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Fundamental Solutions for Linear Retarded Functional Differential Equations in Banach Space

By

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§ 1. Introduction

The object of this paper is to construct the fundamental solution in the sense of S. Nakagiri [3], [4] to the linear retarded functional differential equation

$$\frac{d}{dt}u(t) = Au(t) + A_1 u(t-r) + \int_{-r}^{0} a(s)A_2 u(t+s)ds + f(t) \tag{1.1}$$

in a Banach space X. Here A is the infinitesimal generator of an analytic semigroup $T(t)$, A_1 and A_2 are closed linear operators with domains containing that of A, and $a(\cdot)$ is a real valued Hölder continuous function defined in $[-r, 0]$. By definition the fundamental solution $W(t)$ to (1.1) is an operator valued function satisfying

$$\frac{d}{dt}W(t) = AW(t) + A_1 W(t-r) + \int_{-r}^{0} a(s)A_2 W(t+s)ds \tag{1.2}$$

$$W(0) = I, \qquad W(s) = 0 \qquad \text{for } s \in [-r, 0). \tag{1.3}$$

It will be shown that $W(t)$ also satisfies

$$\frac{d}{dt}W(t) = W(t)A + W(t-r)A_1 + \int_{-r}^{0} W(t+s)a(s)A_2 ds \tag{1.4}$$

on $D(A)$, which is considered as the adjoint equation to (1.2).

Equations of the type (1.1) were investigated by G. Di Blasio, K. Kunisch & E. Sinestrari [1], [2] and E. Sinestrari [6]. See also the bibliography of these papers. In [1] the initial value problem for the equations in a Hilbert space H was solved in the space of L^2 functions with values in H. Essential use was made of the maximal regularity result for equations without delay terms there, and the corresponding regularity result was also obtained for the equations with delay terms. In [2] stability results were established for equations in a Hilbert space. In [6] equations in a general Banach space E were investigated without assuming that A is densely defined. The solvability was established in the space of Hölder continuous functions taking values in E together with maximal regularity results.

The fundamental solution enables us to solve the initial value problem for the equation (1.1). It will be shown that the mild solution satisfying the initial condition

$$u(s) = y(s)\,, \qquad s \in [-r, 0) \tag{1.5}$$

expressed by Nakagiri's formula ((2.8) of [3] or (2.7) of [4]) is actually the strict solution of (1.1), (1.5) provided that f is a Hölder continuous function with values in X and y is a Hölder continuous function in $[-r, 0]$ with values in the Banach space $D(A)$ endowed with the graph norm of A, but with no maximal regularity result.

In the proof of the main result we follow J. Prüss' method of constructing the resolvent operators for the integrodifferential equations of Volterra type ([5]: Theorem 2) since the equation (1.1) is of this type in each subinterval $[nr, (n+1)r]$, $n = 0, 1, 2, \dots$.

The contents of this paper were announced in [7].

§2. Assumptions and main results

Let X be a complex Banach space with norm $\|\ \|$. We assume that

(i) A is a densely defined closed linear operator which generates an analytic semigroup $T(t)$ in X.

(ii) A_1 and A_2 are closed linear operators with domains $D(A_1)$ and $D(A_2)$ containing the domain $D(A)$ of A.

(iii) $a(\cdot)$ is a real valued Hölder continuous function defined in the interval $[-r, 0]$, where r is a fixed positive number.

We denote by ρ the order of Hölder continuity of $a(\cdot)$. For the sake of simplicity we assume that A has an everywhere defined bounded inverse.

Let $W(t)$ be the fundamental solution of (1.1). $W(t)$ is a bounded operator valued function satisfying (1.2) and (1.3). According to Duhamel's principle the problem (1.2), (1.3) is transformed to the integral equation

(2.1)
$$W(t) = \begin{cases} T(t) + \displaystyle\int_0^t T(t-s)\left\{A_1 W(s-r) + \int_{-r}^0 a(\tau) A_2 W(s+\tau)d\tau\right\}ds\,, & t \geqq 0 \\ 0 & t < 0 \end{cases}.$$

With the aid of the change of the variable $\tau \to \tau - s$ and noting that $W(t) = 0$ for $t < 0$ we get

$$W(t) = T(t) + \int_0^t T(t-s) \int_0^s a(\tau - s) A_2 W(\tau) d\tau ds \tag{2.2}$$

in $[0, r]$, and

$$W(t) = T(t) + \int_r^t T(t-s) A_1 W(s-r) ds \tag{2.3}$$
$$+ \int_0^t T(t-s) \int_{s-r}^s a(\tau - s) A_2 W(\tau) d\tau ds$$

in (r, ∞). The exchange of the order of integration yields

$$W(t) = T(t) + \int_0^t \int_\tau^t T(t-s) a(\tau - s) ds A_2 W(\tau) d\tau \tag{2.4}$$

in $(0, r]$, and

$$W(t) = T(t) + \int_r^t T(t-s) A_1 W(s-r) ds \tag{2.5}$$
$$+ \int_0^{t-r} \int_\tau^{\tau+r} T(t-s) a(\tau - s) ds A_2 W(\tau) d\tau$$
$$+ \int_{t-r}^t \int_\tau^t T(t-s) a(\tau - s) ds A_2 W(\tau) d\tau$$

in $(0, \infty)$. It will turn out that $AW(t)$ has singularities at $t = nr$, $n = 0, 1, 2, \ldots$, and

$$\|AW(t)\| \leq C_n/(t - nr), \qquad nr < t < (n+1)r.$$

Hence, the integrals in the right sides of (1.2), (2.1), ..., (2.5) should be understood in the improper sense:

$$\int_{-r}^0 a(s) A_2 W(t+s) ds = \lim_{\varepsilon \to +0} \left(\int_{-r}^{nr-t} + \int_{nr-t+\varepsilon}^0 \right) a(s) A_2 W(t+s) ds$$

in $(nr, (n+1)r]$, $n = 0, 1, 2, \ldots$,

$$W(t) = T(t) + \lim_{\varepsilon \to +0} \int_\varepsilon^t T(t-s) \int_\varepsilon^s a(\tau - s) A_2 W(\tau) d\tau$$
$$= T(t) + \lim_{\varepsilon \to +0} \int_\varepsilon^t \int_\tau^t T(t-s) a(\tau - s) ds A_2 W(\tau) d\tau$$

in $(0, r]$,

$$W(t) = T(t) + \lim_{\varepsilon \to +0} \left[\left(\sum_{j=1}^{n-1} \int_{jr+\varepsilon}^{(j+1)r} + \int_{nr+\varepsilon}^{t} \right) T(t-s) A_1 W(s-r) ds \right.$$
$$+ \sum_{j=0}^{n-1} \int_{jr}^{(j+1)r} T(t-s) \left(\int_{s-r}^{jr} + \int_{jr+\varepsilon}^{s} \right) a(\tau - s) A_2 W(\tau) d\tau ds$$
$$\left. + \int_{nr}^{t} T(t-s) \left(\int_{s-r}^{nr} + \int_{nr+\varepsilon}^{s} \right) a(\tau - s) A_2 W(\tau) d\tau ds \right]$$
$$= T(t) + \lim_{\varepsilon \to +0} \left[\left(\sum_{j=1}^{n-1} \int_{jr+\varepsilon}^{(j+1)r} + \int_{nr+\varepsilon}^{t} \right) T(t-s) A_1 W(s-r) ds \right.$$
$$+ \left(\sum_{j=0}^{n-2} \int_{jr+\varepsilon}^{(j+1)r} + \int_{(n-1)r+\varepsilon}^{t-r} \right) \int_{\tau}^{\tau+r} T(t-s) a(\tau - s) ds A_2 W(\tau) d\tau$$
$$\left. + \left(\int_{t-r}^{nr} + \int_{nr+\varepsilon}^{t} \right) \int_{\tau}^{t} T(t-s) a(\tau - s) ds A_2 W(\tau) d\tau \right]$$

in $(nr, (n+1)r]$, $n = 1, 2, \dots$, and similarly for (2.1).

In what follows we make $D(A)$ a Banach space endowing it with the graph norm of A.

Theorem 1. *The fundamental solution $W(t)$ to* (1.1) *exists and is unique. It also satisfies* (1.4) *on $D(A)$. The functions $AW(t)$ and $dW(t)/dt$ are strongly continuous except at $t = nr$, $n = 0, 1, 2, \dots$, and the following inequalities hold: for $i = 0, 1, 2$ with $A_0 = A$ and $n = 0, 1, 2, \dots$*

$$\|A_i W(t)\| \le C_n/(t - nr), \tag{2.6}$$

$$\|dW(t)/dt\| \le C_n/(t - nr), \tag{2.7}$$

$$\|A_i W(t) A^{-1}\| \le C_n \tag{2.8}$$

in $(nr, (n+1)r)$,

$$\left\| \int_t^{t'} A_i W(\tau) d\tau \right\| \le C_n \tag{2.9}$$

for $nr \le t < t' \le (n+1)r$,

$$\|W(t') - W(t)\| \le C_{n,\kappa} (t'-t)^{\kappa} (t - nr)^{-\kappa}, \tag{2.10}$$

$$\|A_i(W(t') - W(t))\| \le C_{n,\kappa} (t'-t)^{\kappa} (t - nr)^{-\kappa-1}, \tag{2.11}$$

$$\|A_i(W(t') - W(t)) A^{-1}\| \le C_{n,\kappa} (t'-t)^{\kappa} (t - nr)^{-\kappa}, \tag{2.12}$$

for $nr < t < t' < (n+1)r$ and $0 < \kappa < \rho$, where C_n and $C_{n,\kappa}$ are constants dependent on n and n, κ respectively but not on t and t'.

Theorem 2. *If y is a Hölder continuous function in $[-r, 0]$ with values in $D(A)$ and f is a Hölder continuous function in $[0, T]$ with values in X, then*

$$u(t) = W(t)y(0) + \int_{-r}^{0} U_t(s)y(s)ds + \int_{0}^{t} W(t-s)f(s)ds, \tag{2.13}$$

where

$$U_t(s) = W(t-s-r)A_1 + \int_{-r}^{s} W(t-s+\tau)a(\tau)A_2 d\tau, \tag{2.14}$$

is a unique solution of (1.1) *in* $[0, T]$ *satisfying the initial condition*

$$u(s) = y(s), \qquad s \in [-r, 0]. \tag{2.15}$$

The formula (2.14) is due to S. Nakagiri (cf. (2.8) of [3] or (2.7) of [4]).

§ 3. Proof of Theorem 1

For the sake of simplicity we assume that $T(t)$ is uniformly bounded and the following inequalities hold in $(0, \infty)$:

$$\|T(t)\| \le M, \qquad \|AT(t)\| \le M/t, \qquad \|A^2T(t)\| \le M/t^2. \tag{3.1}$$

The following lemma is easily shown.

Lemma 3.1. *For* $0 < t < t' < \infty$

$$\|T(t') - T(t)\| \le M \log(t'/t), \tag{3.2}$$

$$\|AT(t') - AT(t)\| \le M(t'-t)/(t't). \tag{3.3}$$

It is elementary to see that

$$\log(1 + a) \le a^\alpha/\alpha \tag{3.4}$$

for $a > 0$, $0 < \alpha < 1$. Hence, from (3.2) it follows that

$$\|T(t') - T(t)\| \le \frac{M}{\alpha}\left(\frac{t'-t}{t}\right)^\alpha \tag{3.5}$$

for $0 < t < t' < \infty$ and $0 < \alpha < 1$. Noting that $(t'-t)/t' \le ((t'-t)/t')^\kappa$ for $0 < t < t' < \infty$ and $0 < \kappa < 1$, we see from (3.3) that

$$\|AT(t') - AT(t)\| \le M(t'-t)^\kappa t^{-\kappa-1}. \tag{3.6}$$

Set

$$\|a\|_\infty = \max_{s \in [-r,0]} |a(s)|, \qquad |a|_\rho = \sup_{s,\tau \in [-r,0], s \ne \tau} \frac{|a(s) - a(\tau)|}{|s-\tau|^\rho}, \tag{3.7}$$

$$(3.8)\qquad B(t)=\int_0^t AT(t-s)a(-s)ds=\lim_{\varepsilon\to+0}\int_\varepsilon^t AT(t-s)a(-s)ds\,.$$

Lemma 3.2. *$B(t)$ is uniformly bounded, and Hölder continuous in* $(0, r]$:

$$(3.9)\qquad \|B(t')-B(t)\|\le C_\kappa(t'-t)^\kappa t^{-\kappa}\,,\qquad 0<t<t'\le r\,,\qquad 0<\kappa<\rho\,.$$

Proof. It follows from (3.1), (3.7) and

$$(3.10)\qquad B(t)=\int_0^t AT(t-s)(a(-s)-a(-t))ds+(T(t)-I)a(-t)$$

that

$$\|B(t)\|\le M|a|_\rho t^\rho/\rho+(M+1)\|a\|_\infty\,.$$

Hence, $B(t)$ is uniformly bounded:

$$(3.11)\qquad \|B\|_\infty=\sup_{0\le t\le r}\|B(t)\|\le M|a|_\rho r^\rho/\rho+(M+1)\|a\|_\infty<\infty\,.$$

Using (3.10) we have

$$\begin{aligned}(3.12)\quad B(t')-B(t)&=\int_t^{t'} AT(t'-s)(a(-s)-a(-t'))ds\\&\quad+(T(t')-T(t'-t))(a(-t)-a(-t'))\\&\quad+\int_0^t (AT(t'-s)-AT(t-s))(a(-s)-a(-t))ds\\&\quad+(T(t')-T(t))a(-t')+(T(t)-I)(a(-t')-a(-t))\\&=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}\,.\end{aligned}$$

With the aid of (3.1), (3.5), (3.6), (3.7) we get for $0<\kappa<\rho$

$$\|\mathrm{I}\|\le\int_t^{t'} M(t'-s)^{-1}|a|_\rho(t'-s)^\rho ds=M|a|_\rho(t'-t)^\rho/\rho\,,$$

$$\|\mathrm{II}\|\le 2M|a|_\rho(t'-t)^\rho\,,$$

$$\begin{aligned}\|\mathrm{III}\|&\le\int_0^t M(t'-t)^\kappa(t-s)^{-\kappa-1}|a|_\rho(t-s)^\rho ds\\&=M|a|_\rho(t'-t)^\kappa\int_0^t (t-s)^{\rho-\kappa-1}ds=M|a|_\rho(t'-t)^\kappa t^{\rho-\kappa}/(\rho-\kappa)\,,\end{aligned}$$

$$\|\mathrm{IV}\|\le M\|a\|_\infty\frac{1}{\kappa}\left(\frac{t'-t}{t}\right)^\kappa,$$

$$\|V\| \le (M+1)|a|_\rho (t'-t)^\rho ,$$

from which (3.9) follows.

3.1. Construction of fundamental solution in $[0, r]$

Following J. Prüss [5] we set

$$V(t) = A(W(t) - T(t)) \tag{3.13}$$

in $[0, r]$. The integral equation to be satisfied by $V(t)$ is

$$V(t) = V_0(t) + \int_0^t B(t-\tau)A_2A^{-1}V(\tau)d\tau , \tag{3.14}$$

where

$$\begin{aligned} V_0(t) &= \int_0^t B(t-\tau)A_2T(\tau)d\tau \\ &= \int_0^t (B(t-\tau) - B(t))A_2T(\tau)d\tau + B(t)A_2A^{-1}(T(t) - I) . \end{aligned} \tag{3.15}$$

In view of (3.1), (3.9), (3.11)

$$\begin{aligned} \|V_0(t)\| &\le \int_0^t C_\kappa \tau^\kappa (t-\tau)^{-\kappa} \|A_2A^{-1}\| M\tau^{-1}d\tau + \|B(t)\|\,\|A_2A^{-1}\|(M+1) \\ &\le C_\kappa M\|A_2A^{-1}\|B(1-\kappa, \kappa) + (M+1)\|B\|_\infty \|A_2A^{-1}\| . \end{aligned}$$

Hence, $V_0(t)$ is strongly continuous and uniformly bounded. The integral equation (3.14) can be solved by successive approximation and the solution is strongly continuous and uniformly bounded:

$$\sup_{0 \le t \le r} \|V(t)\| < \infty . \tag{3.16}$$

According to Theorem 2 of [5] $W(t)$ satisfies the equation (1.2) and the initial condition $W(0) = I$. We put $W(s) = 0$ for $s \in [-r, 0]$. The inequalities (2.6), (2.7), (2.8) for $n = 0$ are immediate consequences of

$$AW(t) = AT(t) + V(t) , \qquad AW(t)A^{-1} = T(t) + V(t)A^{-1} , \tag{3.17}$$

(1.2), (3.1), (3.7), (3.16) and

$$\begin{aligned} \int_{-t}^0 a(s)A_2W(t+s)ds &= \int_{-t}^0 (a(s) - a(-t))A_2T(t+s)ds \\ &\quad + a(-t)A_2A^{-1}(T(t) - I) + \int_{-t}^0 a(s)A_2A^{-1}V(t+s)ds . \end{aligned}$$

For $0<t<t'<r$

$$\int_t^{t'} AW(\tau)d\tau = T(t') - T(t) + \int_t^{t'} V(\tau)d\tau\,,$$

from which (2.9) for $n=0$ follows. In order to establish (2.10), (2.11), (2.12) we first estimate

$$(3.18)\quad V_0(t') - V_0(t) = \int_t^{t'} B(t'-\tau)A_2T(\tau)d\tau$$
$$+ \int_0^t (B(t'-\tau) - B(t-\tau) - B(t') + B(t))A_2T(\tau)d\tau$$
$$+ (B(t') - B(t))A_2A^{-1}(T(t) - I)\,.$$

By Lemma 3.2 we have for $0<\kappa<\mu<\rho$

$$\|B(t'-\tau) - B(t-\tau) - B(t') + B(t)\| \le \|B(t'-\tau) - B(t-\tau)\| + \|B(t') - B(t)\|$$
$$\le C_\mu(t'-t)^\mu(t-\tau)^{-\mu} + C_\mu(t'-t)^\mu t^{-\mu}$$
$$\le 2C_\mu(t'-t)^\mu(t-\tau)^{-\mu}\,,$$
$$\|B(t'-\tau) - B(t-\tau) - B(t') + B(t)\| \le \|B(t'-\tau) - B(t')\| + \|B(t-\tau) - B(t)\|$$
$$\le C_\mu\tau^\mu(t'-\tau)^{-\mu} + C_\mu\tau^\mu(t-\tau)^{-\mu}$$
$$\le 2C_\mu\tau^\mu(t-\tau)^{-\mu}\,.$$

Hence

$$(3.19)\qquad \|B(t'-\tau) - B(t-\tau) - B(t') + B(t)\|$$
$$\le \{2C_\mu(t'-t)^\mu(t-\tau)^{-\mu}\}^{\kappa/\mu}\{2C_\mu\tau^\mu(t-\tau)^{-\mu}\}^{(\mu-\kappa)/\mu}$$
$$= 2C_\mu(t'-t)^\kappa(t-\tau)^{-\mu}\tau^{\mu-\kappa}\,.$$

From (3.1), (3.4), (3.11), (3.18), (3.19), Lemma 3.2 it follows that

$$\|V_0(t') - V_0(t)\| \le \int_t^{t'} \|B\|_\infty\|A_2A^{-1}\|M\tau^{-1}d\tau$$
$$+ \int_0^t 2C_\mu(t'-t)^\kappa(t-\tau)^{-\mu}\tau^{\mu-\kappa}\|A_2A^{-1}\|M\tau^{-1}d\tau$$
$$+ C_\kappa(t'-t)^\kappa t^{-\kappa}\|A_2A^{-1}\|(M+1)$$
$$\le M\|B\|_\infty\|A_2A^{-1}\|\log(t'/t)$$

$$+ 2MC_\mu \|A_2 A^{-1}\| (t'-t)^\kappa \int_0^t (t-\tau)^{-\mu} \tau^{\mu-\kappa-1} d\tau$$

$$+ (M+1)C_\kappa \|A_2 A^{-1}\| (t'-t)^\kappa t^{-\kappa}$$

$$\leq \{M\|B\|_\infty \|A_2 A^{-1}\|/\kappa + 2MC_\mu \|A_2 A^{-1}\| B(1-\mu, \mu-\kappa)$$

$$+ (M+1)C_\kappa \|A_2 A^{-1}\|\}(t'-t)^\kappa t^{-\kappa} .$$

Hence, we get

$$\|V_0(t') - V_0(t)\| \leq \text{const.}\,(t'-t)^\kappa t^{-\kappa} ,$$

from which the following inequality easily follows:

$$\|V(t') - V(t)\| \leq \text{const.}\,(t'-t)^\kappa t^{-\kappa} . \tag{3.20}$$

The inequalities (2.10), (2.11), (2.12) for $n = 0$ are immediate consequences of (3.5), (3.6), (3.17), (3.20).

3.2. Construction of fundamental solution in $(nr, (n+1)r]$, $n > 0$

Suppose that the fundamental solution $W(t)$ is constructed in the interval $[0, nr]$ and the estimates (2.6), ..., (2.12) are established for $0, \ldots, n-1$ in place of n. In view of (2.10), (2.11)

$$AW(jr-0) = \lim_{t \to jr-0} AW(t)$$

exists for $j = 1, \ldots, n$ in the uniform operator topology. In the interval $[nr, (n+1)r]$ the integral equation to be satisfied by

$$V(t) = A(W(t) - \int_{nr}^t T(t-s)A_1 W(s-r)ds) \tag{3.21}$$

is

$$V(t) = V_0(t) + \int_{nr}^t B(t-\tau)A_2 A^{-1} V(\tau)d\tau , \tag{3.22}$$

where

$$V_0(t) = AT(t) + A\int_r^{nr} T(t-s)A_1 W(s-r)ds \tag{3.23}$$

$$+ \int_0^{t-r} T(t-r-\tau)B(r)A_2 W(\tau)d\tau + \int_{t-r}^{nr} B(t-\tau)A_2 W(\tau)d\tau$$

$$+ \int_{nr}^t B(t-\tau)A_2 \int_{nr}^\tau T(\tau-s)A_1 W(s-r)dsd\tau .$$

We estimate each term of the right hand side of (3.23).

$$(3.24)\quad A\int_r^{nr} T(t-s)A_1W(s-r)ds=\sum_{j=1}^{n-1} A\int_{jr}^{(j+1)} T(t-s)A_1W(s-r)ds\,.$$

In view of (3.1), Lemma 3.1 and the induction hypothesis we have for $j=1, \dots, n-2$

$$(3.25)\quad \left\|A\int_{jr}^{(j+1)r} T(t-s)A_1W(s-r)ds\right\|$$

$$=\left\|\int_{jr}^{(j+1)r}(AT(t-s)-AT(t-jr))A_1W(s-r)ds\right.$$

$$\left.+AT(t-jr)\int_{jr}^{(j+1)r}A_1W(s-r)ds\right\|$$

$$\leq\int_{jr}^{(j+1)r}M\frac{s-jr}{(t-s)(t-jr)}\frac{C_{j-1}}{s-jr}ds+\frac{MC_{j-1}}{t-jr}$$

$$+\frac{MC_{j-1}}{t-jr}\log\frac{t-jr}{t-(j+1)r}+\frac{MC_{j-1}}{t-jr}$$

$$\leq\frac{MC_{j-1}}{(n-j)r}\left(\log\frac{n-j}{n-j-1}+1\right),$$

and for $j=n-1$

$$(3.26)\quad \left\|A\int_{(n-1)r}^{nr}T(t-s)A_1W(s-r)ds\right\|$$

$$=\left\|\int_{(n-1)r}^{(n-1/2)r}(AT(t-s)-AT(t-(n-1)r))A_1W(s-r)ds\right.$$

$$+AT(t-(n-1)r)\int_{(n-1)r}^{(n-1/2)r}A_1W(s-r)ds$$

$$+\int_{(n-1/2)r}^{nr}AT(t-s)(A_1W(s-r)-A_1W(nr-r-0))ds$$

$$\left.+(T(t-(n-1/2)r)-T(t-nr))A_1W(nr-r-0)\right\|$$

$$\leq\int_{(n-1)r}^{(n-1/2)r}\frac{M(s-(n-1)r)}{(t-s)(t-(n-1)r)}\frac{C_{j-1}}{s-(n-1)r}ds+\frac{MC_{n-2}}{t-(n-1)r}$$

$$+\int_{(n-1/2)r}^{nr}\frac{M}{t-s}C_{n-2,\kappa}(nr-s)^{\kappa}(s-(n-1)r)^{-\kappa-1}ds+\frac{2M}{r}C_{n-2}$$

$$\le\frac{MC_{n-2}}{t-(n-1)r}\log\frac{t-(n-1)r}{t-(n-1/2)r}+\frac{MC_{n-2}}{t-(n-1)r}$$

$$+MC_{n-2,\kappa}\int_{(n-1)r}^{nr}(nr-s)^{\kappa-1}(s-(n-1)r)^{-\kappa}ds\frac{2}{r}+\frac{2M}{r}C_{n-2}$$

$$\le M\{C_{n-2}\log 2+3C_{n-2}+2C_{n-2,\kappa}B(\kappa-1,\kappa)\}/r\,.$$

Combining (3.24), (3.25), (3.26) we get

$$\left\|A\int_{r}^{nr}T(t-s)A_1W(s-r)ds\right\| \tag{3.27}$$

$$\le\sum_{j=1}^{n-2}\frac{MC_{j-1}}{(n-j)r}\left(\log\frac{n-j}{n-j-1}+1\right)$$

$$+M\{C_{n-2}\log 2+3C_{n-2}+2C_{n-2,\kappa}B(\kappa,1-\kappa)\}/r\,.$$

We write the third term on the right of (3.23) as

$$\int_0^{t-r}T(t-\tau-r)B(r)A_2W(\tau)d\tau \tag{3.28}$$

$$=\left(\sum_{j=0}^{n-2}\int_{jr}^{(j+1)r}+\int_{(n-1)r}^{t-r}\right)T(t-\tau-r)B(r)A_2W(\tau)d\tau\,.$$

For $j=0,\ \dots,\ n-2$

$$\left\|\int_{jr}^{(j+1)r}T(t-\tau-r)B(r)A_2W(\tau)d\tau\right\| \tag{3.29}$$

$$=\left\|\int_{jr}^{(j+1)r}(T(t-\tau-r)-T(t-jr-r))B(r)A_2W(\tau)d\tau\right.$$

$$\left.+T(t-jr-r)B(r)\int_{jr}^{(j+1)r}A_2W(\tau)d\tau\right\|$$

$$\le\int_{jr}^{(j+1)r}M\log\frac{t-jr-r}{t-\tau-r}\|B(r)\|\frac{C_j}{\tau-jr}d\tau+M\|B(r)\|C_j$$

(by the change of the variable $\tau=(t-jr-r)\sigma+jr$)

$$=M\|B(r)\|C_j\int_0^{r/(t-jr-r)}\log\frac{1}{1-\sigma}\frac{d\sigma}{\sigma}+M\|B(r)\|C_j$$

$$\leq M\|B(r)\|C_j(c_0+1),$$

where

$$c_0=\int_0^1 \log\frac{1}{1-\sigma}\frac{d\sigma}{\sigma}. \tag{3.30}$$

Similarly,

$$\begin{aligned}
&\left\|\int_{(n-1)r}^{t-r} T(t-\tau-r)B(r)A_2W(\tau)d\tau\right\| \\
&=\left\|\int_{(n-1)r}^{t-r}(T(t-\tau-r)-T(t-nr))B(r)A_2W(\tau)d\tau\right. \\
&\qquad\left.+\,T(t-nr)B(r)\int_{(n-1)r}^{t-r}A_2W(\tau)d\tau\right\| \\
&\leq \int_{(n-1)r}^{t-r} M\log\frac{t-nr}{t-\tau-r}\|B(r)\|\frac{C_{n-1}}{\tau-(n-1)r}d\tau+M\|B(r)\|C_{n-1} \\
&\leq M\|B(r)\|C_{n-1}(c_0+1).
\end{aligned} \tag{3.31}$$

By virtue of (3.28), (3.29), (3.31) we obtain

$$\left\|\int_0^{t-r} T(t-\tau-r)B(r)A_2W(\tau)d\tau\right\| \leq M\|B(r)\|(c_0+1)\sum_{j=0}^{n-1} C_j. \tag{3.32}$$

The fourth term of the right member of (3.23) is estimated as

$$\begin{aligned}
&\left\|\int_{t-r}^{nr} B(t-\tau)A_2W(\tau)d\tau\right\| \\
&=\left\|\int_{t-r}^{nr}(B(t-\tau)-B(r))A_2W(\tau)d\tau+B(r)\int_{t-r}^{nr}A_2W(\tau)d\tau\right\| \\
&\leq \int_{t-r}^{nr} C_\kappa(r-t+\tau)^\kappa(t-\tau)^{-\kappa}\frac{C_{n-1}}{\tau-(n-1)r}d\tau+\|B(r)\|C_{n-1} \\
&\leq C_\kappa C_{n-1}\int_{t-r}^{t}(t-\tau)^{-\kappa}(\tau-t+r)^{\kappa-1}d\tau+\|B(r)\|C_{n-1} \\
&=(C_\kappa B(1-\kappa,\kappa)+\|B(r)\|)C_{n-1}.
\end{aligned} \tag{3.33}$$

Let $\sigma=(\tau+nr)/2$ for $nr<\tau<(n+1)r$.

(3.34)

$$\left\| A\int_{nr}^{\tau} T(\tau-s)A_1W(s-r)ds \right\|$$

$$= \left\| \int_{\sigma}^{\tau} AT(\tau-s)(A_1W(s-r)-A_1W(\tau-r))ds \right.$$

$$+ (T((\tau-nr)/2)-I)A_1W(\tau-r)$$

$$+ \int_{nr}^{\sigma} (AT(\tau-s)-AT(\tau-nr))A_1W(s-r)ds$$

$$\left. + AT(\tau-nr)\int_{nr}^{\sigma} A_1W(s-r)ds \right\|$$

$$\le \int_{\sigma}^{\tau} \frac{M}{\tau-\sigma} C_{n-1,\kappa}(\tau-s)^{\kappa}(s-nr)^{-\kappa-1}ds + (M+1)\frac{C_{n-1}}{\tau-nr}$$

$$+ \int_{nr}^{\sigma} \frac{M(s-nr)}{(\tau-s)(\tau-nr)}\frac{C_{n-1}}{s-nr}ds + \frac{MC_{n-1}}{\tau-nr}$$

$$\le MC_{n-1,\kappa}\int_{nr}^{\tau}(\tau-s)^{\kappa-1}(s-nr)^{-\kappa}ds\frac{2}{\tau-nr} + \frac{(2M+1)C_{n-1}}{\tau-nr} + \frac{MC_{n-1}}{\tau-nr}\log 2$$

$$= \{2MC_{n-1,\kappa}B(\kappa, 1-\kappa) + (2M+1+M\log 2)C_{n-1}\}/(\tau-nr) \equiv C'_{n,\kappa}/(\tau-nr)\,.$$

Since $A_1W(s-r)$ is Hölder continuous in $(nr, (n+1)r]$

(3.35)

$$\frac{d}{d\tau}\int_{nr}^{\tau} T(\tau-s)A_1W(s-r)ds = A_1W(\tau-r) + A\int_{nr}^{\tau} T(\tau-s)A_1W(s-r)ds\,.$$

By (3.1), (3.2) and the induction hypothesis

(3.36)

$$\left\| \int_{nr}^{\tau} T(\tau-s)A_1W(s-r)ds \right\|$$

$$= \left\| \int_{nr}^{\tau} (T(\tau-s)-T(\tau-nr))A_1W(s-r)ds + T(\tau-nr)\int_{nr}^{\tau} A_1W(s-r)ds \right\|$$

$$\le \int_{nr}^{\tau} M\log\frac{\tau-nr}{\tau-s}\frac{C_{n-1}}{s-nr}ds + MC_{n-1}$$

$$\le MC_{n-1}c_0 + MC_{n-1}\,,$$

where c_0 is the constant defined by (3.30). Hence $\int_{nr}^{\tau} T(\tau-s)A_1W(s-r)ds$ is uniformly bounded in $(nr, (n+1)r]$, and converges to 0 as $\tau \to nr$ at any element of $D(A)$ in view of (2.8). Consequently,

$$\lim_{\tau \to nr} \int_{nr}^{\tau} T(\tau - s) A_1 W(s - r) ds = 0$$

in the strong operator topology. Thus, integrating (3.35) from nr to t

$$\int_{nr}^{t} A \int_{nr}^{\tau} T(\tau - s) A_1 W(s - r) ds d\tau = \int_{nr}^{t} T(t - s) A_1 W(s - r) ds - \int_{nr}^{t} A_1 W(\tau - r) d\tau . \tag{3.37}$$

With the aid of (2.9), (3.36), (3.37) we get

$$\left\| \int_{nr}^{t} A \int_{nr}^{\tau} T(\tau - s) A_1 W(s - r) ds d\tau \right\| \le (Mc_0 + M + 1) C_{n-1} . \tag{3.38}$$

By virtue of (3.9), (3.34), (3.38) the final term of (3.23) is estimated as

$$\begin{aligned} &\left\| \int_{nr}^{t} B(t - \tau) A_2 \int_{nr}^{\tau} T(\tau - s) A_1 W(s - r) ds d\tau \right\| \\ &= \left\| \int_{nr}^{t} (B(t - \tau) - B(t - nr)) A_2 \int_{nr}^{\tau} T(\tau - s) A_1 W(s - r) ds d\tau \right. \\ &\quad \left. + B(t - nr) \int_{nr}^{t} A_2 \int_{nr}^{\tau} T(\tau - s) A_1 W(s - r) ds d\tau \right\| \\ &\le \int_{nr}^{t} C_\kappa (\tau - nr)^{\kappa} (t - \tau)^{-\kappa} \| A_2 A^{-1} \| C'_{n,\kappa} (\tau - nr)^{-1} d\tau \\ &\quad + \| B(t - nr) \| \, \| A_2 A^{-1} \| (Mc_0 + M + 1) C_{n-1} \\ &\le C_\kappa C'_{n,\kappa} \| A_2 A^{-1} \| B(1 - \kappa, \kappa) + \| B \|_\infty \| A_2 A^{-1} \| (Mc_0 + M + 1) C_{n-1} . \end{aligned} \tag{3.39}$$

Combining (3.23), (3.1), (3.27), (3.32), (3.33), (3.39) we see that $V_0(t)$, and hence $V(t)$ is uniformly bounded in $[nr, (n+1)r]$.

The inequality (2.6) follows from (3.34) and

$$AW(t) = A \int_{nr}^{t} T(t - s) A_1 W(s - r) ds + V(t) . \tag{3.40}$$

In view of (3.1), the induction hypothesis, and the uniform boundedness of $V(t)$ and

$$(3.41)\qquad AW(t)A^{-1} = \int_{nr}^{t} AT(t-s)A_1(W(s-r) - W(t-r))A^{-1}ds + (T(t-nr) - I)A_1 W(t-r)A^{-1} + V(t)A^{-1}$$

we immediately obtain (2.8). In view of (3.38) and (3.40)

$$\int_{nr}^{t} AW(\tau)d\tau = \int_{nr}^{t} A \int_{nr}^{\tau} T(\tau - s)A_1 W(s-r)dsd\tau + \int_{nr}^{t} V(\tau)d\tau$$

is uniformly bounded in $[nr, (n+1)r]$. This implies (2.9).

Since $a(\cdot)$ is Hölder continuous, $A^{-1}B(t) = \int_0^t T(t-s)a(-s)ds$ is differentiable and

$$(3.42)\qquad \frac{d}{dt}A^{-1}B(t) = a(-t) + B(t)\,.$$

Noting $B(0) = 0$ and using (3.42) we can show without difficulty

$$(3.43)\qquad \frac{d}{dt}A^{-1}V(t) = V(t) + \int_{t-r}^{t} a(\tau - t)A_2 W(\tau)d\tau\,.$$

With the aid of (3.13), (3.35) and (3.43) it is not difficult to show that $W(t)$ is differentiable in $(nr, (n+1)r)$ and (1.2) holds. The inequality (2.7) is a simple consequence of (1.2), (2.6), (2.9), (3.7) and

$$\int_{-r}^{0} a(s)A_2 W(t+s)ds = \int_{-r}^{nr-t} (a(s) - a(-r))A_2 W(t+s)ds + a(-r)\int_{-r}^{nr-t} A_2 W(t+s)ds + \int_{nr-t}^{0} (a(s) - a(nr-t))A_2 W(t+s)ds + a(nr-t)\int_{nr-t}^{0} A_2 W(t+s)ds\,.$$

It is not difficult to show $W(nr-0) = W(nr+0)$. Hence, $W(t)$ is strongly continuous at $t = nr$.

3.3. Hölder continuity of $V(t)$ in $(nr, (n+1)r)$, $n > 0$

In order to establish (2.10), (2.11), (2.12) we first show the Hölder continuity of $V(t)$. Let $nr < t < t' < (n+1)r$. By (3.5) and (3.27)

$$(3.44)\quad \left\| A\int_r^{nr} T(t'-s)A_1W(s-r)ds - A\int_r^{nr} T(t-s)A_1W(s-r)ds \right\|$$
$$= \left\| (T(t'-nr)-T(t-nr))\, A\int_r^{nr} T(nr-s)A_1W(s-r)ds \right\|$$
$$\leq \text{const.}\, (t'-t)^{\alpha}(t-nr)^{-\alpha}$$

for $0<\alpha<1$. As for the third term on the right of (3.23)

(3.45)
$$\int_0^{t'-r} T(t'-r-\tau)B(r)A_2W(\tau)d\tau - \int_0^{t-r} T(t-r-\tau)B(r)A_2W(\tau)d\tau$$
$$= \int_{(n-1)r}^{t'-r} T(t'-r-\tau)B(r)A_2W(\tau)d\tau - \int_{(n-1)r}^{t-r} T(t-r-\tau)B(r)A_2W(\tau)d\tau$$
$$+ (T(t'-nr)-T(t-nr))\int_0^{(n-1)r} T((n-1)r-\tau)B(r)A_2W(\tau)d\tau$$
$$= \int_{t-r}^{t'-r} T(t'-r-\tau)B(r)A_2W(\tau)d\tau$$
$$+ \int_{(n-1)r}^{t-r} (T(t'-r-\tau)-T(t-r-\tau)-T(t'-nr)$$
$$+ T(t-nr))B(r)A_2W(\tau)d\tau + (T(t'-nr)-T(t-nr))B(r)\int_{(n-1)r}^{t-r} A_2W(\tau)d\tau$$
$$+ (T(t'-nr)-T(t-nr))\int_0^{(n-1)r} T((n-1)r-\tau)B(r)A_2W(\tau)d\tau$$
$$= I + II + III + IV.$$

By (2.6), (3.1), (3.4)

$$(3.46)\quad \|I\| \leq \int_{t-r}^{t'-r} M\|B(r)\|\frac{C_{n-1}}{\tau-(n-1)r}d\tau \leq M\|B(r)\|C_{n-1}\frac{1}{\alpha}\left(\frac{t'-t}{t-nr}\right)^{\alpha}.$$

By (3.5)

$$\|T(t'-r-\tau)-T(t-r-\tau)-T(t'-nr)+T(t-nr)\|$$
$$\leq \|T(t'-r-\tau)-T(t-r-\tau)\| + \|T(t'-nr)-T(t-nr)\|$$
$$\leq \frac{M}{\alpha}\left(\frac{t'-t}{t-r-\tau}\right)^{\alpha} + \frac{M}{\alpha}\left(\frac{t'-t}{t-nr}\right)^{\alpha} \leq \frac{2M}{\alpha}\left(\frac{t'-t}{t-r-\tau}\right)^{\alpha}$$

for $(n-1)r<\tau<t-r$. Similarly

$$\|T(t'-r-\tau)-T(t-r-\tau)-T(t'-nr)+T(t-nr)\|$$
$$\le \|T(t'-r-\tau)-T(t'-nr)\|+\|T(t-r-\tau)-T(t-nr)\|$$
$$\le \frac{M}{\alpha}\left(\frac{\tau-(n-1)r}{t'-r-\tau}\right)^{\alpha}+\frac{M}{\alpha}\left(\frac{\tau-(n-1)r}{t-r-\tau}\right)^{\alpha}\le \frac{2M}{\alpha}\left(\frac{\tau-(n-1)r}{t-r-\tau}\right)^{\alpha}.$$

Combining these two inequalities we get

$$\|T(t'-r-\tau)-T(t-r-\tau)-T(t'-nr)+T(t-nr)\| \le 2M(t'-t)^{\alpha\beta}(t-r-\tau)^{-\alpha}(\tau-(n-1)r)^{\alpha(1-\beta)}/\alpha \tag{3.47}$$

for $\alpha\in(0,1)$, $\beta\in(0,1)$. With the aid of (3.47), (2.6) we get

(3.48)
$$\|II\| \le \int_{(n-1)r}^{t-r}\frac{2M}{\alpha}(t'-t)^{\alpha\beta}(t-r-\tau)^{-\alpha}(\tau-(n-1)r)^{\alpha(1-\beta)}\|B(r)\|\frac{C_{n-1}}{\tau-(n-1)r}d\tau$$
$$=\frac{2M}{\alpha}\|B(r)\|C_{n-1}(t'-t)^{\alpha\beta}\int_{(n-1)r}^{t-r}(t-r-\tau)^{-\alpha}(\tau-(n-1)r)^{\alpha(1-\beta)-1}d\tau$$
$$=(2M/\alpha)\|B(r)\|C_{n-1}B(1-\alpha,\alpha(1-\beta))(t'-t)^{\alpha\beta}(t-nr)^{-\alpha\beta}.$$

By (3.5), (2.9), (3.32)

$$\|III\|\le M\|B(r)\|C_{n-1}\frac{1}{\alpha}\left(\frac{t'-t}{t-nr}\right)^{\alpha}, \tag{3.49}$$

$$\|IV\|\le M^2\|B(r)\|(c_0+1)\sum_{j=0}^{n-1}C_j\frac{1}{\alpha}\left(\frac{t'-t}{t-nr}\right)^{\alpha}. \tag{3.50}$$

Combining (3.45), (3.46), (3.48), (3.49), (3.50) we conclude

$$\left\|\int_0^{t'-r}T(t'-r-\tau)B(r)A_2W(\tau)d\tau-\int_0^{t-r}T(t-r-\tau)B(r)A_2W(\tau)d\tau\right\| \le \text{const.}\,(t'-t)^{\alpha}(t-nr)^{-\alpha} \tag{3.51}$$

for any $\alpha\in(0,1)$.

Let $(n-1)r<t-r<t'-r<\tau<nr$. By Lemma 3.2

$$\|B(t'-\tau)-B(t-\tau)\|\le C_\mu(t'-t)^{\mu}(t-\tau)^{-\mu}$$

and

$$\|B(t'-\tau)-B(t-\tau)\|\le \|B(t'-\tau)-B(r)\|+\|B(r)-B(t-\tau)\|$$
$$\le C_\mu(r-t'+\tau)^{\mu}(t'-\tau)^{-\mu}+C_\mu(r-t+\tau)^{\mu}(t-\tau)^{-\mu}$$
$$\le 2C_\mu(r-t+\tau)^{\mu}(t-\tau)^{-\mu}.$$

Hence,

$$\|B(t'-\tau)-B(t-\tau)\| \leq 2C_\mu(t'-t)^\kappa(t-\tau)^{-\mu}(r-t+\tau)^{\mu-\kappa} \tag{3.52}$$

for $0<\kappa<\mu<\rho$. Using (2.6) and the inequality we get

$$\left\| \int_{t'-r}^{nr} (B(t'-\tau)-B(t-\tau))A_2W(\tau)d\tau \right\| \tag{3.53}$$

$$\leq \int_{t'-r}^{nr} 2C_\mu(t'-t)^\kappa(t-\tau)^{-\mu}(r-t+\tau)^{\mu-\kappa}\frac{C_{n-1}}{\tau-(n-1)r}d\tau$$

$$\leq 2C_\mu C_{n-1}(t'-t)^\kappa \int_{(n-1)r}^{nr} (nr-\tau)^{-\mu}(\tau-(n-1)r)^{\mu-\kappa-1}d\tau$$

$$= 2C_\mu C_{n-1}(t'-t)^\kappa B(1-\mu, \mu-\kappa)r^{-\kappa},$$

$$\left\| \int_{t-r}^{t'-r} B(t-\tau)A_2W(\tau)d\tau \right\| \leq \|B\|_\infty C_{n-1}\log\frac{t'-nr}{t-nr}. \tag{3.54}$$

Combination of (3.53), (3.54) and (3.4) yields

$$\left\| \int_{t'-r}^{nr} B(t'-\tau)A_2W(\tau)d\tau - \int_{t-r}^{nr} B(t-\tau)A_2W(\tau)d\tau \right\| \tag{3.55}$$

$$= \left\| \int_{t'-r}^{nr} (B(t'-\tau)-B(t-\tau))A_2W(\tau)d\tau - \int_{t-r}^{t'-r} B(t-\tau)A_2W(\tau)d\tau \right\|$$

$$\leq \text{const.}\,(t'-t)^\kappa(t-nr)^{-\kappa}.$$

Analogously to (3.19)

(3.56)

$$\|B(t'-\tau)-B(t-\tau)-B(t'-nr)+B(t-nr)\| \leq 2C_\mu(t'-t)^\kappa(t-\tau)^{-\mu}(\tau-nr)^{\mu-\kappa}$$

for $nr<\tau<t<t'<(n+1)r$ and $0<\kappa<\mu<\rho$. Hence with the aid of (3.34), (3.9), (3.38), (3.4), (3.56)

$$\left\| \int_{nr}^{t'} B(t'-\tau)A_2\int_{nr}^{\tau} T(\tau-s)A_1W(s-r)dsd\tau \right. \tag{3.57}$$

$$\left. - \int_{nr}^{t} B(t-\tau)A_2\int_{nr}^{\tau} T(\tau-s)A_1W(s-r)dsd\tau \right\|$$

$$\leq \left\| \int_{t}^{t'} B(t'-\tau)A_2\int_{nr}^{\tau} T(\tau-s)A_1W(s-r)dsd\tau \right.$$

$$+ \int_{nr}^{t} (B(t'-\tau)-B(t-\tau)-B(t'-nr)+B(t-nr))$$

$$\times A_2 \int_{nr}^{\tau} T(\tau - s)A_1 W(s - r)dsd\tau$$

$$+ (B(t' - nr) - B(t - nr)) \int_{nr}^{t} A_2 \int_{nr}^{\tau} T(\tau - s)A_1 W(s - r)dsd\tau \Big\|$$

$$\leq \int_{t}^{t'} \|B\|_\infty \|A_2 A^{-1}\| C'_{n,\kappa}(\tau - nr)^{-1} d\tau$$

$$+ \int_{nr}^{t} 2C_\mu (t' - t)^\kappa (t - \tau)^{-\mu} (\tau - nr)^{\mu - \kappa} \|A_2 A^{-1}\| C'_{n,\kappa}(\tau - nr)^{-1} d\tau$$

$$+ C_\kappa (t' - t)^\kappa (t - nr)^{-\kappa} \|A_2 A^{-1}\| (Mc_0 + M + 1) C_{n-1}$$

$$\leq \text{const.}\ (t' - t)^\kappa (t - nr)^{-\kappa}\,.$$

Combining (3.6), (3.23), (3.44), (3.51), (3.55), (3.57) we conclude

$$\|V_0(t') - V_0(t)\| \leq \text{const.}\ (t' - t)^\kappa (t - nr)^{-\kappa}\,.$$

Hence,

$$\|V(t') - V(t)\| \leq \text{const.}\ (t' - t)^\kappa (t - nr)^{-\kappa} \tag{3.58}$$

for $nr < t < t' < (n + 1)r$ and $0 < \kappa < \rho$.

3.4. Proof of (2.10), (2.11), (2.12) for $n > 0$

We first show (2.11) for $nr < t < t' < (n + 1)r$. Set $\tau = (t + nr)/2$.

(3.59)

$$A \int_{nr}^{t'} T(t' - s)A_1 W(s - r)ds - A \int_{nr}^{t} T(t - s)A_1 W(s - r)ds$$

$$= A \int_{\tau}^{t'} T(t' - s)A_1 W(s - r)ds + A \int_{nr}^{\tau} T(t' - s)A_1 W(s - r)ds$$

$$- A \int_{\tau}^{t} T(t - s)A_1 W(s - r)ds - A \int_{nr}^{\tau} T(t - s)A_1 W(s - r)ds$$

$$= \int_{\tau}^{t'} AT(t' - s)(A_1 W(s - r) - A_1 W(t' - r))ds + (T(t' - \tau) - I)A_1 W(t' - r)$$

$$- \int_{\tau}^{t} AT(t - s)(A_1 W(s - r) - A_1 W(t - r))ds - (T(t - \tau) - I)A_1 W(t - r)$$

$$+ \int_{nr}^{\tau} (AT(t' - s) - AT(t' - nr))A_1 W(s - r)ds$$

$$+ AT(t'-nr)\int_{nr}^{\tau} A_1 W(s-r)ds$$

$$-\int_{nr}^{\tau}(AT(t-s)-AT(t-nr))A_1W(s-r)ds$$

$$-AT(t-nr)\int_{nr}^{\tau} A_1 W(s-r)ds\,.$$

With the aid of (3.1), (3.5), (3.6) and the induction hypothesis

(3.60)
$$\left\|\int_{\tau}^{t'} AT(t'-s)(A_1W(s-r)-A_1W(t'-r))ds\right.$$
$$\left.-\int_{\tau}^{t} AT(t-s)(A_1W(s-r)-A_1W(t-r))ds\right\|$$
$$=\left\|\int_{t}^{t'} AT(t'-s)(A_1W(s-r)-A_1W(t'-r))ds\right.$$
$$+(T(t'-\tau)-T(t'-t))(A_1W(t-r)-A_1W(t'-r))$$
$$\left.+\int_{\tau}^{t}(AT(t'-s)-AT(t-s))(A_1W(s-r)-A_1W(t-r))ds\right\|$$
$$\leq MC_{n-1,\kappa}\int_{t}^{t'}(t'-s)^{\kappa-1}(s-nr)^{-\kappa-1}ds+2MC_{n-1,\kappa}(t'-t)^{\kappa}(t-nr)^{-\kappa-1}$$
$$+MC_{n-1,\mu}\int_{\tau}^{t}(t'-t)^{\kappa}(t-s)^{\mu-\kappa-1}(s-nr)^{-\mu-1}ds$$
$$\leq MC_{n-1,\kappa}(t'-t)^{\kappa}(t-nr)^{-\kappa-1}/\kappa+2MC_{n-1,\kappa}(t'-t)^{\kappa}(t-nr)^{-\kappa-1}$$
$$+MC_{n-1,\kappa}(t'-t)^{\kappa}\int_{nr}^{t}(t-s)^{\mu-\kappa-1}(s-nr)^{-\mu}ds\frac{2}{t-nr}$$
$$=\{MC_{n-1,\kappa}(1/\kappa+2)+2MC_{n-1,\mu}B(\mu-\kappa,1-\mu)\}(t'-t)^{\kappa}(t-nr)^{-\kappa-1}\,,$$

where $0<\kappa<\mu<\rho$, and

(3.61)
$$\|(T(t'-\tau)-I)A_1W(t'-r)-(T(t-\tau)-I)A_1W(t-r)\|$$
$$\leq\|(T(t'-\tau)-I)(A_1W(t'-r)-A_1W(t-r))\|$$
$$+\|(T(t'-\tau)-T(t-\tau))A_1W(t-r)\|$$
$$\leq(M+1)C_{n-1,\kappa}(t'-t)^{\kappa}(t-nr)^{-\kappa-1}+2^{\kappa}MC_{n-1}(t'-t)^{\kappa}(t-nr)^{-\kappa-1}/\kappa\,.$$

Since

$$\|AT(t'-s) - AT(t'-nr) - AT(t-s) + AT(t-nr)\|$$

$$\leq \begin{cases} M\dfrac{s-nr}{(t'-s)(t'-nr)} + M\dfrac{s-nr}{(t-s)(t-nr)} \leq 2M\dfrac{s-nr}{(t-s)(t-nr)} \\[2ex] M\dfrac{t'-t}{(t'-s)(t-s)} + M\dfrac{t'-t}{(t'-nr)(t-nr)} \leq 2M\dfrac{t'-t}{(t'-s)(t-s)}, \end{cases}$$

for $nr < s < \tau$, we have

$$\|AT(t'-s) - AT(t'-nr) - AT(t-s) + AT(t-nr)\|$$

$$\leq 2M\frac{(t'-t)^{\kappa}(s-nr)^{1-\kappa}}{(t'-s)^{\kappa}(t-s)(t-nr)^{1-\kappa}}.$$

Hence, making use of the induction hypothesis

(3.62)

$$\left\| \int_{nr}^{\tau} (AT(t'-s) - AT(t'-nr))A_1W(s-r)ds \right.$$

$$\left. - \int_{nr}^{\tau} (AT(t-s) - AT(t-nr))A_1W(s-r)ds \right\|$$

$$= \left\| \int_{nr}^{\tau} (AT(t'-s) - AT(t'-nr) - AT(t-s) + AT(t-nr))A_1W(s-r)ds \right\|$$

$$\leq \int_{nr}^{\tau} 2M\frac{(t'-t)^{\kappa}(s-nr)^{1-\kappa}}{(t'-s)^{\kappa}(t-s)(t-nr)^{1-\kappa}}\frac{C_{n-1}}{s-nr}ds$$

$$\leq 2MC_{n-1}(t'-t)^{\kappa}(t-nr)^{\kappa-1}\left(\frac{2}{t-nr}\right)^{1+\kappa}\frac{1}{1-\kappa}\left(\frac{t-nr}{2}\right)^{1-\kappa}$$

$$= 2^{2\kappa+1}MC_{n-1}(t'-t)^{\kappa}(t-nr)^{-\kappa-1}/(1-\kappa).$$

By (3.6) and the induction hypothesis we get

$$\left\| AT(t'-nr)\int_{nr}^{\tau} A_1W(s-r)ds - AT(t-nr)\int_{nr}^{\tau} A_1W(s-r)ds \right\| \tag{3.63}$$

$$= \left\| (AT(t'-nr) - AT(t-nr))\int_{nr}^{\tau} A_1W(s-r)ds \right\|$$

$$\leq MC_{n-1}(t'-t)^{\kappa}(t-nr)^{-\kappa-1}.$$

With the aid of (3.59), (3.60), (3.61), (3.62), (3.63) we otain

$$\left\| A\int_{nr}^{t'} T(t'-s)A_1 W(s-r)ds - A\int_{nr}^{\tau} T(t-s)A_1 W(s-r)ds \right\| \tag{3.64}$$
$$\leq \text{const.}\,(t'-t)^{\kappa}(t-nr)^{-\kappa-1}\,.$$

Combining (3.21), (3.58), (3.64) we conclude (2.11).

Noting

$$\left\| \int_{nr}^{t'} T(t'-s)A_1 W(s-r)ds - \int_{nr}^{t} T(t-s)A_1 W(s-r)ds \right\|$$
$$\leq \text{const.}\,(t'-t)^{\alpha}(t-nr)^{-\alpha}\,, \qquad 0<\alpha<1\,,$$

which is easier to show than (3.64), we immediately obtain (2.10). Applying (3.1), (3.5), (3.6), (3.58) and the induction hypothesis to the right side of

$$\begin{aligned} &AW(t')A^{-1} - AW(t)A^{-1} \\ &= \int_{t}^{t'} AT(t'-s)A_1(W(s-r)-W(t'-r))A^{-1}ds \\ &\quad + \int_{nr}^{t} AT(t'-s)A_1(W(t-r)-W(t'-r))A^{-1}ds \\ &\quad + \int_{nr}^{t} (AT(t'-s)-AT(t-s))A_1(W(s-r)-W(t-r))A^{-1}ds \\ &\quad + (T(t'-nr)-T(t-nr))A_1 W(t'-r)A^{-1} \\ &\quad + (T(t-nr)-I)A_1(W(t'-r)-W(t-r))A^{-1} + (V(t')-V(t))A^{-1} \end{aligned}$$

which is an immediate consequence of (3.41), we obtain (2.12).

3.5. Proof of (1.4)

Letting $Z(t)$ be an operator valued function satisfying

$$\frac{d}{dt}Z(t) = Z(t)A + Z(t-r)A_1 + \int_{-r}^{0} Z(t+s)a(s)A_2 ds\,, \tag{3.65}$$

$$Z(0)=I\,, \qquad Z(s)=0 \qquad \text{for } s\in[-r,0)\,, \tag{3.66}$$

we will show that

$$Z(t)=W(t)\,. \tag{3.67}$$

$Z(t)$ is determined as the solution of the integral equation

(3.68) $$Z(t) = T(t) + \int_0^t Z(\tau) \int_\tau^t a(\tau - s) A_2 T(t - s) ds d\tau$$

in $[0, r]$, and

(3.69) $$\begin{aligned} Z(t) = T(t) &+ \int_r^t Z(s - r) A_1 T(t - s) ds \\ &+ \int_0^{t-r} Z(\tau) \int_\tau^{\tau + r} a(\tau - s) A_2 T(t - s) ds d\tau \\ &+ \int_{t-r}^t Z(\tau) \int_\tau^t a(\tau - s) A_2 T(t - s) ds d\tau \end{aligned}$$

in (r, ∞). Set

(3.70) $$\begin{aligned} G(t) &= \int_0^t a(-s) A_2 T(t - s) ds \\ &= \int_0^t (a(-s) - a(-t)) A_2 T(t - s) ds + a(-t) A_2 A^{-1} (T(t) - I) . \end{aligned}$$

By virtue of (3.1), (3.7)

(3.71) $$\|G(t)\| \leq M |a|_\rho \|A_2 A^{-1}\| t^\rho / \rho + (M + 1) \|a\|_\infty \|A_2 A^{-1}\| .$$

Hence, $Z(t)$ is the solution of

(3.72) $$Z(t) = T(t) + \int_0^t Z(\tau) G(t - \tau) d\tau$$

in $[0, r]$, and

(3.73) $$\begin{aligned} Z(t) = T(t) &+ \int_r^t Z(s - r) A_1 T(t - s) ds \\ &+ \int_0^{t-r} Z(\tau) G(r) T(t - \tau - r) d\tau + \int_{t-r}^{nr} Z(\tau) G(t - \tau) d\tau \\ &+ \int_{nr}^t Z(\tau) G(t - \tau) d\tau \end{aligned}$$

in $[nr, (n + 1)r]$, $n > 0$.

The equation (3.72) is solved by successive approximation. Since $G(0) = 0$ and

$$dG(t)/dt = a(-t) A_2 + G(t) A$$

on $D(A)$, we easily see that

$$\frac{d}{dt}Z(t) = Z(t)A + \int_{-t}^{0} Z(t+s)a(s)dsA_2 \tag{3.74}$$

$$= Z(t)A + \int_{-r}^{0} Z(t+s)a(s)dsA_2$$

on $D(A)$. Here we used that $Z(t) = 0$ for $t < 0$. Let $x \in D(A)$ and $t \in (0, r]$. In view of (1.2), (1.3), (3.74)

$$\int_0^t \frac{\partial}{\partial\sigma}(Z(t-\sigma)W(\sigma)x)d\sigma = \int_0^t \left\{ -\int_{-r}^{0} Z(t-\sigma+s)a(s)dsA_2 W(\sigma)x \right. \tag{3.75}$$

$$\left. + Z(t-\sigma)\int_{-r}^{0} a(s)A_2 W(\sigma+s)xds \right\} d\sigma .$$

By the exchange of the order of integration and the change of the variable $\sigma \to \sigma + s$

(3.76)

$$\int_0^t \int_{-r}^{0} Z(t-\sigma+s)a(s)dsA_2 W(\sigma)xd\sigma = \int_{-r}^{0}\int_{-s}^{t-s} Z(t-\sigma)a(s)A_2 W(\sigma+s)xd\sigma ds .$$

Noting that $Z(t) = W(t) = 0$ for $t < 0$ we may write $\int_0^t$ in place of $\int_{-s}^{t-s}$ in the right side of (3.76). Again changing the order of integration we see that the right side of (3.76) is equal to

$$\int_0^t Z(t-\sigma)\int_{-r}^{0} a(s)A_2 W(\sigma+s)xdsd\sigma .$$

Combining this with (3.75) we get

$$\int_0^t \frac{\partial}{\partial\sigma}(Z(t-\sigma)W(\sigma)x)d\sigma = 0 ,$$

which implies $Z(t)x = W(t)x$ for $x \in D(A)$ and $t \in [0, r]$. Thus (3.67) holds in $[0, r]$.

Suppose that (3.67) is shown to hold in $[0, nr]$. Then, all Z except the last one may be replaced by W in the right hand side of (3.73). Since

$$\int_{nr}^{t} Z(s-r)A_1 T(t-s)ds = \int_{nr}^{t} (Z(s-r) - Z(t-r))A_1 T(t-s)ds$$

$$+ Z(t-r)A_1 A^{-1}(T(t-nr) - I) ,$$

and by (2.10)

$$\|Z(s-r)-Z(t-r)\| = \|W(s-r)-W(t-r)\| \le C_{\kappa,n-1}(t-s)^{\kappa}(s-nr)^{-\kappa},$$

the integral of the second term of the right side of (3.73) exists and is strongly continuous in $[nr,(n+1)r]$. Hence, the integral equation (3.73) can be solved by successive approximation, and it is easy to show that (3.65) holds on $D(A)$.

By virtue of (1.2), (3.65) for $x \in D(A)$

$$(3.77)\quad \int_0^t \frac{\partial}{\partial\sigma}(Z(t-\sigma)W(\sigma)x)d\sigma$$
$$= \int_0^t \Big\{-Z(t-\sigma-r)A_1W(\sigma)x - \int_{-r}^0 Z(t-\sigma+s)a(s)A_2W(\sigma)xds$$
$$+ Z(t-\sigma)A_1W(\sigma-r)x + Z(t-\sigma)\int_{-r}^0 a(s)A_2W(\sigma+s)xds\Big\}d\sigma$$

in $(nr,(n+1)r]$. Since $Z(t)=W(t)=0$ for $t<0$

$$(3.78)\quad \int_0^t Z(t-\sigma)A_1W(\sigma-r)xd\sigma = \int_r^t Z(t-\sigma)A_1W(\sigma-r)xd\sigma$$
$$= \int_0^{t-r} Z(t-\sigma-r)A_1W(\sigma)xd\sigma$$
$$= \int_0^t Z(t-\sigma-r)A_1W(\sigma)xd\sigma,$$

(3.79)

$$\int_0^t Z(t-\sigma)\int_{-r}^0 a(s)A_2W(\sigma+s)xdsd\sigma = \int_{-r}^0\int_0^t Z(t-\sigma)a(s)A_2W(\sigma+s)xd\sigma ds$$
$$= \int_{-r}^0\int_s^{t+s} Z(t-\sigma+s)a(s)A_2W(\sigma)xd\sigma ds$$
$$= \int_{-r}^0\int_0^t Z(t-\sigma+s)a(s)A_2W(\sigma)xd\sigma ds$$
$$= \int_0^t\int_{-r}^0 Z(t-\sigma+s)a(s)A_2W(\sigma)xdsd\sigma.$$

By (3.77), (3.78), (3.79) we get

$$\int_0^t \frac{\partial}{\partial\sigma}(Z(t-\sigma)W(\sigma)x)d\sigma = 0.$$

Thus (3.67) holds in $[nr,(n+1)r]$.

3.6. Uniqueness of fundamental solution

The uniqueness of the fundamental solution can be proved by the argument of the previous subsection.

§ 4. Proof of Theorem 2

Suppose that $Ay(s)$ and $f(t)$ are Hölder continuous functions of orders α and β respectively, and set

$$\|y\|_\infty = \max_{s\in[-r,0]} \|Ay(s)\|\,, \qquad |y|_\alpha = \sup_{\substack{s,\tau\in[-r,0]\\ s\neq\tau}} \frac{\|Ay(s)-Ay(\tau)\|}{|s-\tau|^\beta}\,, \tag{4.1}$$

$$\|f\|_\infty = \max_{t\in[0,T]} \|f(t)\|\,, \qquad |f|_\beta = \sup_{\substack{t,\tau\in[0,T]\\ t\neq\tau}} \frac{\|f(t)-f(\tau)\|}{|t-\tau|^\beta}\,. \tag{4.2}$$

We estimate each term in the right hand side of

$$\begin{aligned}Au(t) = AW(t)y(0) &+ A\int_{-r}^{0} W(t-s-r)A_1 y(s)ds \\ &+ A\int_{-r}^{0}\int_{-r}^{s} W(t-s+\tau)a(\tau)d\tau A_2 y(s)ds + A\int_0^t W(t-s)f(s)ds\,.\end{aligned} \tag{4.3}$$

In view of (2.8)

$$\|AW(t)y(0)\| \le C_n\|Ay(0)\| \tag{4.4}$$

in $[nr, (n+1)r)$. For $0 < nr \le t < (n+1)r$

(4.5)

$$\begin{aligned}A\int_{-r}^{0} W(t-s-r)A_1 y(s)ds = &\int_{-r}^{t-(n+1)r} AW(t-s-r)A_1(y(s)-y(t-(n+1)r)ds \\ &+ \int_{nr}^{t} AW(s)dsA_1 y(t-(n+1)r) \\ &+ \int_{t-(n+1)r}^{0} AW(t-s-r)A_1(y(s)-y(0))ds \\ &+ \int_{t-r}^{nr} AW(s)dsA_1 y(0)\,.\end{aligned}$$

The norm of the third term in the right side of (4.5) does not exceed

$$\int_{t-(n+1)r}^{0} \frac{C_{n-1}}{t-s-nr} \|A_1A^{-1}\| |y|_\alpha (-s)^\alpha ds$$

$$\leq C_{n-1} \|A_1A^{-1}\| |y|_\alpha \int_{t-(n+1)r}^{0} (-s)^{\alpha-1} ds = C_{n-1} \|A_1A^{-1}\| |y|_\alpha ((n+1)r-t)^\alpha/\alpha .$$

Estimating other terms similarly we obtain

$$\left\| A \int_{-r}^{0} W(t-s-r)A_1 y(s) ds \right\| \tag{4.6}$$

$$\leq C_n \|A_1A^{-1}\| |y|_\alpha (t-nr)^\alpha/\alpha + C_n \|A_1 y(t-(n+1)r)\|$$

$$+ C_{n-1} \|A_1A^{-1}\| |y|_\alpha ((n+1)-t)^\alpha/\alpha + C_{n-1} \|A_1 y(0)\| .$$

Let $0 \leq nr \leq t < (n+1)r$.

$$A \int_{-r}^{0} \int_{-r}^{s} W(t-s+\tau)a(\tau)d\tau A_2 y(s) ds$$

$$= \left(\int_{-r}^{t-(n+1)r} \int_{-r}^{s} + \int_{t-(n+1)r}^{0} \int_{-r}^{nr-t+s} + \int_{t-(n+1)r}^{0} \int_{nr-t+s}^{s} \right)$$

$$AW(t-s+\tau)a(\tau)d\tau A_2 y(s) ds = I + II + III .$$

If $n = 0$, II vanishes. In view of (2.6), (2.9), (3.7), (4.1)

$$\|I\| = \left\| \int_{-r}^{t-(n+1)r} \int_{-r}^{s} AW(t-s+\tau)(a(\tau)-a(-r))d\tau A_2 y(s) ds \right. \tag{4.7}$$

$$\left. + \int_{-r}^{t-(n+1)r} \int_{t-s-r}^{t} AW(\tau)d\tau a(-r) A_2 y(s) ds \right\|$$

$$\leq \int_{-r}^{t-(n+1)r} \int_{-r}^{s} \frac{C_n}{t-s+\tau-nr} |a|_\rho (\tau+r)^\rho d\tau \|A_2 y(s)\| ds$$

$$+ C_n |a(-r)| \|A_2A^{-1}\| \|y\|_\infty (t-nr)$$

$$\leq C_n |a|_\rho \|A_2A^{-1}\| \|y\|_\infty \int_{-r}^{t-(n+1)r} \int_{-r}^{s} (\tau+r)^{\rho-1} d\tau ds$$

$$+ C_n |a(-r)| \|A_2A^{-1}\| \|y\|_\infty (t-nr)$$

$$\leq C_n \|A_2A^{-1}\| \|y\|_\infty (|a|_\rho (t-nr)^\rho/\rho(\rho+1) + |a(-r)|)(t-nr) .$$

Similarly

$$(4.8)\quad \|II\| = \left\| \int_{t-(n+1)r}^{0} \int_{-r}^{nr-t+s} AW(t-s+\tau)(a(\tau)-a(-r))d\tau A_2 y(s)ds \right.$$

$$\left. + \int_{t-(n+1)r}^{0} \int_{t-s-r}^{nr} AW(\tau)d\tau a(-r)A_2 y(s)ds \right\|$$

$$\leq C_{n-1}\|A_2A^{-1}\|\,\|y\|_\infty\{|a|_\rho((n+1)r-t)^\rho/\rho(\rho+1)$$

$$+ |a(-r)|\}((n+1)r-t),$$

$$(4.9)\quad \|III\| = \left\| \int_{t-(n+1)r}^{0} \int_{nr-t+s}^{s} AW(t-s+\tau)(a(\tau)-a(nr-t+s))d\tau A_2 y(s)ds \right.$$

$$\left. + \int_{t-(n+1)r}^{0} \int_{nr}^{t} AW(\tau)d\tau a(nr-t+s)A_2 y(s)ds \right\|$$

$$\leq C_n\|A_2A^{-1}\|\,\|y\|_\infty\{|a|_\rho((t-nr)^\rho/\rho + \|a\|_\infty\}((n+1)r-t).$$

Writing the final term of (4.3) as

$$A\int_0^t W(t-s)f(s)ds = A\left(\int_0^{t-nr} + \sum_{j=0}^{n-1}\int_{t-(j+1)r}^{t-jr}\right)W(t-s)f(s)ds$$

$$= \int_0^{t-nr} AW(t-s)(f(s)-f(t-nr))ds + \int_{nr}^{t} AW(s)dsf(t-nr)$$

$$+ \sum_{j=0}^{n-1}\int_{t-(j+1)r}^{t-jr} AW(t-s)(f(s)-f(t-jr))ds$$

$$+ \sum_{j=0}^{n-1}\int_{jr}^{(j+1)r} AW(s)dsf(t-jr)$$

we can show

$$(4.10)\quad \left\|A\int_0^t W(t-s)f(s)ds\right\| \leq C_n|f|_\beta(t-nr)^\beta/\beta + \sum_{j=0}^{n-1} C_j|f|_\beta r^\beta/\beta$$

$$+ \sum_{j=0}^{n} C_j\|f(t-jr)\|.$$

From (4.3), (4.4), (4.6), (4.7), (4.8), (4.9), (4.10) it follows that $Au(t)$ is bounded in $[0, T]$. It is not difficult to show that $Au(t)$ is strongly continuous. The remaining part of the proof of the theorem is routine, and may be omitted.

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