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Oscillation Results for Second Order Neutral Differential Equations*

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Introduction

In this paper we deal with the oscillatory behaviour of the neutral differential equation

$$[y(t) - cy(t-\tau)]'' + p(t)f(y(t-\sigma(t))) = 0 \tag{1}$$

under the assumption

(H) c and τ are positive numbers;
p and $\sigma \in C(\boldsymbol{R}_+, \boldsymbol{R}_+)$, $p(t) \not\equiv 0$, $t - \sigma(t)$ is increasing and tends to ∞ as $t \to \infty$, $\sigma(t) > \tau$;
$f \in C(\boldsymbol{R}, \boldsymbol{R})$ is increasing, $f(-x) = -f(x)$, $f(xy) \geq f(x)f(y)$, $xy > 0$,
$f(\infty) = \infty$, and $\dfrac{f(y)}{y} \to \infty$ or 1 as $y \to 0$.

The oscillation problem of equation (1) has received wide attention [1, 2, 4-9, 11, 12]. Much work has been done for the case where $c < 0$. In [7, 9, 11], the case $c > 0$ was studied for linear equations with constant coefficients and constant delay, some conclusions and conjectures were given, but the oscillation result specialized to the case where $c > 1$ is only a sufficient condition which guarantees that equation (1) has no bounded nonoscillatory solutions. In [4] the oscillatory problem of (1) was considered for the general form of equations, but the results still do not apply to the case $c \geq 1$.

The aim of this paper is to obtain some oscillation criteria for equation (1) for the case where $c \geq 1$ under the assumptions (H) and, along the way, we establish the conjectures in [11]. The results obtained in this paper can be easily extended to the more general form of equations

$$[r(t)(y(t) - cy(t-\tau))']' + p(t)f(y(t-\sigma(t))) = 0.$$

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Main results

For comparison purposes we mention the results for the case $0 < c < 1$ obtained in [4]:

Lemma 0. *Under the assumptions* (H), *if the equation*

$$z'' + p(t)f\left(\frac{\lambda(t-\sigma(t))}{t} z(t)\right) = 0 \tag{2}$$

is oscillatory for some $0 < \lambda < 1$, *then the nonoscillatory solutions of eq.* (1) *tend to zero as* $t \to \infty$.

Theorem 0. *In addition to the conditions of Lemma* 0, *assume further that*

$$\limsup_{t\to\infty} \int_{t-\sigma(t)+\tau}^{t} (u-(t-\sigma(t)+\tau))p(u)\,du > \begin{cases} c & \text{if } \dfrac{f(y)}{y} \longrightarrow 1, \ y \longrightarrow 0 \\ 0 & \text{if } \dfrac{f(y)}{y} \longrightarrow \infty, \ y \longrightarrow 0. \end{cases} \tag{3}$$

Then equation (1) *is oscillatory.*

Now we state our results below.

Definition: Let E be a subset of $\boldsymbol{R}_+$. Define

$$\rho_t(E) = \frac{\mu\{E \cap [0, t]\}}{t}, \quad \text{and} \quad \rho(E) = \limsup_{t\to\infty} \rho_t(E)$$

where μ is the Lebesgue measure.

Lemma 1. *Assume* (H) *holds and* $c = 1$. *Then the nonoscillatory solutions* $y(t)$ *of eq.* (1) *are bounded provided the equation*

$$z''(t) + p(t)f(Q(t)z(t)) = 0 \tag{4}$$

is oscillatory, where $Q(t) = \dfrac{1}{3\tau t}(t-\sigma(t))^2$.

Lemma 2. *Assume* (H) *holds and* $c > 1$. *Then the nonoscillatory solutions* $y(t)$ *of eq.* (1) *satisfy* $y(t) < cy(t-\tau)$ *eventually provided the following conditions hold*:

i) $$z''(t) + p(t)f(R(t,\lambda)z(t)) = 0 \tag{5}$$

is oscillatory for all $0 < \lambda < 1$, *where* $R(t, \lambda) = \dfrac{\lambda}{t} c^{\frac{t-\sigma(t)}{\tau}}$; *and one of the* *following*:

ii) $\int_0^\infty p(u)f(u-\sigma(u)+\tau)\,du=\infty$, *and* (6)

$$\limsup_{\substack{t\to\infty\\ t\notin E}} c_1^{-t/\tau}\int_0^t (t-u)p(u)f(u-\sigma(u)+\tau)\,du>0 \tag{7}$$

holds for some $c_1>c$ *and any set* E *with* $\rho(E)=0$; *or*

ii′) $\limsup_{\substack{t\to\infty\\ t\notin E}} c_1^{-t/\tau}\int_0^t (t-u)p(u)f(u-\sigma(u)+\tau)\,du=\infty$ (8)

holds for some $c_1>c$ *and any set* E *with* $\rho(E)=0$.

Corollary 1. *In addition to the assumptions of Lemma* 1, *assume further that* σ *is a positive constant, and*

$$\sum_{i=0}^{\infty}\int_{T+i\tau}^{T+i\tau+\alpha}(u-T)p(u)\,du=\infty \tag{9}$$

holds for any $T\in \boldsymbol{R}_+$, *and* $0<\alpha\le\tau$, *then all nonoscillatory solutions of eq.* (1) *tend to zero as* $t\to\infty$.

Corollary 2. *In addition to the assumptions of Lemma* 2, *assume further that* σ *is a positive constant,*

$$\int_t^\infty (u-t)p(u)\,du=\infty$$

and

$$\sum_{i=0}^{\infty}f(c^i)\int_{T+i\tau}^{T+i\tau+\alpha}(u-T)p(u)\,du=\infty \tag{10}$$

holds for any $T\in \boldsymbol{R}_+$ *and* $0<\alpha\le\tau$, *then all nonoscillatory solutions of eq.* (1) *tend to zero as* $t\to\infty$.

Remark: Corollaries 1 and 2 establish the conjectures in [11] for $n=2$ since (9) and (10) are true for the case that $p(t)$ is a positive constant. In general (i.e. for even order equations) the conjectures can be established by similar arguments as in this paper.

Theorem. *Assume* (H) *holds and* $c\ge 1$. *In addition to the conditions of Lemmas* 1 *and* 2 *for the cases* $c=1$ *and* $c>1$, *respectively, we assume* (3) *holds. Then eq.* (1) *is oscillatory.*

Proofs

Proof of Lemma 1: Assume the contrary, and without loss of generality let $y(t)$ be an eventually positive solution of eq. (1). Let $z(t) = y(t) - y(t-\tau)$. Then (1) becomes

$$z''(t) + p(t)f(y(t-\sigma(t))) = 0 \tag{11}$$

and $z''(t) \leq 0$, $t \geq t_0 \geq 0$. We claim $z'(t) > 0$, $t \geq t_0$. Otherwise, $z'(t) < 0$, $t \geq t_1 \geq t_0$. Then $z'(t) < -l < 0$, $t \geq t_1$. This gives that $z(t) = y(t) - y(t-\tau) \to -\infty$, contradicting that $y(t)$ is eventually positive.

a) Assume $z(t) > 0$, $t \geq t_2 \geq t_1$. From Erbe's lemma [3] we see that for any $0 < k < 1$ and $i = 0, 1, 2, \ldots$, there exists $T_i \geq t_0$ such that

$$z(t-\sigma(t)-i\tau) \geq \frac{k(t-\sigma(t)-i\tau)}{t} z(t), \quad t-\sigma(t) \geq T_i. \tag{12}$$

Without loss of generality we may assume $t_0 = T_0$. Then we can choose $T_i = T_0 + i\tau$ for a common k. In fact, from the proof of the lemma, it suffices to show that for $i = 0, 1, 2, \ldots$,

$$(1-k)(t-\sigma(t)-i\tau) \geq \tilde{T}_0 \triangleq (1-k)T_0, \quad t-\sigma(t) \geq T_i. \tag{13}$$

(13) is obviously true for $i = 0$. And if (13) is true for some i, then for $t - \sigma(t) \geq T_{i+1} = T_i + \tau$, we have

$$(1-k)(t-\sigma(t)-(i+1)\tau) \geq (1-k)(T_i - i\tau) = (1-k)T_0 = \tilde{T}_0.$$

Denote $R_{T_0} = \{t;\, t+\sigma(t) \geq T_0\}$. Then for any $t \in R_{T_0}$, there is a positive integer n satisfying

$$T_0 \leq t - \sigma(t) - n\tau < T_0 + \tau.$$

Since

$$\begin{aligned} y(t-\sigma(t)) &= \sum_{i=0}^{n-1} z(t-\sigma(t)-i\tau) + y(t-\sigma(t)-n\tau) \\ &\geq \sum_{i=0}^{n} z(t-\sigma(t)-i\tau) \end{aligned}$$

(here $\sum_{i=0}^{-1} = 0$), from eq. (1) we have

$$z''(t) + p(t)f\left(\sum_{i=0}^{n} z(t-\sigma(t)-i\tau)\right) \leq 0.$$

Using (12) we get

$$z''(t) + p(t)f\left(\frac{k}{t}\sum_{i=0}^{n}(t - \sigma(t) - i\tau)z(t)\right) \le 0,$$

i.e.,

$$z''(t) + p(t)f\left(\frac{k}{t}(n+1)\left(t - \sigma(t) - \frac{n}{2}\tau\right)z(t)\right) \le 0.$$

Since $n + 1 > \dfrac{t - \sigma(t) - T_0}{\tau}$, $n\tau \le t - \sigma(t) - T_0$, we get

$$z''(t) + p(t)f\left(\frac{k}{2\tau t}[(t - \sigma(t))^2 - T_0^2]z(t)\right) \le 0.$$

Choose $T \ge T_0$ large enough, then it becomes

$$z''(t) + p(t)f\left(\frac{1}{3\tau t}(t - \sigma(t))^2 z(t)\right) \le 0, \quad t \ge T.$$

Noting that $z(t), z(T)$ are upper and lower solutions of eq. (1), respectively, and using Theorem 7.4 in [10], we see there is a solution $y(t)$ satisfying $z(T) \le y(t) \le z(t)$, contradicting the fact that eq. (4) is oscillatory.

b) Assume $z(t) < 0$, $t \ge t_2 \ge t_1$. Then $y(t) - y(t - \tau) < 0$, $t \ge t_2$. It is obvious that $y(t)$ is a bounded solution since $y(t)$ is eventually positive.

For the proof of Lema 2 we shall need the following lemma.

Lemma 3. *Assume set $E \subset \boldsymbol{R}_+$ and $\rho(E) = \rho > 0$. Then for any $t_0 \in \boldsymbol{R}_+$ and integer n, there exists a $T \in [t_0, t_0 + \tau)$ such that $\{T + i\tau\}_{i=1}^{\infty}$ intersects E at least n times.*

Proof: Assume that the contrary holds, i.e., there exist a $t_0 \in \boldsymbol{R}_+$ and an integer N, such that $\{T + i\tau\}_{i=1}^{\infty}$ intersects E at most N times for any $T \in [t_0, t_0 + \tau)$. This implies that $\mu\{E\} < \infty$. But $\rho(E) = \rho > 0$ means there exist $t_n \to \infty$ such that $\rho_{t_n}(E) \ge \frac{\rho}{2} > 0$. Thus

$$\mu\{E \cap [0, t_n]\} \ge \frac{\rho}{2} t_n \longrightarrow \infty, \quad n \longrightarrow \infty,$$

and this is impossible.

Proof of Lemma 2: Assume the contrary, and without loss of generality let $y(t)$ be an eventually positive solution of eq. (1). Let $z(t) = y(t) - cy(t-\tau)$. Then (1) becomes (11) and $z''(t) \le 0$ eventually. There are three possibilities:

a) $z'(t) > 0,\ z(t) > 0$, b) $z'(t) < 0,\ z(t) < 0$, c) $z'(t) > 0,\ z(t) < 0$

eventually.

a) Assume $z'(t) > 0,\ z(t) > 0,\ t \ge t_0 \ge 0$. Then (12) holds and for any $t \in R_{T_0}$ defined as in the proof of Lemma 1, there is also a positive integer n satisfying

$$T_0 \le t - \sigma(t) - n\tau < T_0 + \tau.$$

Since

$$y(t-\sigma(t)) = \sum_{i=0}^{n-1} c^i z(t-\sigma(t)-i\tau) + c^n y(t-\sigma(t)-n\tau)$$
$$\ge \sum_{i=0}^{n} c^i z(t-\sigma(t)-i\tau),$$

from eq. (1) we have

$$z''(t) + p(t)f(\sum_{i=0}^{n} c^i z(t-\sigma(t)-i\tau)) \le 0.$$

(12) gives that

$$z''(t) + p(t)f\left(\frac{k}{t}\sum_{i=0}^{n} c^i (t-\sigma(t)-i\tau) z(t)\right) \le 0,$$

i.e.,

$$z''(t) + p(t)f\left[\left(\frac{k}{t}(t-\sigma(t))\frac{c^{n+1}-1}{c-1} - \frac{k\tau}{t}\sum_{i=1}^{n} ic^i\right)z(t)\right] \le 0. \tag{14}$$

Since

$$\sum_{i=1}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n-1} + c}{(c-1)^2},$$

we have

$$\frac{k}{t}(t-\sigma(t))\frac{c^{n+1}-1}{c-1} - \frac{k\tau}{t}\sum_{i=1}^{n} ic^i$$
$$= \frac{k}{(c-1)^2 t}[(t-\sigma(t))(c^{n+2} - c^{n+1} - c + 1)$$

$$
\begin{aligned}
&- \tau(nc^{n+2} - (n+1)c^{n+1} + c)] \\
&= \frac{k}{(c-1)^2 t}[(t - \sigma(t) - n\tau)c^{n+2} \\
&- (t - \sigma(t) - (n+1)\tau)c^{n+1} - (t - \sigma(t) + \tau)c + (t - \sigma(t))] \\
&\geq \frac{k}{(c-1)^2 t}[T_0 c^{n+2} - T_0 c^{n+1} - (t - \sigma(t) + \tau)c + (t - \sigma(t))] \\
&\geq \frac{1}{t} c^{n+2} \geq \frac{1}{t} c^{\frac{t-\sigma(t)-T_0+\tau}{\tau}} = \frac{\lambda}{t} c^{\frac{t-\sigma(t)}{\tau}}
\end{aligned} \tag{15}
$$

for some $0 < \lambda < 1$ if T_0 and t are sufficiently large. Substituting (15) into (14) we have

$$
z''(t) + p(t) f\left(\frac{\lambda}{t} c^{\frac{t-\sigma(t)}{\tau}} z(t)\right) \leq 0.
$$

Noting that $z(t)$, $z(T_0)$ are upper and lower solutions of (5), respectively, and using Theorem 7.4 in [10] we see there is a solution $y(t)$ satisfying $z(T_0) \leq y(t) \leq z(t)$, contradicting the fact that eq. (5) is oscillatory for all $\lambda > 0$.

b) Assume $z'(t) < 0$, $z(t) < 0$, $t \geq t_0 \geq 0$. Then $z(t) \leq -lt$, $t \geq t_0$, for some $l > 0$.

We claim $z(t) \geq -c_1^{t/\tau}$ essentially, where $c_1 > c$ is arbitrary, i.e., if $E = \{t: z(t) < -c_1^{t/\tau}\}$, then $\rho(E) = 0$. Otherwise, $\rho(E) = \rho > 0$. By Lemma 3, for any n, there exists a $T_1 \in [t_0, t_0 + \tau)$ such that $\{T_1 + i\tau\}_{i=1}^{\infty}$ intersects E at least n times. Assume $M = \max_{t \in [t_0, t_0+\tau]} \{y(t)\}$. Then if n is sufficiently large,

$$
\begin{aligned}
y(T_1 + n\tau) &\leq c^n y(T_1) + z(T_1 + n\tau) \leq c^n M - c_1^{\frac{T_1 + n\tau}{\tau}} \\
&= c^n M - c_1^{n + \frac{T_1}{\tau}} < 0,
\end{aligned}
$$

contradicting that $y(t) > 0$ eventually.

If ii) holds, then we can show that $z'(t) < -\mu$ for all $\mu > 0$ eventually. For otherwise, there exists a $\mu > 0$ such that $z' \geq -\mu$, $t \geq T_2$. From (11) and $y(t - \tau) \geq -\frac{1}{c} z(t)$, we get

$$
\begin{gathered}
z''(t) + p(t) f\left(-\frac{1}{c} z(t - \sigma(t) + \tau)\right) \leq 0 \\
z'(t) + \int_{T_2}^{t} p(u) f\left(-\frac{1}{c} z(u - \sigma(u) + \tau)\right) du \leq 0.
\end{gathered} \tag{16}
$$

Noting that $z(t-\sigma(t)+\tau) \le -l(t-\sigma(t)+\tau)$ we have

$$z'(t) + \int_{T_2}^{t} p(u) f\left(\frac{l}{c}(u-\sigma(u)+\tau)\right) du \le 0,$$

$$f\left(\frac{l}{c}\right) \int_{T_2}^{t} p(u) f(u-\sigma(u)+\tau)\, du \le -z'(t) \le \mu,$$

which is in contradiction with (6). Hence from (16) we see that for any $\mu > 0$, there is a T_μ such that

$$z(t) + \int_{T_\mu}^{t} (t-u) p(u) f\left(\frac{\mu}{c}(u-\sigma(u)+\tau)\right) du \le 0.$$

On $E^c \cap [T_\mu, \infty)$

$$-c_1^{t/\tau} + \int_{T_\mu}^{t} (t-u) p(u) f\left(\frac{\mu}{c}(u-\sigma(u)+\tau)\right) du \le 0,$$

$$f\left(\frac{\mu}{c}\right) c_1^{-\frac{t}{\tau}} \int_{T_\mu}^{t} (t-u) p(u) f(u-\sigma(u)+\tau)\, du \le 1.$$

Hence

$$c_1^{-\frac{t}{\tau}} \int_{T_\mu}^{t} (t-u) p(u) f(u-\sigma(u)+\tau)\, du \le \frac{1}{f\left(\frac{\mu}{c}\right)}. \tag{17}$$

contradicting (7) since $\mu > 0$ is arbitrary and $f(0) = 0$.

If ii′) holds. Then (17) holds with $\mu = l$, contradicting (8). The proof is complete.

c) Assume $z'(t) < 0$, $z(t) < 0$, $t \ge t_0 \ge 0$. Then $y(t) < cy(t-\tau)$ is obvious.

Proof of Corollary 1: If not, there exists an eventually positive solution $y(t)$ satisfying $\limsup_{t\to\infty} y(t) > 0$, and this can only occur when $z'' \le 0$, $z'(t) > 0$, and $z(t) < 0$, $t \ge t_0 \ge 0$, hence $z'(t) \to 0$, $z(t) \to 0$ as $t \to \infty$. If $\liminf_{t\to\infty} y(t) > 0$, then $y(t) \ge a > 0$, $t \ge t_1 \ge t_0$. Integrating (11) twice we get

$$z(t) + \int_{t}^{\infty} (u-t) p(u) f(a)\, du < 0.$$

Taking super limits on both sides as $t \to \infty$ we have

$$\limsup_{t\to\infty} \int_{t}^{\infty} (u-t) p(u)\, du \le 0,$$

which is in contradiction with (9). So

$$\limsup_{t\to\infty} y(t) > 0 \quad \text{and} \quad \liminf_{t\to\infty} y(t) = 0. \tag{18}$$

Then we can choose $t_2 > t_1 \geq t_0$ such that $y(t_2 - \sigma) > y(t_1 - \sigma)$. We claim

$$\liminf_{n\to\infty} y(t_2 - \sigma + n\tau) > 0. \tag{19}$$

In fact

$$y(t_2 - \sigma + n\tau) = \sum_{i=1}^{n} z(t_2 - \sigma + i\tau) + y(t_2 - \sigma)$$

and

$$y(t_1 - \sigma + n\tau) = \sum_{i=1}^{n} z(t_1 - \sigma + i\tau) + y(t_1 - \sigma).$$

Since $z(t_2 - \sigma + i\tau) \geq z(t_1 - \sigma + i\tau)$ for $i = 1, 2, \ldots, n$, and

$$\liminf_{n\to\infty} y(t_1 - \sigma + n\tau) \geq 0,$$

we have

$$\liminf_{n\to\infty} y(t_2 - \sigma + n\tau) \geq y(t_2 - \sigma) - y(t_1 - \sigma) > 0.$$

Now, choose $t_0 \leq t_1 < t_2 < t_3$ such that for any $T \in [t_2, t_3]$,

$$y(t_1 - \sigma) < y(t_2 - \sigma) \leq y(T - \sigma).$$

From the above discussion, we see that (19) holds, i.e., there exists a $\mu > 0$ such that $y(t_2 - \sigma + n\tau) \geq \mu$ for all n. It is easy to see that for $T \in [t_2, t_3]$.

$$\begin{aligned} y(T - \sigma + n\tau) &= \sum_{i=1}^{n} z(T - \sigma + i\tau) + y(T - \sigma) \\ &\geq \sum_{i=1}^{n} z(t_2 - \sigma + i\tau) + y(t_2 - \sigma) \\ &= y(t_2 - \sigma + n\tau) \geq \mu. \end{aligned}$$

From (11) we have

$$- z'(s) + \int_s^t p(u)f(y(u-\sigma))\,du \le 0, \qquad t_0 \le s \le t,$$

$$z(t_0) + \int_{t_0}^t (u - t_0)p(u)f(y(u-\sigma))\,du \le 0, \qquad t_0 < t.$$

Hence

$$z(t_0) + f(\mu)\sum_{i=0}^n \int_{t_2+i\tau}^{t_3+i\tau} (u - t_0)p(u)\,du \le 0,$$

and then

$$z(t_0) + f(\mu)\sum_{i=0}^n \int_{t_2+i\tau}^{t_3+i\tau} (u - t_2)p(u)\,du \le 0,$$

contradicting (9).

Proof of Corollary 2: If not, similar to the proof of Corollary 1 we see there exists an eventually positive solution $y(t)$ satisfying (18). From the proof of lemma 2 we see this can only occur when $z''(t) \le 0$, $z'(t) > 0$ and $z(t) < 0$, $t \ge t_0 \ge 0$, choose $t_2 > t_1 \ge t_0$ such that $y(t_2 - \sigma) > y(t_1 - \sigma)$. Since

$$y(t_2 - \sigma + n\tau) = \sum_{i=1}^n c^{n-i} z(t_2 - \sigma + i\tau) + c^n y(t_2 - \sigma)$$

$$y(t_1 - \sigma + n\tau) = \sum_{i=1}^n c^{n-i} z(t_1 - \sigma + i\tau) + c^n y(t_1 - \sigma)$$

$$z(t_2 - \sigma + i\tau) \ge z(t_1 - \sigma + i\tau), \qquad i = 1, 2, \ldots, n$$

and $y(t_1 - \sigma + n\tau) > 0$, $n = 0, 1, \ldots,$ we see

$$y(t_2 - \sigma + n\tau) \ge c^n[y(t_2 - \sigma) - y(t_1 - \sigma)] \triangleq Ac^n. \tag{20}$$

Similar to the proof of Corollary 1, we can show that there is an interval $[t_2, t_3]$ such that

$$y(T - \sigma + nt) \ge Ac^n$$

for $T \in [t_2, t_3]$ and all n. From (13) we get

$$z(t_0) + f(A)\sum_{i=0}^n \int_{t_2+i\tau}^{t_3+i\tau} (u - t_2)p(u)f(c^n)\,du \le 0,$$

contradicting (10).

Proof of The Theorem: According to the proofs of Lemmas 1 and 2 we have $z'(t) > 0$, $z(t) < 0$ eventually. The remainder of the proof is similar to that of Lemma 2.2 in [4]. We omit it here.

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