x^*(t) = \frac{3(1+c)K}{4} - cx^*(t - \tau) + \int_{T}^{t} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x^*(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1} \quad \text{for } t \geqslant T.$$

It is easy to see that $x^*(t)$ is a solution of (1), By $\frac{K}{2} \leqslant x^*(t) \leqslant K$, $x^*(t)$ is a bounded nonoscillatory solution of (1).

(b) Case $|c| > 1$. Take T enough large such that $t - \tau > t_0$, $t - \sigma(t) > t_0$ for $t \geqslant T$, and

$$\int_T^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, K)\, ds\, ds_1 \cdots\cdots ds_{n-1} < \frac{1}{4}(|c| - 1)K. \tag{7}$$

Let $T^* = \inf\{t - \tau, t - \sigma(t) \colon t \geqslant T\}$.

$$C_B[T^*, +\infty) = \{x;\, x \in C([T^*, +\infty), R),\, \sup_{t \geqslant T^*} |x(t)| < +\infty\},$$

$$\|x\| = \sup_{t \geqslant T^*} |x(t)|,$$

$$X = \left\{x \in C_B[T^*, +\infty),\, \frac{K}{2} \leqslant x(t) \leqslant K\right\}.$$

Define operators U and $S\colon X \to C_B[T^*, +\infty)$ by

$$(Ux)(t) = \begin{cases} \dfrac{3(1 + c)K}{4c} - \dfrac{1}{c} x(T + \tau), & T^* \leqslant t < T \\ \dfrac{3(1 + c)K}{4c} - \dfrac{1}{c} x(t + \tau), & t \geqslant T \end{cases}$$

and

$$(Sx)(t) = \begin{cases} 0, & T^* \leqslant t < T \\ \dfrac{1}{c} \displaystyle\int_{T+\tau}^{t+\tau} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}, & t \geqslant T \end{cases}$$

for n even,

$$(Sx)(t) = \begin{cases} \dfrac{1}{c} \displaystyle\int_{T+\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}, & T^* \leqslant t < T \\ \dfrac{1}{c} \displaystyle\int_{t+\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}, & t \geqslant T \end{cases}$$

for n odd.

By similar arguments as in case (a), we can prove that, for n even, there is $x^* \in X$ such that $(U + S)x^* = x^*$. Then we have

$$x^*(t) = \frac{3(1 + c)K}{4c} - \frac{1}{c} x^*(t + \tau)$$
$$+ \frac{1}{c} \int_{T+\tau}^{t+\tau} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1} \qquad \text{for } t \geqslant T,$$

or

$$cx^*(t-\tau) = \frac{3(1+c)K}{4} - x^*(t)$$

$$+ \int_{T+\tau}^{t} \int_{s_{n-1}}^{+\infty} \cdots\cdots \int_{s_1}^{+\infty} f(s, x(s, \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1} \qquad \text{for } t \geqslant T.$$

It follows that $x^*(t)$ is a solution of (1), By $\frac{K}{2} \leqslant x^*(t) \leqslant K$, $x^*(t)$ is bounded and nonoscillatory. For n odd, we can similarly prove that (1) has a bounded nonoscillatory solution. The proof of Theorem 1 is complete.

Remark. Since (2) and $\int^{+\infty} s^{n-1}|f(s, K)|\, ds < +\infty$ are equivalent, (2) of Theorem 1 can be replaced by $\int^{+\infty} s^{n-1}|f(s, K)|\, ds < +\infty$.

Theorem 2. *Suppose that* $|c| \neq 1$, $\sigma(t) \leqslant \sigma < +\infty$, $xf(t, x) > 0$ $(x \neq 0)$ *and* $|f(t, x)| \leqslant |f(t, y)|$ *for* $|x| \leqslant |y|$, $xy > 0$. *If*

$$\int^{+\infty} |f(s, KR(s - \sigma(s)))|\, ds < +\infty \tag{8}$$

for some $K \neq 0$, *where* $R(t) = t^{n-1}$, *then* (1) *has an unbounded nonoscillatory solution.*

Proof. Without loss of generality, we assume that $K > 0$.

(a) Case $0 < |c| < 1$. Take $0 < c_1 < 1$ such that $0 < |c| < c_1 < 1$. Since

$$\lim_{t\to+\infty}\left(|c|\frac{R(t)}{R(t-\tau-\sigma)}\right) = |c| < c_1,$$

$$\lim_{t\to+\infty}\frac{R(t-\tau)}{R(t)} = 1 > 1 - \frac{1}{4|c|}(1-|c|)$$

and (8) holds, we can choose T enough large such that $T > |t_0| + 1 + \sigma + \tau$,

$$|c|\frac{R(t)}{R(t-\tau-\sigma)} < c_1 \qquad \text{for } t \geqslant T, \tag{9}$$

$$\frac{R(t-\tau)}{R(t)} > 1 - \frac{1}{4|c|}(1-|c|) \qquad \text{for } t \geqslant T \tag{10}$$

and

$$\int_{T}^{+\infty} f(s, KR(s-\sigma(s)))\,ds < \frac{1}{8}(1-|c|)K. \tag{11}$$

Let $T^* = T - \tau - \sigma$. By (9), we have

$$|c|\frac{R(T)}{R(T^*)} < c_1. \tag{9'}$$

We introduce the Banach space

$$C_R[T^*, +\infty) = \left\{x: x \in C([T^*, +\infty), R),\ \sup_{t \geqslant T^*} \frac{|x(t)|}{R^2(t)} < +\infty\right\}.$$

$$\|x\|_R = \sup_{t \geqslant T^*} \frac{|x(t)|}{R^2(t)}$$

Set $X = \left\{x: x \in C_R[T^*, +\infty), \frac{1}{2}KR(t) \leqslant x(t) \leqslant KR(t)\right\}$.

Then X is a bounded convex and closed subset of $C_R[T^*, +\infty)$. Define two operators U and $S: X \to C_R[T^*, +\infty)$ by

$$(Ux)(t) = \begin{cases} \dfrac{3(1+c)}{4}KR(t) - \dfrac{cx(T-\tau)}{R(T)}R(t), & T^* \leqslant t < T \\ \dfrac{3(1+c)}{4}KR(t) - cx(t-\tau), & t \geqslant T; \end{cases}$$

$$(Sx)(t) = \begin{cases} 0, & T^* \leqslant t < T \\ \displaystyle\int_T^t \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \int_{s_1}^{+\infty} f(s, x(s-\sigma(s)))\,ds\,ds_1 \cdots\cdots ds_{n-1}, & t \geqslant T. \end{cases}$$

(for $n = 2$,

$$(Sx)(t) = \begin{cases} 0, & T^* \leqslant t < T \\ \displaystyle\int_T^t \int_{s_1}^{+\infty} f(s, x(s-\sigma(s)))\,ds\,ds_1, & t \geqslant T \end{cases}$$
)

We are going to demonstrate that $U + S$ has a fixed point in X. For that, we show following three properties:

(i) $Ux + Sy \in X$ for any $x, y \in X$.

Since $x, y \in X$, $\frac{1}{2}KR(t) \leqslant x(t) \leqslant KR(t)$, $\frac{1}{2}KR(t) \leqslant y(t) \leqslant KR(t)$. By the conditions of Theorem 2, we have

$$f(t, x(t-\sigma(t))) \leqslant f(t, KR(t-\sigma(t))), \quad \text{for } t \geqslant T; \tag{12}$$

(13) $$f(t, y(t-\sigma(t))) \leqslant f(t, KR(t-\sigma(t))) \quad \text{for } t \geqslant T.$$

For $t \geqslant T$, $0 < c < 1$, by (8)–(12), we obtain

$$(Ux)(t) + (Sy)(t) \geqslant \frac{3(1+c)}{4} KR(t) - cx(t-\tau) \geqslant \frac{3(1+c)}{4} KR(t) - cKR(t-\tau)$$

$$\geqslant \frac{3(1+c)}{4} KR(t) - cKR(t) = \left(\frac{3}{4} - \frac{c}{4}\right) KR(t) \geqslant \frac{1}{2} KR(t).$$

$$(Ux)(t) + (Sy)(t) \leqslant \frac{3(1+c)}{4} KR(t) - \frac{c}{2} KR(t-\tau)$$

$$+ \int_T^t \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \frac{1}{8}(1-c) K ds_1 \cdots\cdots ds_{n-1}$$

$$\leqslant \frac{3(1+c)}{4} KR(t) - \frac{c}{2} K \cdot \frac{R(t-\tau)}{R(t)} R(t) + \frac{1}{8} \cdot (1-c) K \frac{(t-T)^{n-1}}{(n-1)!}$$

$$\leqslant \frac{3(1+c)}{4} KR(t) - \frac{c}{2} K \left(1 - \frac{1}{4c}(1-c)\right) R(t) + \frac{1}{8}(1-c) KR(t) = KR(t).$$

Similarly, for $t \geqslant T$, $-1 < c < 0$, by (8)–(12), we have

$$(Ux)(t) + (Sy)(t) \geqslant \frac{3(1+c)}{4} KR(t) - \frac{c}{2} KR(t-\tau)$$

$$\geqslant \frac{3(1+c)}{4} KR(t) - \frac{c}{2} K \cdot \frac{R(t-\tau)}{R(t)} R(t)$$

$$\geqslant \frac{3(1+c)}{4} KR(t) - \frac{c}{2} K \left(1 + \frac{1}{4c}(1+c)\right) R(t) = \left(\frac{5}{8} + \frac{c}{8}\right) KR(t)$$

$$\geqslant \frac{1}{2} KR(t).$$

$$(Ux)(t) + (Sy)(t) \leqslant \frac{3(1+c)}{4} KR(t) - cKR(t-\tau)$$

$$+ \int_T^t \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \frac{1}{8}(1+c) K ds_1 \cdots\cdots ds_{n-1}$$

$$\leqslant \frac{3(1+c)}{4} KR(t) - cKR(t) + \frac{1}{8}(1+c) K \cdot \frac{(t-T)^{n-1}}{(n-1)!}$$

$$\leqslant \frac{3(1+c)}{4} KR(t) - cKR(t) + \frac{1}{8}(1+c) KR(t) = \left(\frac{7}{8} - \frac{c}{8}\right) KR(t) \leqslant KR(t).$$

For $T^* \leqslant t < T$, $0 < |c| < 1$, noting that $\frac{1}{2}KR(t) \leqslant (Ux)(t) + (Sy)(t) \leqslant KR(t)$ ($t \geqslant T$), we obtain

$$(Ux)(t) + (Sy)(t) = ((Ux)(T) + (Sy)(T)) \cdot \frac{R(t)}{R(T)} \leqslant \frac{1}{2}KR(T) \cdot \frac{R(t)}{R(T)} = \frac{1}{2}KR(t),$$

$$(Ux)(t) + (Sy)(t) = ((Ux)(T) + (Sy)(T)) \cdot \frac{R(t)}{R(T)} \geqslant KR(T) \cdot \frac{R(t)}{R(T)} = KR(t).$$

Then $Ux + Sy \in X$.

(ii) U is a contraction mapping.

Let $x, y \in X$. Then, for $T^* \leqslant t < T$, by (9′), we have

$$\frac{|(Ux)(t) - (Uy)(t)|}{R^2(t)} = |c| \cdot \frac{|x(T-\tau) - y(T-\tau)|}{R(t)R(T)}$$
$$= |c| \frac{|x(t-\tau) - y(T-\tau)|}{R^2(T-\tau)} \cdot \frac{R^2(T-\tau)}{R(t)R(T)} \leqslant |c| \frac{|x(T-\tau) - y(T-\tau)|}{R^2(T-\tau)} \cdot \frac{R(T)}{R(T^*)}$$
$$\leqslant c_1 \sup_{t \geqslant T^*} \frac{|x(t) - y(t)|}{R^2(t)}.$$

For $t \geqslant T$, we have

$$\frac{|(Ux)(t) - (Uy)(t)|}{R^2(t)} = |c| \cdot \frac{|x(t-\tau) - y(t-\tau)|}{R^2(t)} \leqslant |c| \frac{|x(t-\tau) - y(t-\tau)|}{R^2(t-\tau)}$$
$$\leqslant c_1 \sup_{t \geqslant T^*} \frac{|x(t) - y(t)|}{R^2(t)}.$$

Then $\|Ux - Uy\|_R \leqslant c_1 \|x - y\|_R$. Since $0 < c_1 < 1$, U is a contraction mapping.

(iii) S is completely continuous.

(The proof of this result is basically the same as the corresponding proof of Theorem 1 in [3].)

Let $x_k \in X$, $\|x_k - x\|_R \to 0$ ($k \to +\infty$). Then $x \in X$ and

$$|(Sx_k)(t) - (Sx)(t)| \begin{cases} = 0, & T^* \leqslant t < T \\ \leqslant \int_T^t \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \int_{s_1}^{+\infty} |G_k(s)|\, ds\, ds_1 \cdots\cdots ds_{n-1}, & t \geqslant T, \end{cases}$$

where

(14) $\quad |G_k(s)| = |f(s, x_k(s - \sigma(s))) - f(s, x(s - \sigma(s)))| \leqslant 2f(s, KR(s - \sigma(s)))$.

Thus, for $t \geqslant T$,

$$|(Sx_k)(t) - (Sx)(t)| \leqslant \int_T^t \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \int_T^{+\infty} |G_k(s)|\, ds\, ds_1 \cdots\cdots ds_{n-1}$$

$$\leqslant \left(\int_T^{+\infty} |G_k(s)|\, ds \right) \cdot \frac{(t-T)^{n-1}}{(n-1)!} \leqslant R(t) \int_T^{+\infty} |G_k(s)|\, ds;$$

for $T^* \leqslant t < T$, $|(Sx_k)(t) - (Sx)(t)| = 0$. It follows that

$$\| Sx_k - Sx \|_R \leqslant \sup_{t \geqslant T^*} R^{-1}(t) \int_T^{+\infty} |G_k(s)|\, ds.$$

Using the Lebesgue dominated convergence theorem and noting (14) hold, we get

$$\lim_{k \to +\infty} \| Sx_k - Sx \|_R = 0$$

and S is continuous. Next, we prove that SX is relatively compact. It suffices to show that the family of functions $\{R^{-2}Sx : x \in X\}$ is uniformly bounded and equicontinuous on $[T^*, +\infty)$. The uniform boundedness is trivial. We need only to show the equicontinuity. For any $\varepsilon > 0$, take $T' \geqslant T$ enough large such that, for $t \geqslant T'$,

$$\frac{1}{R(t)} < \frac{\varepsilon}{(1 - |c|)K}.$$

Then, by (11), (12), for $t_2 > t_1 \geqslant T'$,

$$|(R^{-2}Sx)(t_2) - (R^{-2}Sx)(t_1)| \leqslant (R^{-2}Sx)(t_2) + (R^{-2}Sx)(t_1)$$

$$\leqslant \sum_{i=1}^{2} \frac{1}{R^2(t_i)} \cdot \int_T^{t_i} \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \frac{1}{8}(1 - |c|)K\, ds\, ds_1 \cdots\cdots ds_{n-1}$$

$$\leqslant \sum_{i=1}^{2} \frac{1}{R^2(t_i)} \frac{1}{8}(1 - |c|)KR(t_i) = \frac{1}{8}(1 - |c|)K \sum_{i=1}^{2} \frac{1}{R(t_i)} < \frac{\varepsilon}{4} < \varepsilon.$$

For $T \leqslant t_1 < t_2 < T' + 1$,

$$|(R^{-2}Sx)(t_2) - (R^{-2}Sx)(t_1)|$$

$$\leqslant \left| \frac{1}{R^2(t_2)} - \frac{1}{R^2(t_1)} \right| \int_T^{t_2} \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \int_{s_1}^{+\infty} f(s, KR(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}$$

$$+ \frac{1}{R^2(t_1)} \int_{t_1}^{t_2} \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \int_{s_1}^{+\infty} f(s, KR(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}$$

$$\leqslant \left| \frac{1}{R^2(t_2)} - \frac{1}{R^2(t_1)} \right| \cdot \int_T^{t_2} \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \frac{1}{8}(1 - |c|)K\, ds_1 \cdots\cdots ds_{n-1}$$

$$+\frac{1}{R^2(t_1)}\cdot\int_{t_1}^{t_2}\int_T^{s_{n-1}}\cdots\cdots\int_T^{s_2}\frac{1}{8}(1-|c|)K\,ds_1\cdots\cdots ds_{n-1}.$$

For $t_1 < T < t_2 < T' + 1$,

$$|(R^{-2}Sx)(t_2)-(R^{-2}Sx)(t_1)| = |(R^{-2}Sx)(t_2)-0|$$
$$\leqslant \frac{1}{R^2(t_2)}\cdot\int_T^{t_2}\int_T^{s_{n-1}}\cdots\cdots\int_T^{s_2}\frac{1}{8}(1-|c|)K\,ds_1\cdots\cdots ds_{n-1}.$$

For $T^* \leqslant t_1 < t_2 < T$,

$$|(R^{-2}Sx)(t_2)-(R^{-2}Sx)(t_1)| = 0.$$

Thus, there is $\delta > 0$ such that for any $x\in X$

$$|(R^{-2}Sx)(t_2)-(R^{-2}Sx)(t_1)| < \varepsilon, \qquad \text{if } 0 < t_2 - t_1 < \delta.$$

If follows that $\{R^{-2}Sx\colon x\in X\}$ is equicontinuous on $[T^*, +\infty)$ and SX is relatively compact. Since S is continuous and SX is relatively compact, S is completely continuous. By Theorem A, there is $x^*\in X$ such that $(U+S)x^* = x^*$. It is easy to show $x^*(t)$ is an unbounded nonoscillatory solution of (1).

(b) Case $c = 0$. Take T enough large such that $T > |t_0| + 1 + \sigma + \tau$ and

$$\int_T^{+\infty} f(s, KR(s-\sigma(s)))\,ds < \frac{1}{4}K.$$

Let $T^* = T - \tau - \sigma$,

$$C_R[T^*, +\infty) = \left\{x\colon x\in C([T^*, +\infty), R),\ \sup_{t\geqslant T^*}\frac{|x(t)|}{R^2(t)} < +\infty\right\},$$

$$\|x\|_R = \sup_{t\geqslant T^*}\frac{|x(t)|}{R^2(t)},$$

$$X = \left\{x\colon x\in C_R[T^*, +\infty),\ \frac{1}{2}KR(t)\leqslant x(t)\leqslant KR(t)\right\}.$$

Define operators U and $S\colon X\to C_R[T^*, +\infty)$ by

$$(Ux)(t) = \frac{3}{4}KR(t);$$

$$(Sx)(t)$$
$$=\begin{cases} 0, & T^*\leqslant t < T\\ \displaystyle\int_T^{t}\int_T^{s_{n-1}}\cdots\cdots\int_T^{s_2}\int_{s_1}^{+\infty} f(s, x(s-\sigma(s)))\,ds\,ds_1\cdots\cdots ds_{n-1}, & t\geqslant T.\end{cases}$$

By similar arguments as in the case (a), we can prove that there is $x^* \in X$ such that $(U + S)x^* = x^*$. It is easy to show $x^*(t)$ is an unbounded solution of (1).

(c) Case $|c| > 1$. Take c_1 such that $0 < \frac{1}{|c|} < c_1 < 1$ and take T enough large such that $T > |t_0| + 1 + \sigma + \tau$ and

$$\frac{1}{|c|} \frac{R^2(t+\tau)}{R^2(t-\tau-\sigma)} < c_1 < 1 \qquad \text{for } t \geqslant T,$$

$$\frac{R(t+\tau)}{R(t)} < 1 + \frac{1}{4}(|c| - 1) \qquad \text{for } t \geqslant T,$$

$$\int_T^{+\infty} f(s, KR(s - \sigma(s)))\, ds < \frac{1}{8}(|c| - 1) K.$$

Let $T^* = T - \tau - \sigma$.

$$C_R[T^*, +\infty) = \left\{x: x \in C([T^*, +\infty), R),\ \sup_{t \geqslant T^*} \frac{|x(t)|}{R^2(t)} < +\infty\right\}.$$

$$\|x\|_R = \sup_{t \geqslant T^*} \frac{|x(t)|}{R^2(t)},$$

$$X = \left\{x: x \in C_R[T^*, +\infty),\ \frac{1}{2} KR(t) \leqslant x(t) \leqslant KR(t)\right\}.$$

Define operators U and $S: X \to C_R[T^*, +\infty)$ by

$$(Ux)(t) = \begin{cases} \dfrac{3(1+c)}{4c} KR(t) - \dfrac{1}{c} \dfrac{x(T+\tau)}{R(T)} R(t), & T^* \leqslant t < T \\ \dfrac{3(1+c)}{4c} KR(t) - \dfrac{1}{c} x(t+\tau), & t \geqslant T; \end{cases}$$

$$(Sx)(t) = \begin{cases} 0, & T^* \leqslant t < T \\ \dfrac{1}{c} \displaystyle\int_{T+\tau}^{t+\tau} \int_T^{s_{n-1}} \cdots\cdots \int_T^{s_2} \int_{s_1}^{+\infty} f(s, x(s - \sigma(s)))\, ds\, ds_1 \cdots\cdots ds_{n-1}, & t \geqslant T. \end{cases}$$

By a similar arguments as in the case (a) of the proof of this Theorem and as in the case (b) of the proof of Theorem 1, we can prove that there is $x^* \in X$ such that $(U + S)x^* = x^*$. It is easy to show $x^*(t)$ is an unbounded nonoscillatory solution of (1). The proof of Theorem 2 is complete.

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