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On the Hammerstein Integral Equation with Weakly Singular Kernel

By

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1. Introduction

Let E, F be Banach spaces and let D be a compact subset in the Euclidean space $\boldsymbol{R}^m$. Denote by $L^p(D, E)$ $(p > 1)$ the space of all strongly measurable functions $x: D \to E$ with $\int_D \|x(t)\|^p \, dt < \infty$, provided with the norm $\|x\|_p = \left(\int_D \|x(t)\|^p \, dt\right)^{1/p}$.

In this paper we give sufficient conditions for the existence of a solution $x \in L^p(D, E)$ of the integral equation

$$x(t) = f(t) + \lambda \int_D K(t, s) g(s, x(s)) \, ds \tag{1}$$

with the kernel

$$K(t, s) = A(t, s)|t - s|^{-r} \qquad (t, s \in D, \ t \neq s),$$

where $0 < r < m$ and A is a bounded strongly measurable function from $D \times D$ into the space of continuous linear mappings $F \to E$.

Throughout this paper we shall assume that

1° $f \in L^p(D, E)$;

2° $(s, x) \to g(s, x)$ is a function from $D \times E$ into F such that

(i) g is strongly measurable in s and continuous in x;

(ii) $\|g(s, x)\| \leqq a(s) + b\|x\|$ for $s \in D$ and $x \in E$, where $a \in L^p(D, \boldsymbol{R})$ and $b \geqq 0$.

In contrast to the case $E = \boldsymbol{R}^n$, the above conditions are not sufficient for the existence of a solution of (1) when E is infinite dimensional. Therefore one has imposed additional conditions on g. We shall show that (1) has an L^p-solution whenever g satisfies a Lipschitz condition expressed in terms of Kuratowski's measure of noncompactness. Let us recall that another existence theorems for L^p-solutions of (1), but with a kernel $K \in L^p$, were proved in the papers [6] and [7]. Obviously, in general, a weakly singular kernel is not p-

integrable.

2. Basic lemmas

It is well known that

$$\int_D |t-s|^r\,ds \leqq Q \qquad \text{for all} \quad t\in D, \tag{2}$$

where $Q = 2\pi^{m/2}(\operatorname{diam} D)^{m-r}/(m-r)\Gamma(m/2)$.

Put $c = \max\{\|A(t,s)\| : s, t\in D\}$, $L^p = L^p(D, E)$ and

$$(Sx)(t) = \int_D K(t,s)x(s)\,ds \qquad (x\in L^p,\ t\in D).$$

Lemma 1. *S is a continuous linear mapping of L^p into itself and*

$$\|S\| \leq cQ.$$

Proof. By (2) for each $z\in L^1(D, \boldsymbol{R})$ we have

$$\iint_{D\times D} |t-s|^{-r}|z(s)|\,dsdt = \int_D\Big(\int_D |t-s|^{-r}\,dt\Big)|z(s)|\,ds \leqq Q\int_D |z(s)|\,ds, \tag{3}$$

and therefore for almost every $t\in D$ there exists the integral

$$\int_D |t-s|^{-r}|z(s)|\,ds.$$

This shows that S is well defined. Let $q = p/(p-1)$. If $x\in L^p$, then by the Hölder inequality

$$\begin{aligned}\|(Sx)(t)\| &\leqq \int_D \|A(t,s)\|\,\|x(s)\|\,|t-s|^{-r/p}|t-s|^{-r/q}\,ds\\ &\leqq c\Big(\int_D \|x(s)\|^p|t-s|^{-r}\,ds\Big)^{1/p}\Big(\int_D |t-s|^{-r}\,ds\Big)^{1/q}\\ &\leqq cQ^{1/q}\Big(\int_D \|x(s)\|^p|t-s|^{-r}\,ds\Big)^{1/p}.\end{aligned}$$

Thus

$$\begin{aligned}\int_D \|(Sx)(t)\|^p\,dt &\leqq c^pQ^{p/q}\int_D\Big(\int_D \|x(s)\|^p|t-s|^{-r}\,ds\Big)dt\\ &= c^pQ^{p/q}\int_D\Big(\int_D |t-s|^{-r}\,dt\Big)\|x(s)\|^p\,ds\end{aligned}$$

$$\leqq c^p Q^p \int_D \|x(s)\|^p \, ds,$$

so that

$$\|Sx\|_p \leqq cQ\|x\|_p.$$

■

Lemma 2. *Put* $\tilde{g}(x)(s) = g(s, x(s))$ *for* $x \in L^p$ *and* $s \in D$. *Then* $\tilde{g}$ *is a continuous mapping of* L^p *into itself.*

Proof. Let x_n, $x_0 \in L^p$ and $\lim_{n\to\infty} \|x_n - x_0\|_p = 0$. Suppose that $\|\tilde{g}(x_n) - \tilde{g}(x_0)\|_p$ does not converge to 0 as $n \to \infty$. Then there are $\varepsilon > 0$ and a subsequence $\{x_{n_j}\}$ such that

(4) $$\|\tilde{g}(x_{n_j}) - \tilde{g}(x_0)\|_p > \varepsilon \qquad \text{for } j = 1, 2, \ldots$$

and $\lim_{j\to\infty} x_{n_j}(s) = x_0(s)$ for a.e. $s \in D$. By 2° (i) we have $\lim_{j\to\infty} \|g(s, x_{n_j}(s)) - g(s, x_0(s))\| = 0$ for a.e. $s \in D$. Moreover, as $\lim_{n\to\infty} \|x_n - x_0\|_p = 0$ implies that the sequence $\{x_n\}$ has equiabsolutely continuous norms in L^p, from 2° (ii) it follows that the functions $\|g(\cdot, x_n) - g(\cdot, x_0)\|^p$ $(n = 1, 2, \ldots)$ are equi-integrable on D. Hence, by the Vitali convergence theorem, $\lim_{j\to\infty} \|g(\cdot, x_{n_j}) - g(\cdot, x_0)\|_p = 0$. This contradicts (4). ■

Denote by α and α_1 the Kuratowski measures of noncompactness in E and $L^1(D, E)$, respectively. The next lemma clarifies the relation between α and α_1. For any set V of functions belonging to $L^1(D, E)$ denote by v the function defined by $v(t) = \alpha(V(t))$ for $t \in D$ (under the convention that $\alpha(X) = \infty$ if X is unbounded), where $V(t) = \{x(t): x \in V\}$.

Lemma 3. *Assume that* V *is a countable set of strongly measurable functions* $D \to E$ *and there exists an integrable function* μ *such that* $\|x(t)\| \leqq \mu(t)$ *for all* $x \in V$ *and* $t \in D$. *Then the corresponding function* v *is integrable on* D *and*

$$\alpha\left(\left\{\int_D x(t)\,dt : x \in V\right\}\right) \leqq 2\int_D v(t)\,dt.$$

If, in addition $\limsup_{h\to 0} \sup_{x\in V} \int_D \|x(t+h) - x(t)\|\,dt = 0$, *then*

$$\alpha_1(V) \leqq 2\int_D v(t)\,dt.$$

(*cf.* [1], *Th.* 2.1 *and* [8], *Th.* 1).

3. Existence theorems

Let $H: D \to \boldsymbol{R}_+$ be a measurable function such that the function $(t, s) \to \|A(t, s)\| H(s)$ is bounded on $D \times D$. Put

$$\eta = \sup\{\|A(t, s)\| H(s) : t, s \in D\}.$$

Theorem 1. *If*

$$\alpha(g(s, X)) \leqq H(s)\alpha(X) \tag{5}$$

for $s \in D$ *and for each bounded subset* X *of* E, *then for each* $\lambda \in \boldsymbol{R}$ *such that*

$$|\lambda| bcQ < 1 \quad \text{and} \quad 2|\lambda|\eta Q < 1 \tag{6}$$

there exists a solution $x \in L^p(D, E)$ *of* (1).

Proof. As $|\lambda| bcQ < 1$, by Lemma 1 and the Banach fixed point theorem there exists a nonnegative solution $u \in L^p(D, \boldsymbol{R})$ of the integral equation

$$u(t) = \|f(t)\| + |\lambda| \int_D \|K(t, s)\| a(s) ds + |\lambda| b \int_D \|K(t, s)\| u(s) ds.$$

Put $B = \{x \in L^p : \|x(t)\| \leq u(t) \text{ for a.e. } t \in D\}$ and

$$G(x) = f + \lambda S\tilde{g}(x) \qquad \text{for} \quad x \in B.$$

Since

$$\begin{aligned} \|G(x)(t)\| &\leqq \|f(t)\| + |\lambda| \int_D \|K(t, s)\| (a(s) + b\|x(s)\|) ds \\ &\leqq \|f(t)\| + |\lambda| \int_D \|K(t, s)\| a(s) ds + |\lambda| b \int_D \|K(t, s)\| u(s) ds \\ &= u(t) \end{aligned}$$

for $x \in B$ and $t \in D$, Lemmas 1 and 2 prove that G is a continuous mapping $B \to B$. Moreover,

$$\|G(x)(t + h) - G(x)(t)\| \leqq d(t, h) \qquad \text{for} \quad x \in B,\ t \in D \text{ and small } |h|, \tag{7}$$

where

$$d(t, h) = \begin{cases} u(t) & \text{if } t \in D \text{ and } t + h \notin D, \\ \|f(t + h) - f(t)\| & \\ \quad + |\lambda| \int_D \|K(t + h, s) - K(t, s)\| (a(s) + bu(s)) ds & \text{if } t, t + h \in D. \end{cases}$$

In view of (3) the function $(t, s) \to W(t, s) = K(t, s)(a(s) + bu(s))$ is integrable on $D \times D$. Therefore

$$\lim_{h \to 0} \int_D \left(\int_D \| K(t+h, s) - K(t, s) \| (a(s) + bu(s))\, ds \right) dt$$
$$= \lim_{h \to 0} \iint_{D \times D} \| W(t+h, s) - W(t, s) \| \, ds dt$$
$$= 0,$$

and consequently

$$\lim_{h \to 0} \int_D d(t, h)\, dt = 0 \qquad \text{for} \quad t \in D. \tag{8}$$

This fact, plus (7), implies that

$$\lim_{h \to 0} \sup_{x \in B} \int_D \| G(x)(t+h) - G(x)(t) \| \, dt = 0. \tag{9}$$

Let V be a countable subset of B such that

$$V \subset \overline{\text{conv}}\,(G(V) \cup \{0\}). \tag{10}$$

Then $V(t) \subset \overline{\text{conv}\,(G(V)(t) \cup \{0\}}$ for a.e. $t \in D$, so that

$$\alpha(V(t)) \leqq \alpha(G(V)(t)) \qquad \text{for a.e. } t \in D. \tag{11}$$

Put $v(t) = \alpha(V(t))$ for $t \in D$. From (9) and (10) it is clear that

$$\lim_{h \to 0} \sup_{x \in V} \int_D \| x(t+h) - x(t) \| \, dt = 0.$$

Moreover, $\| x(t) \| \leqq u(t)$ for all $x \in V$ and a.e. $t \in D$. Hence, by Lemma 3, $v \in L^p(D, \boldsymbol{R})$ and

$$\alpha_1(V) \leqq 2 \int_D v(t)\, dt. \tag{12}$$

From (3) it follows that

$$\int_D |t - s|^{-r}(a(s) + bu(s))\, ds < \infty \qquad \text{for} \quad \text{a.e. } t \in D. \tag{13}$$

Fix now $t \in D$ such that the integral (13) is finite. Since

$$\| K(t, s) g(s, x(s)) \| \leqq c|t - s|^{-r}(a(s) + bu(s)) \qquad \text{for} \quad x \in B \text{ and } s \in D,$$

owing to (11), (5) and Lemma 3 we get

$$\alpha(V(t)) \leqq |\lambda| \alpha\left(\left\{\int_D K(t, s) g(s, x(s)) ds : x \in V\right\}\right)$$
$$\leqq 2|\lambda| \int_D \alpha\left(\left\{K(t, s) g(s, x(s)) : x \in V\right\}\right) ds$$
$$\leqq 2|\lambda| \int_D \| K(t, s) \| H(s) \alpha(V(s)) \, ds,$$

i.e.

$$v(t) \leqq 2|\lambda| \int_D \| K(t, s) \| H(s) v(s) \, ds.$$

As the last inequality holds for a.e. $t \in D$, from (2) it follows that

$$\int_D v(t)\, dt \leqq 2|\lambda| \int_D \left(\int_D \| A(t, s) \| \, |t-s|^{-r} H(s) v(s)\, ds\right) dt$$
$$\leqq 2|\lambda| \int_D \left(\int_D \eta |t-s|^{-r} dt\right) v(s)\, ds \leqq 2|\lambda| \eta Q \int_D v(s)\, ds.$$

Because $2|\lambda|\eta Q < 1$, this proves that $\int_D v(t)\, dt = 0$. Thus, by (12), $\alpha_1(V) = 0$, so that V is relatively compact in L^1. On the other hand, the set B has equiabsolutely continuous norms in L^p and $V \subseteq B$. Consequently, V is relatively compact in L^p.

Applying now Mönch's fixed point theorem (see [3], Th. 2.1), we conclude that there exists $x \in B$ such that $x = G(x)$. Clearly x is a solution of (1). ■

We shall now present another existence theorem for (1). Using (7) and (8), and applying the method of proof of Theorem in [7], we can prove the following

Theorem 2. *If for any* $\varepsilon > 0$ *and for any bounded subset* X *of* E *there exists a closed subset* D_ε *of* D *such that* $\mathrm{mes}(D \setminus D_\varepsilon) < \varepsilon$ *and*

$$(5') \qquad \alpha(g(T \times X)) \leqq \sup_{s \in T} H(s) \alpha(X)$$

for each closed subset T *of* D_ε, *then for each* $\lambda \in \mathbf{R}$ *such that*

$$(6') \qquad |\lambda| bcQ < 1 \quad \textit{and} \quad |\lambda| \eta Q < 1$$

there exists a solution $x \in L^p(D, E)$ *of* (1).

Let us remark that (5′) is stronger than (5) while (6′) is weaker than (6).

Corollary. *Assume now that the functions A and f are continuous. Then under the assumptions of Theorem* 1 *or Theorem* 2 *the equation* (1) *has a continuous solution.*

Proof. From Theorem 1 or Theorem 2 it is clear that there exists a solution $x \in L^p$ of (1). On the other hand, it is well known that if the function A is continuous, then for each $y \in L^1(D, E)$ the function Sy is continuous. Consequently, the function $x = f + S\tilde{g}(x)$ is continuous. ■

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