Solvability and Smoothing Effect for Semilinear Parabolic Equations

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Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $C^\infty$-boundary $\partial\Omega$. Let

$$L = L(x, D) = \sum_{|\mu| \leq 2m} a_\mu(x) D^\mu$$

be an elliptic operator whose coefficients $a_\mu$ are of class $C^\infty(\overline{\Omega})$, and let

$$B_j = B_j(x, D) = \sum_{|\mu| \leq m_j} b_{j\mu}(x) D^\mu, \quad j = 1, 2, \ldots, m,$$

be a differential operator on $\partial \Omega$, where $b_{j\mu} \in C^\infty(\partial \Omega)$ for each $\mu$ and $j = 1, 2, \ldots, m$. We consider the following initial-boundary value problem for semilinear parabolic equations:

$$(IBP) \quad \begin{cases}
\frac{\partial u}{\partial t} + Lu = f(D^{a_1}u, D^{a_2}u, \ldots, D^{a_n}u), & x \in \Omega, \quad t > 0, \\
B_j u = 0 & (j = 1, 2, \ldots, m), \quad x \in \partial \Omega, \quad t > 0, \\
u(\cdot, 0) = \phi, & x \in \Omega,
\end{cases}$$

where $0 \leq |a_k| < 2m, \ k = 1, 2, \ldots, n$.

It is well known that (IBP) has a smooth (local) solution when initial data $\phi$ is sufficiently smooth and satisfies some compatibility conditions on $\partial \Omega$ (see, e.g., Ladyzenskaja, Solonnikov and Ural'ceva [7]). Moreover, since parabolic equations possess smoothing effects, it is expected that (IBP) has a solution, which is smooth for $t > 0$, even if $\phi$ is not sufficiently smooth.

Our task in this paper is to establish the local existence of solutions to (IBP) for a wide class of non-smooth initial data. We intend to show that such solutions become smooth for $t > 0$ and fulfill a certain kind of singular behavior near $t = 0$. These results are described in the framework of various function spaces (say, the Sobolev space and the space of Hölder continuous functions). We also give some sufficient conditions which assure the global existence for (IBP).
We take some function spaces (say, $L^p(\Omega)$) and treat (IBP) in an abstract setting. Then we will be able to derive, in a systematic manner, useful information about the existence of local solutions, their singular behavior near $t = 0$, the existence of global solutions and their asymptotic behavior as $t \to \infty$. Let $X$ be a Banach space with norm $\| \cdot \|$. We consider the following abstract Cauchy problem:

\begin{align*}
\frac{du}{dt} + Au &= f(u), \quad t > 0, \\
u(0) &= \phi,
\end{align*}

where $A$ is a closed linear operator in $X$ with dense domain $D(A)$ and $f$ is a nonlinear operator in $X$. We assume that $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t \geq 0}$ in $X$.

Throughout this paper we consider the case $\text{Re} \, \sigma(A) > 0$, where $\text{Re} \, \sigma(A)$ stands for the real part of the spectrum of $A$. Since it is possible to define the fractional power $A^\gamma$ of $A$ for each $\gamma \geq 0$ in the standard manner (see, e.g., [3] and [9]), we put some conditions on $f$ and $\phi$ in terms of the fractional powers of $A$.

(F.1) $f(0) = 0$;

(F.2) there exist constants $C_0 > 0$, $v > 1$ and $0 \leq \alpha < 1$ such that

$$
\|f(u) - f(w)\| \leq C_0(\|A^{\alpha}u\| + \|A^{\alpha}w\|)^{v-1}\|A^{\alpha}u - A^{\alpha}w\|,
$$

for all $u, w \in D(A^\alpha)$;

(\Phi.1) $\phi \in D(A^\theta)$ for $0 \leq \theta < 1$.

Many authors, say, Henry [3], Kielhöfer [6], Racke [10], Rothe [11] and von Wahl [14], have discussed the solvability of semilinear parabolic initial-boundary value problems in the form (P). Among them, Henry and Kielhöfer have established the general theory in the case $\alpha \leq \theta$. Therefore, our analysis concentrates on the case $\alpha \geq \theta$, which is closely related to the study of singular behavior of solutions near $t = 0$. (See also the work of von Wahl [14], who has discussed the case $\alpha \geq \theta$.) Our strategy to approach (P) is to reduce it to the corresponding integral equation (see (IE) in §1) and construct a solution of (IE) by Banach’s fixed point theorem. Fractional powers play a crucial role in showing that such a solution has various regularity properties, which eventually allow us to solve (P).

In Section 1 we state our main theorems in the abstract setting; Theorem 1 (existence and uniqueness of solutions to (IE)), Theorem 2 (regularity properties of solutions) and Theorem 3 (existence of global solutions). In Section 2 we prepare two lemmas. Section 3 is devoted to the proofs of Theorems 1, 2 and 3. In Section 4 we apply the abstract results to (IBP).
addition to the solvability of (IBP) for non-smooth initial data, it is proved that every solution has a smoothing effect for \( t > 0 \). We will also give some information about its singular behavior near \( t = 0 \) in various function spaces. Especially, we can say that (IBP) has a classical solution for \( t > 0 \) even if \( \phi \) is not smooth.

Finally, we should refer to the case where \( \sigma(A) \cap \{ z \in \mathbb{C}; \text{Re} \; z \leq 0 \} \) is not empty. In this case, Theorems 1 and 2 are still valid with some modifications. However, Theorem 3 is not correct; we will be required to construct a stable manifold to get global existence results (see Henry [3] and Kielhöfer [6] in the case \( 0 \notin \text{Re} \; \sigma(A) \) and see Ito [5] in the case \( 0 \in \text{Re} \; \sigma(A) \)). Our results in such a case will be discussed elsewhere.

§1. Abstract theory

We will state existence, uniqueness and regularity properties of solutions of (P) in a Banach space \( X \) with norm \( \| \cdot \| \). Let \( A \) be a closed linear operator in \( X \) with dense domain \( D(A) \). Suppose that \( A \) satisfies the following resolvent condition:

\begin{align}
&(A.1) \quad \text{there exist } \psi \in (0, \pi/2), \; a \in \mathbb{R} \text{ and } C > 0 \text{ such that}\]
&\quad \rho(A) \ni S_{a, \psi} \equiv \{ z \in \mathbb{C}; \; \psi \leq |\arg(z - a)| \leq \pi, \; z \neq a \}
&\quad \text{and}\]
&\quad \| (z - A)^{-1} \| \leq C/|z - a| \quad \text{for all } \; z \in S_{a, \psi},
\end{align}

where \( \rho(A) \) means the resolvent set of \( A \).

Then (A.1) assures that \( -A \) generates an analytic semigroup of bounded linear operators \( \{ e^{-tA} \}_{t \geq 0} \) in \( X \) (see, e.g., [2], [3] or [13]). Moreover we assume

\begin{align}
&(A.2) \quad \sigma(A) \subset \{ z \in \mathbb{C}; \; \text{Re} \; z > \lambda > 0 \}.
\end{align}

It follows from (A.1) and (A.2) that the fractional powers \( A^\gamma \) of \( A \) are defined. We give assumptions (F.1), (F.2) on \( f \) and (\( \Phi.1 \)) on \( \phi \) in terms of the fractional powers.

As is standard, we formally integrate (P) to get

\begin{align}
&(IE) \quad u(t) = e^{-tA} \phi + \int_0^t e^{-(t-s)A} f(u(s))ds
\end{align}

and study the solvability of (IE).

\textbf{Theorem 1.} \textit{Suppose that } \phi \in D(A^\theta) \text{ with }

\begin{align}
&(1.1) \quad 0 \leq \theta \leq \alpha, \quad 1 - \alpha v + \theta(v - 1) \geq 0, \quad (\alpha - \theta)v < 1.
\end{align}
(i) (Existence) There exists a positive number \( T \) such that (IE) has a solution \( u \) on \([0, T]\) satisfying
\[
t^{\theta-\gamma}u \in B(0, T]; D(A^{\theta}) \quad \text{and} \quad \lim_{t \to 0} t^{\theta-\gamma} \|A^{\theta}u(t)\| = 0
\]
for every \( \beta \in (\theta, 1) \) and
\[
u \in C([0, T]; D(A^{\gamma})),
\]
for every \( \gamma \in [0, \theta] \). Moreover, for every \( \beta \in [0, 1) \)
\[
\|A^{\theta}u(t)\| \leq C(\beta) t^{-(\beta-\gamma)+} \|A^{\theta}u\|, \quad t \in (0, T],
\]
with some \( C(\beta) > 0 \), where \( \alpha^+ \) stands for \( \alpha^+ = \alpha \) if \( \alpha > 0 \) and \( \alpha^+ = 0 \) if \( \alpha \leq 0 \).

(ii) (Uniqueness) (a) In the case \( 1 - \alpha \theta + \theta (\nu - 1) > 0 \), a solution \( u \) of (IE) on \([0, T]\) is uniquely determined in the class of \( u \) such that \( t^{\nu-\gamma}u \in B((0, T]; D(A^{\nu})) \).

(b) In the case \( 1 - \alpha \theta + \theta (\nu - 1) = 0 \), it is uniquely determined in the class of \( u \) such that \( t^{\nu-\gamma}u \in B((0, T]; D(A^{\nu})) \) and \( \lim_{t \to 0} t^{\nu-\gamma} \|A^{\nu}u(t)\| = 0 \).

Remark 1.1. It is easily seen that (1.1) is equivalent to
\[
\frac{\alpha \nu - 1}{\nu - 1} \leq \theta \leq \alpha \quad \text{if} \quad \alpha \nu > 1, \\
0 < \theta \leq \alpha \quad \text{if} \quad \alpha \nu = 1, \\
0 \leq \theta \leq \alpha \quad \text{if} \quad \alpha \nu < 1.
\]

In this paper, we denote by \( C \) (resp. \( C(\alpha, \beta, \gamma, \ldots) \)) various positive constants (resp. constants depending on \( \alpha, \beta, \gamma, \ldots \)).

Theorem 2. Let \( u \) be the solution of (IE) in Theorem 1. Then
(i) \( t^{1-\theta}u \in B((0, T]; D(A)) \) with
\[
\|Au(t)\| \leq Ct^{-(1-\theta)} \|A^{\theta}u\|, \quad t \in (0, T].
\]
(ii) For every \( \gamma \in [0, 1 - \alpha) \), \( t^{\gamma+1-\theta}du/dt \in B((0, T]; D(A^{\gamma})) \) and
\[
\|A^{\gamma}(du/dt)(t)\| \leq C(\gamma) t^{-(\gamma+1-\theta)} \|A^{\theta}u\|, \quad t \in (0, T].
\]
(iii) \( u \) satisfies (P).

Remark 1.2. In Theorems 1 and 2, the restriction \( \theta \leq \alpha \) is not essential for the solvability. Indeed, it is well known that, for every \( \phi \in D(\alpha^{\theta}) \) with \( \theta \geq \alpha \), (P) has a unique solution \( u \) on \([0, T]\) in the class \( C([0, T]; D(\alpha^{\nu})) \). Moreover, \( u \) satisfies
\[
t^{(\beta-\theta)+}u \in B((0, T]; D(\alpha^{\beta})) \quad \text{for every} \quad \beta \in [0, 1]
\]
and
\[ t^{\gamma+1-\theta}du/dt \in B((0, T]; D(A^\gamma)) \quad \text{for every } \gamma \in [0, 1). \]

For these results, see e.g., Henry [3, Theorems 3.3.3 and 3.5.2]. See also Remark 3.1.

Finally we give a global existence result for (IE) or (P).

**Theorem 3.** Under the same assumptions of Theorem 1, there exists a positive constant \( \delta \) such that, if \( \|A^\theta \phi\| \leq \delta \), then (IE) has a global solution \( u \) on \([0, \infty)\) which \( u \) satisfies
\[ \|A^\beta u(t)\| \leq C(\beta) t^{-(\beta-\theta)} e^{-\lambda t} \|A^\theta \phi\|, \quad t \in (0, \infty) \]
for every \( \beta \in [0, 1) \).

Moreover, \( u \) satisfies (P), \( u \in C((0, \infty); D(A)) \) with
\[ \|Au(t)\| \leq Ct^{-(1-\theta)} e^{-\lambda t} \|A^\theta \phi\|, \quad t \in (0, \infty) \]
and \( du/dt \in C((0, \infty); D(A^\gamma)) \) with
\[ \|A^\gamma (du/dt)(t)\| \leq C(\gamma) t^{-(\gamma+1-\theta)} e^{-\lambda t} \|A^\theta \phi\|, \quad t \in (0, \infty) \]
for every \( \gamma \in [0, 1-\alpha) \).

\[ \]
as is introduced by [11]. The following lemma plays an important role in the proofs of Theorems 1, 2 and 3.

**Lemma 2.2.** If \( G \) satisfies
\[
G(t) = \int_0^t (t-s)^{-a} q(s)^{-b} e^{-cs} ds
\]
with \( a, b \in [0, 1) \) and \( c \in (0, \infty) \), then
\[
(2.3) \quad G(t) \leq C(a, b, c) q(t)^{1-a-b}
\]
with some \( C(a, b, c) > 0 \).

**Proof.** For \( 0 < t \leq 1 \), (2.3) is obvious. For \( t > 1 \), it suffices to show the uniform boundedness of \( G(t) \). We decompose the integral as
\[
\left( \int_0^\delta + \int_\delta^{t-\delta} + \int_{t-\delta}^t \right) q(t-s)^{-a} q(s)^{-b} e^{-cs} ds
\]
with \( 0 < \delta < 1 \); then the uniform boundedness is easily derived.

§3. Proofs of Theorems 1, 2 and 3

(I) **Proof of Theorem 1**

We begin with the proof of existence part. For each \( \beta \in [\theta, 1) \), we introduce the following space:
\[
E_{\beta, T} = \{ u: [0, T] \to X; q(t)^{\beta-\theta} e^{\lambda t} u(t) \in B((0, T]; D(A^\beta)) \}
\]
which becomes a Banach space endowed with norm
\[
\| u \|_{\beta, T} = \sup_{0 < t \leq T} q(t)^{\beta-\theta} e^{\lambda t} \| A^\beta u(t) \|.
\]
Define a closed ball \( B_T \) in \( E_{x,T} \):
\[
B_T = \{ u \in E_{x,T}; \| u \|_{x,T} \leq K \}
\]
where \( K \) and \( T \) are constants to be determined later.

For \( u \in B_T \), set
\[
(3.1) \quad F(u)(t) = e^{-tA}\phi + \int_0^t e^{-(t-s)A} f(u(s)) ds.
\]
If \( u \in E_{x,T} \), then it follows from (F.1), (F.2) and the definition of \( E_{x,T} \) that
\[
(3.2) \quad \| f(u(t)) \| \leq C_0 q(t)^{-(\theta-\theta')e^{-\lambda\nu} \| u \|_{x,T}^\nu}
\]
for all $t \in (0, T]$. Therefore, it follows from (2.1) and (3.2) that
\[
e^{\lambda t} \int_0^t \|A^\gamma e^{-(t-s)A}f(u(s))\|ds \leq C_1(\gamma)C_0 \|u\|_{L^p, T} \int_0^t q(t-s)^{-(\gamma-p)v}e^{-\lambda(v-1)s}ds
\]
for $\gamma \in [0, 1)$. Hence, with use of Lemma 2.2,
\[
(3.3) \quad e^{\lambda t} \int_0^t \|A^\gamma e^{-(t-s)A}f(u(s))\|ds \leq C_3(\gamma)q(t)^{1-(\gamma-p)v}|||u|||_{L^p, T}^{v}
\]
(observe that $(\alpha - \theta)v < 1$ and $v > 1$).

Here we prove the continuity of $A^\beta Fu(t)$ with respect to $t$.

**Lemma 3.1.** Let $\theta \leq \beta < 1$ be fixed. For each $u \in B_T$, $A^\beta Fu(t)$ is Hölder continuous in $[\varepsilon, T]$ for any $\varepsilon > 0$:
\[
(3.4) \quad \|A^\beta Fu(t + h) - A^\beta Fu(t)\| \leq C[(K^\gamma + \|A^\phi\|)h^\delta q(t)^{-(\beta + \delta - \theta)}e^{-\lambda t}
\]
\[
+ K^\gamma h^1 - \beta q(t)^{-(\beta - \theta)v}e^{-\lambda vt}]
\]
for $0 < t < t + h \leq T$ and $\delta \in (0, 1 - \beta)$.

**Proof.** This lemma can be proved along the same line as Pazy [9, p. 197–p. 198]. Let $0 < t < t + h \leq T$. It follows from (3.1) that
\[
\|A^\beta Fu(t + h) - A^\beta Fu(t)\| \leq \|(e^{-hA} - I)A^\beta e^{-tA}\phi\|
\]
\[
+ \int_t^{t+h} \|A^\beta e^{-(t+h-s)A}f(u(s))\|ds
\]
\[
\equiv I_1 + I_2 + I_3.
\]
By (2.1) and (2.2) we find that for every $\delta \in (0, 1)$,
\[
I_1 \leq C_2(\delta)h^\delta \|A^{\beta + \delta}e^{-tA}\phi\|
\]
\[
\leq C_1(\beta + \delta - \theta)C_2(\delta)h^\delta q(t)^{-(\beta + \delta - \theta)}e^{-\lambda t}|||u|||_{L^p, T}^{v}.
\]
In view of (3.2) and the definition of $B_T$, we see
\[
(3.5) \quad \|f(u(t))\| \leq C_0 K^\gamma q(t)^{-(\alpha - \theta)v}e^{-\lambda vt}.
\]
Therefore, on account of (2.1),
\[
I_2 \leq C_1(\beta)C_0 K^\gamma e^{-\lambda(t+h)} \int_t^{t+h} (t + h - s)^{-(\alpha - \theta)v}e^{-\lambda(v-1)s}ds
\]
\[
\leq CK^\gamma h^1 - \beta q(t)^{-(\alpha - \theta)v}e^{-\lambda vt}.
\]
Finally it follows from Lemmas 2.1, 2.2 and (3.5) that for any $\delta \in (0, 1 - \beta)$

\[
I_3 \leq C_2(\delta) h^\delta \int_0^t \| A^{\beta + \delta} e^{-(t-s)A} f(u(s)) \| \, ds \\
\leq C(\beta, \delta) K^\delta q(t)^{1-(\beta + \delta) - (\alpha - \theta) v} e^{-\lambda t} \\
\leq C(\beta, \delta) K^\delta q(t)^{-(\beta + \delta - \theta)} e^{-\lambda t},
\]

where (1.1) is used in the last inequality. Thus the proof is complete.

Now we are ready to show that $F$ is a mapping from $B_T$ into itself; in view of Lemma 3.1 it is sufficient to show $\| Fu \|_{\alpha, T} \leq K$ for every $u \in B_T$. Making use of (3.1) and (3.3), we get for every $\beta \in [0, 1)$

\[
q(t)^{\beta - \theta} e^{\lambda t} \| Au(t) \| \leq q(t)^{\beta - \theta} e^{\lambda t} \| A \| e^{-tA} \phi + q(t)^{\beta - \theta} e^{\lambda t} \int_0^t \| A \| e^{-(t-s)A} f(u(s)) \| \, ds \\
\leq q(t)^{\beta - \theta} e^{\lambda t} \| A \| e^{-tA} \phi + C_3(\beta) q(T)^{\alpha - \theta} e^{-\lambda t} \| u \|_{\alpha, T}.
\]

Since $1 - \alpha v + \theta(v - 1) \geq 0$, we have

\[
(3.6) \quad \| Fu \|_{\beta, T} \leq \sup_{0 < t \leq T} q(t)^{\beta - \theta} e^{\lambda t} \| A \| e^{-tA} \phi + C_3(\beta) q(T)^{1-\alpha v + \theta(v-1)} \| u \|_{\alpha, T}^v.
\]

Therefore, taking $\beta = \alpha$ in (3.6) one can show that, if

\[
(3.7) \quad \sup_{0 < t \leq T} q(t)^{\alpha - \theta} e^{\lambda t} \| A \| e^{-tA} \phi + C_4 q(T)^{-\alpha v + \theta(v-1)} K^v \leq K
\]

then $F$ maps $B_T$ into itself.

The same procedure enables us to prove that $F$ is a strictly contraction mapping. If $v, w \in E_{\alpha, T}$, then it follows from (F.2) and the definition of $E_{\alpha, T}$ that

\[
(3.8) \quad \| f(v(t)) - f(w(t)) \| \leq C_0 q(t)^{-(\alpha - \theta)} e^{-\lambda t} (\| v \|_{\alpha, T} + \| w \|_{\alpha, T})^{v-1} \| v - w \|_{\alpha, T}
\]

for all $t \in \{0, T\}$. Therefore, by (2.1) and (3.8)

\[
e^{\lambda t} \int_0^t \| A \| e^{-(t-s)A} \{ f(v(s)) - f(w(s)) \} \| \, ds \leq C_1(\gamma) C_0 (\| v \|_{\alpha, T} \\
+ \| w \|_{\alpha, T})^{v-1} \| v - w \|_{\alpha, T} \\
\times \int_0^t q(t-s)^{-\gamma} q(s)^{-(\alpha - \theta)} e^{-\lambda(v-1)s} \, ds
\]

for every $\gamma \in [0, 1)$. Hence Lemma 2.2 yields
\begin{align*}
(3.9) \quad & e^{\lambda t} \int_0^t \| A^\gamma e^{-(t-s)A} \{ f(v(s)) - f(w(s)) \} \| \, ds \\
& \leq C_3(\gamma) q(t)^{1-\gamma -(\alpha-\theta)v} \| v \|_{a,T} + \| w \|_{a,T} \| v - w \|_{a,T}^{v-1} \\

text{with the same constant } C_3(\gamma) \text{ as in (3.3). Estimate (3.9) gives} \\
(3.10) \quad & q(t)^{\alpha-\theta} e^{\lambda t} \| A^\gamma Fv(t) - A^\gamma Fw(t) \| \\
& \leq C_4 q(T)^{1-\alpha v + \theta(v-1)} \| v \|_{a,T} + \| w \|_{a,T}^{v-1} \| v - w \|_{a,T} ;
\end{align*}
so that
\begin{align*}
(3.11) \quad & \| Fv - Fw \|_{a,T} \leq C_4 q(T)^{1-\alpha v + \theta(v-1)} (2K)^{v-1} \| v - w \|_{a,T} \\
\text{for all } v, w \in B_T. \text{ Thus (3.11) implies that } F: B_T \to B_T \text{ becomes a strictly contraction mapping provided that} \\
(3.12) \quad & C_4 q(T)^{1-\alpha v + \theta(v-1)} (2K)^{v-1} < 1.
\end{align*}
Here it remains to verify that one can take \( K \) and \( T \) satisfying (3.7) and (3.12). Since
\begin{align*}
q(t)^{\alpha-\theta} e^{\lambda t} \| A^\gamma e^{-tA} \phi \| \leq C_1(\alpha - \theta) \| A^\theta \phi \|
\end{align*}
by (2.1), we note that
\begin{align*}
(3.7)' \quad & C_1(\alpha - \theta) \| A^\theta \phi \| + C_4 q(T)^{1-\alpha v + \theta(v-1)} K^v \leq K
\end{align*}
implies (3.7). In the case where \( 1 - \alpha v + \theta(v-1) > 0 \), we choose \( K = 2C_1(\alpha - \theta) \| A^\theta \phi \| \); then (3.7)' and (3.12) hold true by making \( T \) sufficiently small. In the case where \( 1 - \alpha v + \theta(v-1) = 0 \), it is sufficient to choose \( K \) such that \( C_4(2K)^{v-1} < 1 \) and make \( T \) sufficiently small so that
\begin{align*}
\sup_{0 < t \leq T} q(t)^{\alpha-\theta} e^{\lambda t} \| A^\gamma e^{-tA} A^\theta \phi \| = K - C_4 K^v
\end{align*}
holds, which is possible by virtue of (iii) of Lemma 2.1 (note \( \alpha > \theta \) in this case). Observe that, in both cases, \( K \) satisfies
\begin{align*}
(3.13) \quad & K \leq C \| A^\theta \phi \| .
\end{align*}
Thus we can apply Banach’s fixed point theorem to show that \( F \) has a unique fixed point \( u \) in \( B_T \). Clearly, \( u \) is a solution of (IE) on \([0, T]\). For such \( u \), we obtain, in view of (3.6),
\begin{align*}
(3.14) \quad & \| u \|_{\beta,T} \leq \sup_{0 < t \leq T} q(t)^{\beta-\theta} e^{\lambda t} \| A^\gamma e^{-tA} \phi \| + C_3(\beta) q(T)^{1-\alpha v + \theta(v-1)} \| u \|_{a,T} ;
\end{align*}
so that \( q(t)^{\beta-\theta} e^{\lambda t} u \in B((0, T]; D(A^\beta)) \) for every \( \beta \in [\theta, 1) \) and (1.2) follows with the aid of (2.1) and (3.13).
It remains to study the continuity of \( u \) at \( t = 0 \). For every \( \gamma < 1 \), (3.3) yields
\[
q(t)^{-\theta}e^{\lambda t} \int_0^t \| A^\gamma e^{-(t-s)A}f(u(s)) \| \, ds \leq C_3(\gamma) K^v q(t)^{1-v+\theta(v-1)}.
\]

If \( 1 - \alpha v + \theta(v - 1) > 0 \), then the right-hand side tends to zero as \( t \to 0 \); so that \( \lim_{t \to 0} q(t)^{-\theta}e^{\lambda t}A^\beta u(t) = 0 \) for every \( \beta \in (\theta, 1) \). One can also see that \( u \in C([0, T]; D(A^\theta)) \), which allows us to show \( u \in C([0, T]; D(A^\gamma)) \) for every \( \gamma \in [0, \theta] \). If \( 1 - \alpha v + \theta(v - 1) = 0 \), then we use (3.14) to deduce
\[
\| u \|_{a, T_0} \leq \sup_{0 < t \leq T_0} q(t)^{-\theta}e^{\lambda t} \| A^\gamma e^{-(t-A)}f \| + C_4 K^{v-1} \| u \|_{a, T_0}
\]
for any \( T_0 \in (0, T] \). Because of \( C_4(2K)^{v-1} < 1 \), there is a positive constant \( C_5 > 0 \) such that
\[
\| u \|_{a, T_0} \leq C_5 \sup_{0 < t \leq T_0} q(t)^{-\theta}e^{\lambda t} \| A^\gamma e^{-(t-A)}f \|
\]
which gives, in place of (3.15),
\[
q(t)^{-\theta}e^{\lambda t} \int_0^t \| A^\gamma e^{-(t-s)A}f(u(s)) \| \, ds \leq C(\gamma) C_5 K^{v-1} \sup_{0 < t \leq T_0} q(t)^{-\theta}e^{\lambda t} \| A^\gamma e^{-(t-A)}f \|
\]
for \( t \in (0, T_0] \). Since the right-hand side tends to zero as \( T_0 \to 0 \) by (iii) of Lemma 2.1, it is easy to derive the same conclusion as the case \( 1 - \alpha v + \theta(v - 1) > 0 \).

To show the uniqueness, let \( v, w \in E_{a, T} \) be two solutions of (IE). Similarly to (3.10), it is possible to derive
\[
\| v - w \|_{a, T} \leq C_4 q(T)^{1-\alpha v+\theta(v-1)}(\| v \|_{a, T} + \| w \|_{a, T})^{v-1} \| v - w \|_{a, T},
\]
which enables us to complete the proof of the uniqueness part.

(II) **Proof of Theorem 2**

Let \( u \in B_T \) be the solution of (IE) in Theorem 1. We combine (3.4), (3.13) and (F.2) to get the Hölder continuity of \( f(u(t)) \) in \( [\varepsilon, T] \) with any \( \varepsilon > 0 \); for any \( \delta \in (0, 1 - \alpha) \) and \( 0 < t < t + h \leq T \),
\[
\| f(u(t + h)) - f(u(t)) \| \leq C[h^\delta q(t)^{-\alpha v+\theta(v-1)}e^{-\lambda vt} + h^{1+\delta} q(t)^{-\alpha v+\theta(2v-1)\varepsilon^{-(2v-1)\varepsilon}}] \| A^\theta f \|.
\]

As is well known, the Hölder continuity of \( f(u(t)) \) with respect to \( t \in (0, T] \) assures that the solution \( u \) of (IE) actually satisfies (P) (see, e.g., Henry [3] or Pazy [8]).

Estimates (1.3) and (1.4) in Theorem 2 follow from the following lemma.
Lemma 3.2. Let $u \in B_T$ be the solution of (IE) in Theorem 1, then the following results hold true:

(i) \[ \|Au(t)\| \leq C q(t)^{(1-\theta)} e^{-\lambda t} \|A^\theta \phi\|, \quad t \in (0, T]. \]

(ii) For every $\gamma \in [0, 1 - \alpha)$, $q(t)^{\gamma + 1 - \theta} e^{\lambda t} du/dt \in B((0, T]; D(A^\gamma))$ and \[ \|A^\gamma (du/dt)(t)\| \leq C q(t)^{-(\gamma + 1 - \theta)} e^{-\lambda t} \|A^\theta \phi\|, \quad t \in (0, T]. \]

(iii) $A^\gamma (du/dt)(t)$ is Hölder continuous in $[\xi, T]$ for any $\xi > 0$.

Proof. (i) It follows from (2.1) that \[ q(t)^{1-\theta} e^{\lambda t} \|A e^{-tA} \phi\| \leq C_1 \|A^\theta \phi\|. \]

We use the following identity:

\[
A \int_0^t e^{-(t-s)A} f(u(s)) ds = \int_0^{t/2} A e^{-(t-s)A} f(u(s)) ds + \int_{t/2}^t A e^{-(t-s)A} \{ f(u(s)) - f(u(t)) \} ds + [I - e^{-(t/2)A}] f(u(t)) \\
\equiv I_1 + I_2 + I_3.
\]

(3.17)

It is easy to see from (3.5) and (3.13) that $q(t)^{1-\theta} e^{\lambda t} \|I_3\| \leq C \|A^\theta \phi\|$ with some $C > 0$. Moreover, Lemmas 2.1 and 2.2, together with (3.5) and (3.13), give

\[
q(t)^{1-\theta} e^{\lambda t} \|I_1\| \leq C_1 C_0 K^n q(t)^{1-\theta} e^{\lambda t} \int_0^{t/2} q(t-s)^{-1} e^{-\lambda(t-s)} q(s)^{-(a-\theta)v} e^{-\lambda vs} ds
\]

\[ \leq C q(t)^{1-v+\theta(v-1)} \|A^\theta \phi\| \leq C \|A^\theta \phi\|. \]

Finally it follows from (3.16) that

\[
q(t)^{1-\theta} e^{\lambda t} \|I_2\| \leq C_1 q(t)^{1-\theta} e^{\lambda t} \int_{t/2}^t (t-s)^{-1} e^{-\lambda(t-s)} \|f(u(s)) - f(u(t))\| ds
\]

(3.18)

\[ \leq C q(t)^{1-\theta} \|A^\theta \phi\| \left\{ \int_{t/2}^t (t-s)^{-1+\delta} q(s)^{-(a-\theta)v-\delta} e^{-\lambda(v-1)s} ds + \int_{t/2}^t (t-s)^{-\delta} q(s)^{-(a-\theta)(2v-1)\delta} e^{-2\lambda(v-1)s} ds \right\}. \]
The first integral in the braces of (3.18) is bounded by

\[
q(t/2)^{-(a-\theta)v+\delta} \int_0^t q(t-s)^{-1+\delta} e^{-\lambda(v-1)s} ds \leq Cq(t)^{-\delta} \leq Cq(t)^{\theta - 1}.
\]

The second one is shown to be bounded in the same way. Thus the proof of (i) is complete.

(ii) The differentiation of (IE) with respect to \( t \) gives

\[
\frac{du}{dt}(t) = -Ae^{-tA}\phi + e^{-(t/2)A}f(u(t)) - \int_{t/2}^t Ae^{-(t-s)A}\{f(u(s)) - f(u(t))\}ds.
\]

Operate \( A^\gamma \) with \( 0 \leq \gamma < \delta \) (< 1 - \( \alpha \)) to both sides; then (ii) can be proved in the same way as (i).

(iii) One can show (iii) by applying the technique used in the proof of Lemma 3.1 to (3.19).

Remark 3.1. Stronger result than (ii) of Lemma 3.2 holds true. Indeed one can show that

\[
q(t)^{\gamma + 1 - \theta}e^{2t}du/dt \in B((0, T]; D(A^\gamma))
\]

for every \( \gamma \in [0, 1) \). For the details of the proof, see Hishida [4, Theorem 3] (see also Henry [3, Theorem 3.5.2]).

(III) Proof of Theorem 3

Take any \( T \geq 1 \) in the proof of Theorem 1. The proof of Theorem 1 is also valid to show the existence of a global solution of (IE) if we can choose a sufficiently small number \( K \) satisfying (3.7)' and (3.12) with \( q(T) = 1 \). Therefore, it is sufficient to take \( \phi \) satisfying

\[
C_1(\alpha - \theta)\|A^\theta\phi\| \leq K - C_4K^\gamma.
\]

By (3.13) and (3.14), we get (1.5). The regularity and differentiability properties follow from Lemma 3.2. Moreover, (1.6) and (1.7) are derived with use of (3.17) and (3.19).

Remark 3.2. We have put (F.2) on \( f \) to avoid some technical complication. Our argument remains valid even if (F.2) is replaced by general assumptions, say,

\[
\|f(v) - f(w)\| \leq C \sum_{k=1}^n (\|A^k_v\| + \|A^k_w\|)^{\nu - 1} \|A^\theta_v - A^\theta_w\|
\]
with $\alpha_k, \beta_k \in [0, 1)$ and $\nu_k > 1$ ($k = 1, 2, \ldots, n$). In this context, we should refer to the work of von Wahl [14, Theorem II.3.3], who has put

$$\|f(v) - f(w)\| \leq C \left( \|v\| + \|w\| \right) \|A^\alpha v - A^\alpha w\| + \left( \|A^\alpha v\| + \|A^\alpha w\| \right) \|v - w\|$$

(to study mainly the Navier-Stokes equations).

§4. Applications to (IBP)

In this section we apply the preceding abstract results to (IBP) stated in Introduction. We collect some assumptions on $L, B_j$ ($j = 1, 2, \ldots, m$) and $f$.

Assumption 4.1. (i) $L = \sum_{|\mu| \leq 2m} a_\mu(x)D^\mu$ is a uniformly and strongly elliptic differential operator in $\overline{\Omega}$.

(ii) $B_j = \sum_{|\mu| \leq m_j} b_\mu(x)D^\mu$, $1 \leq j \leq m$, with $m_j < 2m$ is a normal system of boundary operators on $\partial \Omega$.

(iii) For any $x \in \partial \Omega$, $L(x, D)$ and $\{B_j(x, D)\}_{j=1,2,\ldots,m}$ satisfy the complementing condition (for the complementing condition, see e.g., [2] or [13]).

(iv) If $I(x, D_x, D_t) = L(x, D_x) - (-1)^m e^{i\psi}D^m_t$ is defined for $(x, t) \in Q = \Omega \times (-\infty, \infty)$, then $I(x, D_x, D_t)$ and $\{B_j(x, D)\}_{j=1,2,\ldots,m}$ satisfy the complementing condition in $\overline{Q}$ for every $\psi \in [\pi/2, 3\pi/2]$.

Assumption 4.2. (i) For each $k$, $0 \leq |\alpha_k| < 2m$ and $\ell \equiv \max_{k=1,2,\ldots,n} |\alpha_k| < 2m$.

(ii) $f(0) = 0$.

(iii) There exist constants $M > 0$ and $v > 1$ such that

$$|f(v) - f(w)| \leq M \sum_{k=1}^n (|D^{\alpha_k} v| + |D^{\alpha_k} w|)^{-1} |D^{\alpha_k} v - D^{\alpha_k} w|.$$

We take $X = L^p(\Omega)$ ($1 < p < \infty$) with norm $\| \cdot \|_p$ and define an operator $A_p$ in $L^p(\Omega)$ by

$$D(A_p) = \{ u \in W^{2m,p}(\Omega) ; B_j u = 0 \text{ on } \partial \Omega \text{ for } j = 1, 2, \ldots, m \} \quad \text{with} \quad A_p u = Lu \quad \text{for} \quad u \in D(A_p).$$

Due to Assumption 4.1, it is well known that $A_p$ satisfies (A.1) (see, e.g., Tanabe [13]). Moreover, we assume

Assumption 4.3. For a sufficiently large number $\lambda_0$

$$a_0(x) \geq \lambda_0 \quad \text{in} \quad \Omega,$$

where $a_0$ is the coefficient for $\mu = 0$ in $L$.

By Assumption 4.3, $A_p$ will satisfy (A.2). We rewrite (IBP) as (P) with

$$f(u) \equiv f(D^{2\ell} u, D^{2\ell} u, \ldots, D^{2\ell} u).$$
Lemma 4.1. If \(0 \leq \gamma \leq 1\), then
(i) \(D(A_p^\gamma) \subset W^{k,p}(\Omega)\) for \(k - N/p^* \leq 2m\gamma - N/p\) and \(p^* \geq p\),
(ii) \(D(A_p^\gamma) \subset C^\sigma(\overline{\Omega})\) for \(0 \leq \sigma < 2m\gamma - N/p\),
where each imbedding is continuous.

For the proof of this lemma, see [3] or [9].

We take \(p\) satisfying
\[
\max \left\{ \frac{N(v-1)}{(2m-\ell)v}, 1 \right\} < p < \infty
\]
and \(\alpha\) such that
\[
\frac{N(v-1)}{2mpv} + \frac{\ell}{2m} < \alpha < 1.
\]
Note that (4.1) implies \(N(v-1)/(2mpv) + \ell/(2m) < 1\). Since \(\ell - N/(pv) < 2m\alpha - N/p\) by (4.2), (i) of Lemma 4.4 yields \(D(A_p^\alpha) \subset W^{\ell,p}(\Omega)\); so that
\[
\|u\|_{W^{\ell,p}(\Omega)} \leq C\|A_p^\alpha u\|_p
\]
for all \(u \in D(A_p^\alpha)\)
with some \(C > 0\).

Lemma 4.2. \(f\) satisfies (F.1) and (F.2).

Proof. Since \(f\) clearly satisfies (F.1), it is sufficient to show (F.2). By virtue of (iii) of Assumption 4.2, Hölder’s inequality yields
\[
\|f(v) - f(w)\|_p \leq C \sum_{k=1}^n \left[ \|D^{a_k}v\|_{pv} + \|D^{a_k}w\|_{pv} \right]^{\gamma-1} \|D^{a_k}v - D^{a_k}w\|_{pv}
\]
\[
\leq nC\left[ \|v\|_{W^{\ell,p}(\Omega)} + \|w\|_{W^{\ell,p}(\Omega)} \right]^{\gamma-1} \|v - w\|_{W^{\ell,p}(\Omega)}
\]
with some \(C > 0\) for \(v, w \in D(A_p^\alpha)\). Then it is easy to verify (F.2) with use of (4.3).

Observe that by (4.2)
\[
\alpha v - 1 > \frac{N(v-1)}{2mp} + \frac{\ell v}{2m} - 1.
\]
Then we have

Theorem 4.3. Let \(p\) satisfy (4.1) and let \(\theta \in [0, 1)\) satisfy
\[
\theta \geq 0 \quad \text{if} \quad 1 > \frac{N(v-1)}{2mp} + \frac{\ell v}{2m},
\]
\[
\theta > \frac{N}{2mp} + \frac{\ell v}{2m(v-1)} - \frac{1}{v-1} \quad \text{if} \quad 1 \leq \frac{N(v-1)}{2mp} + \frac{\ell v}{2m}.
\]
Then, for every \(\phi \in D(A_p^\theta)\), there exists a positive number \(T\) such that (IBP) has
a unique solution $u$ on $[0, T]$ satisfying $t^{(\beta-\theta)^+}u \in B([0, T]; D(A_p^\alpha))$ with

$$
\|A_p^\beta u(t)\|_p \leq Ct^{-(\beta-\theta)^+}\|A_p^\alpha \phi\|_p , \quad t \in (0, T]
$$

for every $\beta \in [0, 1]$ and $t^{\gamma+1-\theta}u_t \in B([0, T]; D(A_p^\gamma))$ with

$$
\|A_p^\gamma u_t(t)\|_p \leq Ct^{-(\gamma+1-\theta)}\|A_p^\alpha \phi\|_p , \quad t \in (0, T]
$$

for every $\gamma \in [0, 1-N(v-1)/(2mp)-\ell/(2m))$.

Moreover, if $\|A_p^\beta \phi\|_p$ is sufficiently small, then the solution $u$ exists globally in $[0, \infty)$.

**Proof.** If $p$ satisfies $1 > N(v-1)/(2mp) + \ell v/(2m)$, then one can take $\alpha$ satisfying (4.2) and $\alpha v < 1$. On the other hand, if $1 \leq N(v-1)/(2mp) + \ell v/(2m)$, then $\alpha v > 1$ for every $\alpha$ satisfying (4.2). Therefore, the assertions follow from Theorems 1, 2 and 3 (see also Remarks 1.1 and 1.2).

If we are interested in classical solutions of (IBP), it is usual to make $p$ sufficiently large in Theorem 4.3. In fact, if $p > N$, then $u(t) \in D(A_p)$ for $t \in (0, T]$, together with (ii) of Lemma 4.1, implies $u(t) \in C^{2m-1+\sigma}(\overline{\Omega})$ with some $\sigma \in (0, 1)$. Hence $f(u(t)) \in C^\sigma(\overline{\Omega})$ by Assumption 4.2. Moreover, since $u_t(t) \in D(A_p^\gamma)$ with $\gamma \in [0, 1-N(v-1)/(2mp)-\ell/(2m))$, it follows from (ii) of Lemma 4.1 that $u_t(t)$ is Hölder continuous in $\overline{\Omega}$ by taking a large $p$. These results assure the Hölder continuity of $Lu(t) = f(u(t)) - u_t(t)$ in $\overline{\Omega}$; so that, since $a_\mu$ are $C^\infty(\overline{\Omega})$-functions, we get $u(t) \in C^{2m+\sigma'}(\overline{\Omega})$ for some $\sigma' > 0$ by the regularity theory for elliptic equations (see, e.g., Agmon, Douglis and Nirenberg [1]).

However, we can develop the regularity theory for (IBP) even if $p$ is not sufficiently large. In the following steps, we will study smoothing properties for (IBP) in various function spaces. For this purpose, it is convenient to introduce the Sobolev spaces of fractional order $H^{s,p}(\Omega)$ (for the definition, see Triebel [12]). In particular, $H^{k,p}(\Omega)$ is identical with $W^{k,p}(\Omega)$, if $k$ is a non-negative integer.

The following lemma, which gives the characterization of $D(A_p^\alpha)$, is very useful.

**Lemma 4.4.** (i) For $1 < q_1 \leq q_2 < \infty$ and $s_1 \geq s_2$ with $s_2 - N/q_2 \leq s_1 - N/q_1$, the following imbedding relation holds true;

$$H^{s_1,q_1}(\Omega) \subset H^{s_2,q_2}(\Omega).$$

(ii) $D(A_p^\alpha) = \{u \in H^{2m\gamma, p}(\Omega); B_j u = 0$ on $\partial \Omega$ for $j$ satisfying $m_j < 2m\gamma - 1/p$ and for $k$ satisfying $m_k = 2m\gamma - 1/p$, $B_k u \in H^{1/p, p}(R^N)$ with $\text{supp } B_k u \subset \overline{\Omega}$ by extending $b_{k\mu}$ so that $b_{k\mu}$ and their first derivatives are continuous in $\Omega$.\}

For the proof, see Triebel [12].
We will study the asymptotic behavior of the solution $u$ of (IBP) near $t = 0$.

**Theorem 4.5.** Let $p$ satisfy (4.1), (4.2) and
\[
\frac{N(v - 1)}{2mp} + \frac{\ell v}{2m} < 1. 
\]
For every $q > p$, the (local) solution $u$ in Theorem 4.3 satisfies the following estimates with
\[
\rho = \frac{N}{2mp} - \frac{N}{2mq} - \theta; 
\]
(4.7) \[\|A_{p_i}^\delta u(t)\|_q \leq Ct^{-\rho^+}\|A_p^\delta \phi\|_p, \quad t \in (0, T]\]
for every $\beta \in [0, 1]$ and
(4.8) \[\|A_{p_i}^\gamma u_t(t)\|_q \leq Ct^{-\gamma-1-\rho^+}\|A_p^\delta \phi\|_p, \quad t \in (0, T]\]
for every $\gamma \in [0, 1 - N(v - 1)/(2mq\ell) - \ell/(2m))$.

**Proof.** We will begin with the proof of
(4.9) \[\|u(t)\|_q \leq Ct^{-\rho^+}\|A_p^\delta \phi\|_p, \quad t \in (0, T]\]
We first assume $\theta = 0$; so that $\rho^+ = N(1/p - 1/q)/(2m)$. Choose an integer $n \in N$ such that $n \geq N(1/p - 1/q)/(2m)$ and take $\delta \in (0, 1]$ such that $1/q = 1/p - 2mn\delta/N$. If we set
\[
\frac{1}{p_i} = \frac{1}{p_{i-1}} - \frac{2m\delta}{N}, \quad i = 1, 2, \ldots, n
\]
with $p_0 = p$ and $p_n = q$, then Lemma 4.4 gives
(4.10) \[D(A_{p_{i-1}}^\delta) \subset H^{2m\delta, p_i-1}(\Omega) \subset H^{0, p_i}(\Omega) = L^{p_i}(\Omega)
\]
for $1 \leq i \leq n$. We now apply Theorem 4.3 to get
\[
\|A_{p_i}^\delta u(t)\|_p \leq Ct^{-\delta}\|\phi\|_p, \quad t \in (0, T]\]
Therefore, it follows from (4.10) that
\[
\|u(t)\|_{p_i} \leq C\|A_{p_i}^\delta u(t)\|_p \leq Ct^{-\delta}\|\phi\|_p, \quad t \in (0, T]\]
Let any $\tau \in (0, T)$ be fixed and consider (IBP) for $t \geq \tau$ with initial data $u(t) \in L^{p_1}(\Omega)$ at $t = \tau$. Since $1 > N(v - 1)/(2mp_1) + \ell v/(2m)$, Theorem 4.3 with $p = p_1$ enables us to deduce
\[
\|A_{p_1}^\delta u(t)\|_{p_1} \leq C(t - \tau)^{-\delta}\|u(\tau)\|_{p_1} \leq C(t - \tau)^{-\delta\tau^{-\delta}}\|\phi\|_p
\]
for \( t \in (\tau, T] \). Since \( \tau \) is arbitrary, we see by taking \( \tau = t/2 \) in the last inequality that
\[
\| A_{p_1}^\theta u(t) \|_{p_1} \leq C t^{-2\beta} \| \phi \|_p , \quad t \in (0, T],
\]
which, together with (4.10), yields
\[
\| u(t) \|_{p_2} \leq C t^{-2\beta} \| \phi \|_p , \quad t \in (0, T].
\]
Repeating this procedure leads us to
\[
(4.11) \quad \| u(t) \|_q \leq C t^{-n\delta} \| \phi \|_p , \quad t \in (0, T] \quad \text{with} \quad n\delta = \frac{N}{2m} \left( \frac{1}{p} - \frac{1}{q} \right) = \rho
\]
Thus we have shown (4.9) for \( \theta = 0 \).

For general \( \theta > 0 \), if \( -N/q \leq 2m\theta - N/p \) (i.e., \( \rho^+ = 0 \)), it follows from Theorem 4.3 and Lemma 4.4 that
\[
(4.12) \quad \| u(t) \|_q \leq C \| A_{\rho}^\theta u(t) \|_p \leq C \| A_{p}^\theta \phi \|_p;
\]
thus (4.9) holds true. If \( -N/q > 2m\theta - N/p \), choose \( q_1 \in (p, q) \) such that
\[
2m\theta - N/p = -N/q_1.
\]
Then (4.12) implies
\[
\| u(t) \|_{q_1} \leq C \| A_{p}^\theta \phi \|_p
\]
for any \( t \in (0, T) \). We next apply (4.11) with \( \phi = u(\tau) \) and \( p = q_1 \); then
\[
\| u(t) \|_q \leq C (t - \tau)^{-\left(\frac{N}{q_1} - \frac{N}{q}/(2m)\right)} \| u(\tau) \|_{q_1}, \quad t \in (\tau, T]
\]
Since \( N(1/q_1 - 1/q)/(2m) = \rho^+ \), we arrive at (4.9) by taking \( \tau = t/2 \) in the above inequality.

As in the preceding consideration, we take any \( \tau \in (0, T) \) and consider (IBP) for \( t \geq \tau \) by regarding \( u(\tau) \) as an initial data in \( L^q(\Omega) \) at \( t = \tau \). Then it follows from (4.5) that
\[
(4.13) \quad \| A_{q}^\theta u(t) \|_q \leq C (t - \tau)^{-\beta} \| u(\tau) \|_q , \quad t \in (\tau, T]
\]
for every \( \beta \in [0, 1] \). By taking \( \tau = t/2 \), it is easy to derive (4.7) from (4.9) and (4.13). The proof of (4.8) is similarly accomplished with use of (4.6) in place of (4.5).

**Theorem 4.6.** Let \( p \) and \( \theta \) satisfy (4.1), (4.4) and
\[
\frac{N(v - 1)}{2mp} + \frac{\ell v}{2m} \geq 1.
\]
For every \( q > p \), the (local) solution \( u \) in Theorem 4.3 satisfies the following estimates with \( \rho = N/(2mp) - N/(2mq) - \theta \);
\[
(4.14) \quad \| A_{q}^\theta u(t) \|_q \leq C t^{-(\beta + \rho)^+} \| A_{p}^\theta \phi \|_p , \quad t \in (0, T]
\]
for every $\beta \in [0, 1]$ and

\begin{equation}
\| A_q^\beta u(t) \|_q \leq C t^{-\gamma - 1 - \rho} \| A_p^\rho \|_p, \quad t \in (0, T]
\end{equation}

for every $\gamma \in [0, 1 - N(v - 1)/(2mqv) - \ell/(2m))$.

**Proof.** We devide the proof into two cases; (I) $N(v - 1)/(2mq) + \ell v/(2m) \geq 1$ and (II) $N(v - 1)/(2mq) + \ell v/(2m) < 1$.

Case (I). We set

$$\tilde{\theta} = \theta + \frac{N}{2mq} - \frac{N}{2mp} (= - \rho).$$

Since

$$1 > \theta > \frac{N}{2mp} + \frac{\ell v}{2m(v - 1)} - \frac{1}{v - 1},$$

it is easy to see

\begin{equation}
1 > \tilde{\theta} > \frac{N}{2mq} + \frac{\ell v}{2m(v - 1)} - \frac{1}{v - 1} \geq 0.
\end{equation}

Now Lemma 4.4 implies $\phi \in D(A_q^\theta) \subset D(A_q^\theta)$; so that one may consider (IBP) in $L^q(\Omega)$. In view of (4.16), we apply Theorem 4.3 with $p = q$ and $\theta = \tilde{\theta}$ to obtain

\begin{equation}
\| A_q^\beta u(t) \|_q \leq C t^{-(\theta - \tilde{\theta})} \| A_q^\theta \|_q, \quad t \in (0, T],
\end{equation}

for every $\beta \in [0, 1]$. Since $\| A_q^\theta \|_q \leq C \| A_p^\rho \|_p$, (4.14) follows from (4.17). The proof of (4.15) is based on (4.6) with $p = q$ and $\theta = \tilde{\theta}$.

Case (II). If $\rho \leq 0$, then the proof is almost the same as Case (I). Note that the local solvability for (IBP) with $\phi \in D(A_q^\theta)$ is assured for every $\theta \geq 0$ in $L^q(\Omega)$. When $\rho > 0$, we choose $r \in (p, q)$ such that

$$\theta + \frac{N}{2mr} - \frac{N}{2mp} = 0.$$

By Lemma 4.4, $\phi \in D(A_q^\theta)$ belongs to $L^r(\Omega)$. Moreover, since

$$\theta > \frac{N}{2mp} + \frac{\ell v}{2m(v - 1)} - \frac{1}{v - 1}$$

by (4.4), $r$ clearly satisfies $1 > N(v - 1)/(2mr) + \ell v/(2m)$. Therefore, making use of Theorem 4.5 with $p = r$ and $\theta = 0$, we have

$$\| A_q^\beta u(t) \|_q \leq C t^{-\rho} \| \phi \|_r, \quad t \in (0, T],$$
for every $\beta \in [0, 1]$. Since $\|\phi\|_r \leq C\|A_p^\beta \phi\|_p$ and
\[
\beta + \frac{N}{2mr} - \frac{N}{2mq} = \beta - \theta + \frac{N}{2mp} - \frac{N}{2mq} = \beta + \rho ,
\]
(4.14) easily follows. The proof of (4.15) is completed along the same idea.

**Corollary 4.7.** Let $p$ and $\theta$ satisfy (4.1) and (4.4). For every $q > p$ and integer $k \leq 2m$, the solution $u$ in Theorem 4.3 satisfies
\[
\|u(t)\|_{W^{k,q}(\Omega)} \leq \begin{cases} 
Ct^{-k/(2m)-\rho^+} \|A_p^\theta \phi\|_p & \text{if } \frac{N(v-1)}{2mp} + \ell v \leq 1 , \\
Ct^{-k/(2m)+\rho^-} \|A_p^\theta \phi\|_p & \text{if } \frac{N(v-1)}{2mp} + \ell v \geq 1 ,
\end{cases}
\]
with $\rho = N/(2mp) - N/(2mq) - \theta$.

**Proof.** It suffices to combine Lemma 4.4 with Theorem 4.5 or 4.6.

Remark 4.1. When classical solutions are concerned, we have only to take a sufficiently large $q$ in Theorem 4.5 or 4.6. Then (ii) of Lemma 4.1 helps us to get various information about the asymptotic behavior of $u$ near $t = 0$ in $C^\sigma(\Omega)$ with $\sigma < 2m + 1$ (see also the paragraph after Theorem 4.3).

Remark 4.2. If $u$ exists globally in time, when we can study its asymptotic behavior as $t \to \infty$ in various function spaces along the preceding arguments with slight modification.

Remark 4.3. Consider (IBP) with $L = -A$, $B_1u = u$ and $f(u) = |u|^{v-1}u$. For this problem, there is an excellent work by Weissler [15]. His local existence result (Theorem 1) includes ours (Theorem 4.3); indeed, he has obtained the solvability conditions which are slightly weaker than (4.4). However, Theorems 4.5, 4.6 and Corollary 4.7 will give us better understanding about the smoothing effects.

**References**


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