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Oscillation Criteria for Second Order Difference Equation

By

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1. Introduction

There is an extensive literature on the topic of oscillation criteria for the linear second order differential equation

$$u'' + p(x)u = 0, \qquad 0 < x < \infty .$$

Oscillation criteria for second order linear difference equation have also been investigated by a number of authors in recent years [2, 3, 5, 6, 7, 8, 10], but the literature is relatively limited. In this paper, we are concerned with the difference equation

$$\Delta^2 x_{k-1} + b_k x_k = 0, \qquad k = 1, 2, 3, \cdots \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x_k = x_{k+1} - x_k$, and $\{b_k\}_1^\infty$ is a nonnegative sequence with infinitely many positive terms. A nontrivial solution $\{x_k\}_0^\infty$ of (1.1) is said to be oscillatory if for every positive integer N, there exists $n \geq N$ such that $x_n x_{n+1} \leq 0$, and nonoscillatory otherwise. It is known that if (1.1) has an oscillatory solution then all its nontrivial solutions are oscillatory [2, Theorem 7]. Thus we may classify (1.1) as being oscillatory or nonoscillatory according to whether it has an oscillatory solution or not.

We shall derive in this paper a number of oscillation criteria several of which are discrete analogs of those of Nehari [9] and Hille [11]. As we shall see in the following development, some of the ideas behind these analogs are similar to those employed by Nehari, but the details are substantially different due to the discrete nature of our equation (1.1).

In section two, we shall give a number of preparatory results, one of which is based on Wirtinger's type theorems obtained by Cheng [2, 3]. A lemma which bridges discrete and continuous functions is also given here. In section

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three, oscillation and nonoscillation criteria are derived. In the last section, the concept of conditional oscillation [9] is introduced and related oscillation criteria are derived.

2. Preparatory lemmas

We shall make use of several preparatory results the first of which states that a nonoscillatory solution of (1.1) is eventually monotonic.

Lemma 2.1. ([7, Lemma 4.2]) *If* $\{x_k\}_0^\infty$ *is a solution of* (1.1) *such that* $x_k > 0$ *for all* $k \geq M$, *then* $\Delta x_k > 0$ *for* $k \geq M$.

With respect to the real numbers σ and ξ, a real vector $v = (v_0, v_1, \cdots, v_{n+1})$ is said to be admissible if it is nontrivial and satisfies $v_0 + \sigma v_1 = 0$ and $v_{n+1} + \xi v_n = 0$. A result of Cheng [2, 3] states that if $u = (u_0, u_1, \cdots, u_{n+1})$ is an admissible solution of

$$\Delta^2 u_{k-1} + q_k u_k = 0, \qquad k = 1, 2, \cdots, n$$

then

$$(1 + \sigma)u_1^2 + \sum_{k=1}^{n-1} (\Delta u_k)^2 + (1 + \xi)u_n^2 = \sum_{k=1}^{n} q_k u_k^2, \tag{2.1}$$

and if $v = (v_0, v_1, \cdots, v_{n+1})$ is a solution of the above equation such that $v_k > 0$ for $1 \leq k \leq n$ and $v_0 + \sigma v_1 \geq 0$, $v_{n+1} + \xi v_n \geq 0$ with at least one strict inequality, then for any admissible vector $w = (w_0, w_1, \cdots, w_{n+1})$,

$$(1 + \sigma)w_1^2 + \sum_{k=1}^{n-1} (\Delta w_k)^2 + (1 + \xi)w_n^2 > \sum_{k=1}^{n} q_k w_k^2. \tag{2.2}$$

As a consequence, we have the following

Theorem 2.1. *Equation* (1.1) *is nonoscillatory if and only if there is a positive integer* M *such that*

$$\sum_{k=M+1}^{N} q_k y_k^2 < y_{M+1}^2 + \sum_{k=M+1}^{N-1} (\Delta y_k)^2 \tag{2.3}$$

holds for any nontrivial vector $(y_{M+1}, y_{M+2}, \cdots, y_N)$.

Proof. If (1.1) is nonoscillatory, then there is a solution $\{x_k\}_0^\infty$ which satisfies $x_k > 0$ for $k \geq M$. By Lemma 2.1, we have $\Delta x_k > 0$ for $k \geq M$, so that (2.3) holds by (2.2).

Conversely, suppose there exists a positive integer M such that (2.3) holds for any nontrivial vector $(y_{M+1}, y_{M+2}, \cdots, y_N)$. Let $\{x_k\}_0^\infty$ be the solution of

(1.1) determined by the conditions $x_M = 0$ and $x_{M+1} = 1$. We assert that $\Delta x_k > 0$ for $k > M$. Suppose to the contrary N is the first integer greater than M such that $\Delta x_N \leq 0$ and that $x_k > 0$ for $M+1 \leq k \leq N$. Define the vector $(y_{M+1}, y_{M+2}, \cdots, y_N)$ by $y_k = x_k$ for $M+1 \leq k \leq N$. Then according to (2.1),

$$\begin{aligned} y_{M+1}^2 &+ \sum_{k=M+1}^{N-1} (\Delta y_k)^2 - \sum_{k=M+1}^{N} b_k y_k^2 \\ &= x_{M+1}^2 + \sum_{k=M+1}^{N-1} (\Delta x_k)^2 - \sum_{k=M+1}^{N} b_k x_k^2 \\ &= -\left(1 - \frac{x_{N+1}}{x_N}\right) x_N^2 = x_N \Delta x_N \leq 0, \end{aligned}$$

which contradicts our assumption. Q.E.D.

As an illustration of Theorem 2.1, consider the difference equation

$$\Delta^2 x_{k-1} + \frac{1}{(4k+1)(4k+2)} x_k = 0, \qquad k = 1, 2, 3, \cdots. \tag{2.4}$$

We assert that this equation has a nonoscillatory solution. This can be shown directly by finding a positive solution of (2.4) or indirectly by means of Theorem 2.1. First of all, consider the Riccati equation [6],

$$\gamma_k + \frac{1}{\gamma_{k-1}} = 2 - \frac{1}{(4k+1)(4k+2)}, \qquad k = 2, 3, \cdots.$$

Let $\gamma_1 = 5/4$, then we can show by induction that

$$\gamma_k > 1 + \frac{1}{4(k+1)}, \qquad k = 1, 2, \cdots.$$

Indeed, our assertion clearly holds for $k = 1$. Assume our assertion holds for $k = n - 1$, then

$$\begin{aligned} \gamma_n &= 2 - \frac{1}{(4n+1)(4n+2)} - \frac{1}{\gamma_{n-1}} \\ &> 2 - \left(\frac{1}{4n+1} - \frac{1}{4n+2}\right) - 1 + \frac{1}{4n+1} \\ &= 1 + \frac{1}{4n+2} > 1 + \frac{1}{4(n+1)}, \end{aligned}$$

which completes our proof. Now let $x_0 = 43/60$, $x_1 = 1$ and $x_{k+1} = \gamma_1 \gamma_2 \cdots \gamma_k$ for $k \geq 1$, then it is easily seen that $\{x_k\}_0^\infty$ is a positive solution of (2.4). This shows that (2.4) is nonoscillatory.

Next, for any nontrivial vector $(y_{M+1}, y_{M+2},\cdots, y_N)$, we have

$$
\begin{aligned}
y_{M+1}^2 &+ \sum_{k=M+1}^{N-1} (\Delta y_k)^2 - \sum_{k=M+1}^{N} \frac{1}{(4k+1)(4k+2)} y_k^2 \\
&= y_{M+1}^2 + \sum_{k=M+1}^{N-1} \{y_k^2 - 2y_k y_{k+1} + y_{k+1}^2\} - \sum_{k=M+1}^{N} \left\{\frac{1}{4k+1} - \frac{1}{4k+2}\right\} y_k^2 \\
&= \left[1 - \frac{1}{4(M+1)+1}\right] y_{M+1}^2 + \sum_{k=M+1}^{N-1} \left\{\left[1 + \frac{1}{4k+2}\right] y_k^2 - 2y_k y_{k+1}\right. \\
&\quad \left. + \left[1 - \frac{1}{4(k+1)+1}\right] y_{k+1}^2\right\} + \frac{1}{4N+2} y_N^2.
\end{aligned}
$$

Since

$$
\left[1 + \frac{1}{4k+2}\right]\left[1 - \frac{1}{4(k+1)+1}\right] = \frac{16k^2 + 28k + 12}{16k^2 + 28k + 10} > 1,
$$

$$
1 + \frac{1}{4k+2} > 0, \text{ and } 1 - \frac{1}{4(k+1)+1} > 0,
$$

we have

$$
\left[1 + \frac{1}{4k+2}\right] y_k^2 - 2y_k y_{k+1} + \left[1 - \frac{1}{4(k+1)+1}\right] y_{k+1}^2 \geq 0,
$$

and equality holds only if $y_k = y_{k+1} = 0$. Since $(y_{M+1}, y_{M+2},\cdots, y_N)$ is nontrivial, we see that

$$
\sum_{k=M+1}^{N} \frac{1}{(4k+1)(4k+2)} y_k^2 < y_{M+1}^2 + \sum_{k=M+1}^{N-1} (\Delta y_k)^2,
$$

which, by means of Theorem 2.1, shows that (2.4) is nonoscillatory.

As another example, the second order linear difference equation

$$
\Delta^2 x_{k-1} + \frac{1}{2k(2k+1)} x_k = 0, \qquad k = 1, 2, 3, \cdots \tag{2.5}
$$

has a positive solution $\{x_k\}_0^\infty$ defined by $x_0 = 1$ and

$$
x_k = 1 \cdot \frac{2}{1} \cdot \frac{4}{3} \cdots \frac{2k}{2k-1}, \qquad k \geq 1.
$$

This shows that (2.4) is nonoscillatory.

Next, for any nontrivial vector $(y_{M+1}, y_{M+2},\cdots, y_N)$, we have

$$
y_{M+1}^2 + \sum_{k=M+1}^{N-1} (\Delta y_k)^2 - \sum_{k=M+1}^{N} \frac{1}{2k(2k+1)} y_k^2
$$

$$= \frac{2M+1}{2M+2} y_{M+1}^2 + \sum_{k=M+1}^{N} \frac{2k+2}{2k+1}\left[y_k - \left[\frac{2k+1}{2k+2}\right] y_{k+1}\right]^2 + \frac{1}{2N+1} y_N^2 \geq 0,$$

where equality holds only if $y_k = \left[\frac{2k+1}{2k+2}\right] y_{k+1}$ for $k = M+1, \cdots, N-1$, and $y_{M+1} = y_N = 0$. Since $(y_{M+1}, y_{M+2}, \cdots, y_N)$ is a nontrivial vector, therefore, we obtain

$$\sum_{k=M+1}^{N} \frac{1}{2k(2k+1)} y_k^2 < y_{M+1}^2 + \sum_{k=M+1}^{N-1} (\Delta y_k)^2.$$

Lemma 2.3. *If* $\psi(k) = ak + b$, *then*

(i) $\Delta \psi^{\tau+1}(k) \geq a(\tau+1)\psi^{\tau}(k)$ *if* $\tau \geq 0$, $a \geq 0$, $\psi(k) \geq 0$,

(ii) $\Delta \psi^{\tau+1}(k) > a(\tau+1)\psi^{\tau}(k+1)$ *if* $-1 < \tau < 0$, $a > 0$, $\psi(k) \geq 0$,

(iii) $\{\Delta \psi^{\tau/2}(k)\}^2 \leq \dfrac{a\tau^2}{4(\tau-1)} \Delta \psi^{\tau-1}(k-1)$

if $0 \leq \tau \leq 2$, $\tau \neq 1$, $a \geq 0$, $\psi(k-1) > 0$,

(iv) $\{\Delta \psi^{\tau/2}(k)\}^2 < \dfrac{a\tau^2}{4(\tau-1)} \Delta \psi^{\tau-1}(k+1)$ *if* $\tau > 2$, $\psi(k) \geq 0$, $a > 0$,

(v) $a\psi^{\tau}(k) \geq \dfrac{1}{\tau+1} \Delta \psi^{\tau+1}(k-1)$ *if* $\tau \geq 0$, $\psi(k-1) \geq 0$, $a \geq 0$,

(vi) $a\psi^{\tau}(k) > \dfrac{1}{\tau+1} \Delta \psi^{\tau+1}(k)$ *if* $\tau < 0$, $\tau \neq -1$, $\psi(k) > 0$, $a > 0$.

Proof. Suppose $\tau \geq 0$, $a \geq 0$ and $\psi(k) \geq 0$. Then by the mean value theorem, we have

$$\begin{aligned} \Delta \psi^{\tau+1}(k) &= \psi^{\tau+1}(k+1) - \psi^{\tau+1}(k) \\ &= a(\tau+1)\psi^{\tau}(\xi_k) \geq a(\tau+1)\psi^{\tau}(k), \qquad k < \xi_k < k+1, \end{aligned}$$

which proves the validity of (i). The case (ii) is similarly proved.

If $0 \leq \tau \leq 2$, $\tau \neq 1$, $a \geq 0$, and $\psi(k-1) > 0$, then

$$\begin{aligned} \{\Delta \psi^{\tau/2}(k)\}^2 &= \{\psi^{\tau/2}(k+1) - \psi^{\tau/2}(k)\}^2 \\ &= \frac{a\tau^2}{4}\{a\psi^{\tau-2}(\mu_k)\} \qquad k < \mu_k < k+1 \\ &\leq \frac{a\tau^2}{4}\{a\psi^{\tau-2}(k)\} \end{aligned}$$

$$\le \frac{a\tau^2}{4}\int_{\psi(k-1)}^{\psi(k)} x^{\tau-2}dx$$
$$= \frac{a\tau^2}{4(\tau-1)}\Delta\psi^{\tau-1}(k-1),$$

which proves the validity of (iii). The case (iv) is similarly proved.

If $\tau \ge 0$, $a \ge 0$ and $\psi(k-1) \ge 0$, then

$$a\psi^{\tau}(k) = a(ak+b)^{\tau} \ge \int_{\psi(k-1)}^{\psi(k)} x^{\tau}dx$$
$$= \frac{1}{\tau+1}\{\psi^{\tau+1}(k) - \psi^{\tau+1}(k-1)\}$$
$$= \frac{1}{\tau+1}\Delta\psi^{\tau+1}(k-1),$$

which proves the validity of (v). The case (vi) is similarly proved.

Q.E.D.

For sequences $\{x_k\}$ and $\{y_k\}$, the following summation by parts (also known as the Abel's transformation) will be useful:

$$\sum_{k=m}^{n} x_k \Delta y_k = x_{n+1}y_{n+1} - x_m y_m - \sum_{k=m}^{n} y_{k+1}\Delta x_k.$$

3. Oscillation criteria

We first illustrate how Theorem 2.1 can be applied to yield oscillation criteria in a very simple case. If equation (1.1) is nonoscillatory, then there exists a positive integer M such that (2.3) holds for any nontrivial vector $(y_{M+1}, y_{M+2}, \cdots, y_N)$. Let n be a positive integer such that $M < n < N$, and let

$$y_k = \begin{cases} (k-M)/(n-M) & M+1 \le k \le n, \\ 1 & k \ge n. \end{cases}$$

Then

$$y_{M+1}^2 + \sum_{k=M+1}^{N-1} (\Delta y_k)^2 = y_{M+1}^2 + \sum_{k=M+1}^{n-1} (\Delta y_k)^2 + \sum_{k=n}^{N-1} (\Delta y_k)^2 = \frac{1}{n-M}. \tag{3.1}$$

By (2.3), we then have

$$\sum_{k=n+1}^{N} b_k = \sum_{k=n+1}^{N} b_k y_k^2 \le \sum_{k=M+1}^{N} b_k y_k^2 < \frac{1}{n-M}, \qquad N > M+1. \tag{3.2}$$

Since N may be taken arbitrarily large, thus (3.2) implies

$$(n-M)\sum_{k=n+1}^{\infty} b_k \leq 1.$$

Letting

$$b^* = \limsup_{n\to\infty} n \sum_{k=n+1}^{\infty} b_k,$$

we see that if $b^* > 1$, then (1.1) is oscillatory.

More generally, we have the following

Theorem 3.1. *If* (1.1) *is nonoscillatory and* $0 \leq \alpha < 1$, $\beta > 1$, *then for any real number* ω, *there exists a positive integer* $M > \omega$, *such that*

$$\limsup_{n\to\infty}\left\{(n-\omega)^{1-\beta}\sum_{k=M+1}^{n}(k-\omega)^{\beta}b_k + (n-\omega)^{1-\alpha}\sum_{k=n+1}^{\infty}(k-\omega)^{\alpha}b_k\right\} \leq \frac{\beta-\alpha}{4}\left\{1+\frac{1}{(\beta-1)(1-\alpha)}\right\}. \tag{3.3}$$

Proof. Since (1.1) is nonoscillatory, it has a solution $\{x_k\}$ such that $x_k > 0$ for $k \geq M > \omega$. By Lemma 2.2, (2.3) holds for any nontrivial vector $(y_{M+1}, y_{M+2}, \cdots, y_N)$. Let n be a positive integer such that $M < n < N$ and let

$$y_k = \begin{cases} \left[\dfrac{k-\omega}{n-\omega}\right]^{\beta/2} & M < k \leq n, \\ \left[\dfrac{k-\omega}{n-\omega}\right]^{\alpha/2} & n \leq k \leq N. \end{cases}$$

Note first that since

$$\Delta y_n = \left[\frac{n+1-\omega}{n-\omega}\right]^{\alpha/2} - 1 < \left[\frac{n+1-\omega}{n-\omega}\right]^{\beta/2} - 1,$$

it follows that

$$\begin{aligned} y_{M+1}^2 + \sum_{k=M+1}^{N-1}(\Delta y_k)^2 &< y_{M+1}^2 + \sum_{k=M+1}^{n}\left\{\left[\frac{k+1-\omega}{n-\omega}\right]^{\beta/2} - \left[\frac{k-\omega}{n-\omega}\right]^{\beta/2}\right\}^2 \\ &\quad + \sum_{k=n+1}^{N-1}\left\{\left[\frac{k+1-\omega}{n-\omega}\right]^{\alpha/2} - \left[\frac{k-\omega}{n-\omega}\right]^{\alpha/2}\right\}^2. \end{aligned} \tag{3.4}$$

We have two cases to consider. If $1 < \beta \leq 2$, then by Lemma 2.3 (iii) and (3.4), we have

$$(3.5)\quad y_{M+1}^2 + \sum_{k=M+1}^{N-1} (\Delta y_k)^2$$
$$< y_{M+1}^2 + \frac{\beta^2}{4(\beta-1)}\frac{1}{n-\omega}\sum_{k=M+1}^{n}\left\{\left[\frac{k-\omega}{n-\omega}\right]^{\beta-1} - \left[\frac{k-1-\omega}{n-\omega}\right]^{\beta-1}\right\}$$
$$+ \frac{\alpha^2}{4(\alpha-1)}\frac{1}{n-\omega}\sum_{k=n+1}^{N-1}\left\{\left[\frac{k-\omega}{n-\omega}\right]^{\alpha-1} - \left[\frac{k-1-\omega}{n-\omega}\right]^{\alpha-1}\right\}$$
$$= y_{M+1}^2 + \frac{\beta^2}{4(\beta-1)}\frac{1}{n-\omega}\left\{1 - \left[\frac{M-\omega}{n-\omega}\right]^{\beta-1}\right\}$$
$$+ \frac{\alpha^2}{4(\alpha-1)}\frac{1}{n-\omega}\left\{-1 - \left[\frac{N-1-\omega}{n-\omega}\right]^{\alpha-1}\right\}$$
$$< \left[\frac{M+1-\omega}{n-\omega}\right]^{\beta} + \frac{\beta^2}{4(\beta-1)(n-\omega)} + \frac{\alpha^2}{4(1-\alpha)(n-\omega)}.$$

Since

$$(3.6)\quad \sum_{k=M+1}^{N} b_k y_k^2 = (n-\omega)^{-\beta}\sum_{k=M+1}^{n}(k-\omega)^{\beta} b_k + (n-\omega)^{-\alpha}\sum_{k=n+1}^{N}(k-\omega)^{\alpha} b_k,$$

by (2.3) and (3.5), it follows that

$$(n-\omega)^{-\beta}\sum_{k=M+1}^{n}(k-\omega)^{\beta} b_k + (n-\omega)^{-\alpha}\sum_{k=n+1}^{N}(k-\omega)^{\alpha} b_k$$
$$< \left[\frac{M+1-\omega}{n-\omega}\right]^{\beta} + \frac{\beta^2}{4(\beta-1)(n-\omega)} + \frac{\alpha^2}{4(1-\alpha)(n-\omega)}.$$

Multiplying the above inequality by $(n-\omega)$ and letting N approach infinity, we obtain

$$(n-\omega)^{1-\beta}\sum_{k=M+1}^{n}(k-\omega)^{\beta} b_k + (n-\omega)^{1-\alpha}\sum_{k=n+1}^{\infty}(k-\omega)^{\alpha} b_k$$
$$\leq \left[\frac{M+1-\omega}{n-\omega}\right]^{\beta-1}(M+1-\omega) + \frac{\beta^2}{4(\beta-1)} + \frac{\alpha^2}{4(1-\alpha)},$$

which implies (3.3) as desired.

If $\beta > 2$, then by (3.4) and Lemma 3.3 (iii-iv), we have

$$(3.7)\quad y_{M+1}^2 + \sum_{k=M+1}^{N-1}(\Delta y_k)^2$$
$$< y_{M+1}^2 + \frac{\beta^2}{4(\beta-1)}\frac{1}{n-\omega}\sum_{k=M+1}^{n}\left\{\left[\frac{k+2-\omega}{n-\omega}\right]^{\beta-1} - \left[\frac{k+1-\omega}{n-\omega}\right]^{\beta-1}\right\}$$

$$+\frac{\alpha^2}{4(\alpha-1)}\frac{1}{n-\omega}\sum_{k=n+1}^{N-1}\left\{\left[\frac{k-\omega}{n-\omega}\right]^{\alpha-1}-\left[\frac{k-1-\omega}{n-\omega}\right]^{\alpha-1}\right\}$$

$$=\left[\frac{M+1-\omega}{n-\omega}\right]^{\beta}+\frac{\beta^2}{4(\beta-1)}\frac{1}{n-\omega}\left\{\left[\frac{n+2-\omega}{n-\omega}\right]^{\beta-1}-\left[\frac{M+2-\omega}{n-\omega}\right]^{\beta-1}\right\}$$

$$+\frac{\alpha^2}{4(1-\alpha)}\frac{1}{n-\omega}\left\{1-\left[\frac{n-\omega}{N-1-\omega}\right]^{1-\alpha}\right\}$$

$$<\left[\frac{M+1-\omega}{n-\omega}\right]^{\beta}+\frac{\beta^2}{4(\beta-1)(n-\omega)}\left[\frac{n+2-\omega}{n-\omega}\right]^{\beta-1}+\frac{\alpha^2}{4(1-\alpha)(n-\omega)}.$$

By (2.3), (3.6) and (3.7), we then have

$$(n-\omega)^{-\beta}\sum_{k=M+1}^{n}(k-\omega)^{\beta}b_k+(n-\omega)^{-\alpha}\sum_{k=n+1}^{N}(k-\omega)^{\alpha}b_k$$

$$<\left[\frac{M+1-\omega}{n-\omega}\right]^{\beta}+\frac{1}{n-\omega}\left\{\frac{\beta^2}{4(\beta-1)}\left[\frac{n+2-\omega}{n-\omega}\right]^{\beta-1}+\frac{\alpha^2}{4(1-\alpha)}\right\}.$$

Again, multiplying the above inequality by $(n-\omega)$ and then letting N approach infinity, we obtain

$$(n-\omega)^{1-\beta}\sum_{k=M+1}^{n}(k-\omega)^{\beta}b_k+(n-\omega)^{1-\alpha}\sum_{k=n+1}^{\infty}(k-\omega)^{\alpha}b_k$$

$$\leq\left[\frac{M+1-\omega}{n-\omega}\right]^{\beta-1}(M+1-\omega)+\left\{\frac{\beta^2}{4(\beta-1)}\left[\frac{n+2-\omega}{n-\omega}\right]^{\beta-1}+\frac{\alpha^2}{4(1-\alpha)}\right\},$$

which implies (3.3) as desired. Q.E.D.

If we choose $\omega=0$ in the above Theorem, then (3.3) is changed to

$$\limsup_{n\to\infty}\{n^{1-\beta}\sum_{k=M+1}^{n}k^{\beta}b_k+n^{1-\alpha}\sum_{k=n+1}^{\infty}k^{\alpha}b_k\}\leq\frac{\beta-\alpha}{4}\left\{1+\frac{1}{(\beta-1)(1-\alpha)}\right\}. \tag{3.8}$$

Furthermore, since both terms on the left hand side of (3.8) are nonnegative, we may let $\alpha=0$ in (3.8) to obtain

$$\limsup_{n\to\infty}n^{1-\beta}\sum_{k=M+1}^{n}k^{\beta}b_k\leq\frac{\beta^2}{4(\beta-1)}$$

and let $\beta=2$ to obtain

$$\limsup_{n\to\infty}n^{1-\alpha}\sum_{k=n+1}^{\infty}k^{\alpha}b_k\leq\frac{(2-\alpha)^2}{4(1-\alpha)}. \tag{3.9}$$

Note that if we let $\alpha = 0$ in (3.9), then we shall obtain $b^* \leq 1$, which has been shown earlier.

For the next result, we shall need the sequence $\{r_k\}_1^\infty$ defined by

$$r_k = k^{1-\alpha} \sum_{j=k+1}^{\infty} j^\alpha b_j, \tag{3.10}$$

where $0 \leq \alpha < 1$. Note that if (1.1) is nonoscillatory, then the positive sequence $\{r_k\}$ is bounded in view of (3.9).

Theorem 3.2 *Suppose* $0 \leq \alpha < 1$, $\beta > 1$ *and* (1.1) *is nonoscillatory, then there exists a positive integer M such that*

$$\limsup_{n\to\infty} n^{1-\beta} \sum_{k=M+1}^{n-1} k^{\beta-2} r_k \leq \frac{1}{4} + \frac{1}{4(\beta-1)(1-\alpha)}. \tag{3.11}$$

Proof. Note first that if we let

$$T_k = k^{\alpha-1} r_k = \sum_{j=k+1}^{\infty} j^\alpha b_j, \qquad k \geq M,$$

then in view of (3.9), T_k is bounded and

$$\Delta T_k = -(k+1)^\alpha b_{k+1} \leq 0, \qquad k \geq M,$$
$$b_k = -k^{-\alpha} \Delta T_{k-1}, \qquad k \geq M+1.$$

Consider first the case $\beta - \alpha - 1 \geq 0$. Since

$$\sum_{k=M+1}^{n} k^\beta b_k = -\sum_{k=M+1}^{n} k^{\beta-\alpha} \Delta T_{k-1},$$

summing the right hand side by parts, we then have

$$\begin{aligned}
\sum_{k=M+1}^{n} k^\beta b_k &= -(n+1)^{\beta-\alpha} T_n + (M+1)^{\beta-\alpha} T_M + \sum_{k=M+1}^{n} T_k \Delta k^{\beta-\alpha} \\
&= -n^{\beta-\alpha} T_n + (M+1)^{\beta-\alpha} T_M + \sum_{k=M+1}^{n-1} T_k \Delta k^{\beta-\alpha} \\
&> -n^{\beta-\alpha} T_n + \sum_{k=M+1}^{n-1} T_k \Delta k^{\beta-\alpha} \\
&\geq -n^{\beta-\alpha} T_n + \sum_{k=M+1}^{n-1} T_k (\beta-\alpha) k^{\beta-\alpha-1} \\
&= -n^{\beta-1} r_n + (\beta-\alpha) \sum_{k=M+1}^{n-1} r_k k^{\beta-2},
\end{aligned}$$

where Lemma 2.3 (i) has been used to derive the second inequality.

Next we consider the case $\beta - \alpha - 1 < 0$. Since

$$\sum_{k=M+1}^{n} k^{\beta} b_k = - \sum_{k=M+1}^{n} k^{\beta-\alpha} \Delta T_{k-1} \geq - \sum_{k=M+1}^{n} (k-1)^{\beta-\alpha} \Delta T_{k-1},$$

summing the last term by parts again, we obtain

$$\begin{aligned}\sum_{k=M+1}^{n} k^{\beta} b_k &\geq - n^{\beta-\alpha} T_n + M^{\beta-\alpha} T_M + \sum_{k=M+1}^{n} T_k \Delta (k-1)^{\beta-\alpha} \\ &> - n^{\beta-\alpha} T_n + \sum_{k=M+1}^{n-1} T_k (\beta - \alpha) k^{\beta-\alpha-1} \\ &= - n^{\beta-1} r_n + (\beta - \alpha) \sum_{k=M+1}^{n-1} r_k k^{\beta-2},\end{aligned}$$

where Lemma 2.3 (ii) has been used to derive the second inequality.

In either case, it follows that

$$(\beta - \alpha) n^{1-\beta} \sum_{k=M+1}^{n-1} r_k k^{\beta-2} < n^{1-\beta} \sum_{k=M+1}^{n} k^{\beta} b_k + n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha} b_k$$

Our assertion is now clear from Theorem 3.1. Q.E.D.

Theorem 3.3. *If* (1.1) *is nonoscillatory and* $0 \leq \alpha < 1$, *then*

$$\liminf_{n \to \infty} n^{1-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha} b_k \leq \frac{1}{4(1-\alpha)}. \tag{3.12}$$

Proof. If $1 < \beta < 2$, then by Theorem 3.2, there is a positive integer M such that (3.11) holds. Let z be an integer such that $M + 1 < z < n - 1$, then

$$\limsup_{n \to \infty} \{ n^{1-\beta} \sum_{k=M+1}^{z-1} k^{\beta-2} r_k + n^{1-\beta} \sum_{k=z}^{n-1} k^{\beta-2} r_k \} \leq \frac{1}{4} + \frac{1}{4(\beta-1)(1-\alpha)}. \tag{3.13}$$

Let $m_z = \inf\{r_k | k = z, z+1, \cdots\}$, then

$$\begin{aligned} n^{1-\beta} \sum_{k=z}^{n-1} k^{\beta-2} r_k &\geq m_z \sum_{k=z}^{n-1} \left[\frac{k}{n} \right]^{\beta-2} \frac{1}{n} \\ &> m_z \sum_{k=z}^{n-1} \frac{1}{\beta-1} \Delta \left[\frac{k}{n} \right]^{\beta-1} = \frac{m_z}{\beta-1} \left\{ 1 - \left[\frac{z}{n} \right]^{\beta-1} \right\}, \end{aligned}$$

where the second inequality is obtained by means of Lemma 2.3 (vi). It follows that

$$\frac{1}{4} + \frac{1}{4(\beta-1)(1-\alpha)} \geq \limsup_{n \to \infty} n^{1-\beta} \sum_{k=z}^{n-1} k^{\beta-2} r_k \geq \frac{m_z}{\beta-1}.$$

Multiplying throughout by $\beta - 1$ and letting $\beta \to 1$, we have

$$m_z \leq \frac{1}{4(1-\alpha)}$$

which implies (3.12) as desired. Q.E.D.

As an immediate consequence, we may let $\alpha = 0$ in Theorem 3.3 to obtain

Theorem 3.4. *If* (1.1) *is nonoscillatory, then*

$$b_* = \liminf_{n\to\infty} n \sum_{k=n+1}^{\infty} b_k \leq \frac{1}{4}. \tag{3.14}$$

The number $1/4$ *is the best possible.*

To see that (3.14) is sharp, let $b_k = 1/2k(2k+1)$ for $k \geq 1$. As we have seen, the equation (2.5) is nonoscillatory. Also

$$\frac{1}{4} = \liminf_{n\to\infty} n \sum_{k=n+1}^{\infty} \frac{1}{4k(k+1)} \leq \liminf_{n\to\infty} n \sum_{k=n+1}^{\infty} b_k$$
$$= \liminf_{n\to\infty} n \sum_{k=n+1}^{\infty} \frac{1}{2}\left\{\frac{1}{2k-1} - \frac{1}{2k+1}\right\} = \frac{1}{4}.$$

as desired.

A further example can be given. The Riccati equation

$$\gamma_k + \frac{1}{\gamma_{k-1}} = 2 - \frac{1}{4k(k+1)}, \qquad k = 2, 3, \cdots \tag{3.15}$$

has a solution $\{\gamma_k\}_1^\infty$ which satisfies $\gamma_1 = 11/8$ and $\gamma_k > (2k+2)/(2k+1)$ for $k \geq 2$. Thus the difference equation

$$\Delta^2 x_{k-1} + \frac{1}{4k(k+1)} x_k = 0, \qquad k = 1, 2, 3, \cdots \tag{3.16}$$

has a nonoscillatory solution $x_0 = 1/2$, $x_1 = 1$, $x_{k+1} = \gamma_1\gamma_2\cdots\gamma_k$ for $k \geq 1$. Since

$$\liminf_{n\to\infty} n \sum_{k=n+1}^{\infty} \frac{1}{4k(k+1)} = \liminf_{n\to\infty} \frac{n}{4} \sum_{k=n+1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1}\right] = \frac{1}{4},$$

we see that the number $1/4$ in (3.14) is the best possible.

Theorem 3.5. (cf. [5, Theorem 5]) *If* (1.1) *is oscillatory, then*

$$\limsup_{n\to\infty} n \sum_{k=n+1}^{\infty} b_k \geq \frac{1}{4}.$$

The proof is essentially as that of Theorem 5 in [5], and is thus omitted. Note, however, that an additional assumption is imposed in [5, Theorem 5].

As an example, the difference equation

$$\Delta^2 x_{k-1} + \frac{2}{3k(3k+2)} x_k = 0, \qquad k = 1, 2, 3, \cdots$$

is nonoscillatory since

$$\begin{aligned}
\limsup_{n\to\infty} n \sum_{k=n+1}^{\infty} \frac{2}{3k(3k+2)} &= \frac{2}{9} \limsup_{n\to\infty} n \sum_{k=n+1}^{\infty} \frac{1}{k(k+2/3)} \\
&\le \frac{2}{9} \limsup_{n\to\infty} n \sum_{k=n+1}^{\infty} \frac{1}{k(k-1)} = \frac{2}{9} \limsup_{n\to\infty} n \sum_{k=n+1}^{\infty} \left[\frac{1}{k-1} - \frac{1}{k}\right] \\
&= 2/9 < 1/4.
\end{aligned}$$

Indeed, the above equation has a nonoscillatory solution $x_0 = 1$, $x_k = 1 \cdot \frac{3}{2} \cdot \frac{6}{5} \cdots \frac{3k}{3k-1}$ for $k = 1, 2, \cdots$.

4. Conditional oscillation criteria

As in [9], we can divide the class of equations (1.1) according to the following definitions:

(a) The equation (1.1) is said to be strongly oscillatory if the equation

$$\Delta^2 x_{k-1} + \lambda b_k x_k = 0, \qquad k = 1, 2, 3, \cdots \tag{4.1}$$

is oscillatory for all positive values of λ.

(b) The equation (1.1) is said to be strongly nonoscillatory if (4.1) is nonoscillatory for all λ.

(c) The equation (1.1) is said to be conditionally oscillatory if (4.1) is oscillatory for some positive λ and nonoscillatory for some other $\lambda > 0$.

By Cheng's comparison theorem [2, Theorem 6], it follows that in the case (c) there must exist a positive number $\mu[b]$ such that (4.1) is oscillatory for $\lambda > \mu[b]$ and nonoscillatory for $\lambda < \mu[b]$. This number $\mu[b]$ will be called the oscillation constant of $\{b_k\}$.

Theorem 4.1. *Equation* (1.1) *is strongly oscillatory if and only if*

$$\limsup_{n\to\infty} n \sum_{k=n+1}^{\infty} b_k = \infty.$$

Equation (1.1) *is strongly nonoscillatory if and only if*

$$\limsup_{n\to\infty} n \sum_{k=n+1}^{\infty} b_k = 0.$$

In view of Theorems 3.4 and 3.5, the proof is similar to that of Theorem II in [9] and is thus omitted.

By means of the above Theorem, it is not difficult to see that the following equations

$$\Delta^2 x_{k-1} + \frac{(c+1)^2}{c} x_k = 0, \qquad k = 1, 2, 3, \cdots, \qquad c > 0$$

$$\Delta^2 x_{k-1} + k^{-\alpha} x_k = 0, \qquad k = 1, 2, 3, \cdots, \qquad \alpha \leq 1,$$

$$\Delta^2 x_{k-1} + [\cos k^{-1}] x_k = 0, \qquad k = 1, 2, 3, \cdots$$

$$\Delta^2 x_{k-1} + \frac{2^k}{\ln(k+1)} x_k = 0, \qquad k = 1, 2, 3, \cdots$$

are strongly oscillatory; while the following equation

$$\Delta^2 x_{k-1} + c^{-k} x_k = 0, \qquad k = 1, 2, 3, \cdots, \qquad c > 1$$

is strongly nonoscillatory.

Theorem 4.2. *Let* $\{b_k\}_1^\infty$ *and* $\{a_k\}_1^\infty$ *be two nonnegative sequences each of which has infinitely many positive terms, and let* $\mu[a]$ $(0 < \mu[a] < \infty)$ *be the oscillation constant of* $\{a_k\}_1^\infty$. *Let*

$$\Psi = \liminf_{n\to\infty} \frac{\sum_{k=n+1}^{\infty} b_k}{\sum_{k=n+1}^{\infty} a_k}. \tag{4.2}$$

If $\Psi > \mu[a]$, *then* (1.1) *is oscillatory.*

Proof. The difference equation

$$\Delta^2 x_{k-1} + \lambda a_k x_k = 0, \qquad k = 1, 2, 3, \cdots$$

is oscillatory if $\lambda > \mu[a]$. Thus by Theorem 3.5, we have

$$\limsup_{n\to\infty} n \sum_{k=n+1}^{\infty} a_k \geq \frac{1}{4\lambda},$$

so that for any $1/4\mu[a] > \varepsilon > 0$, there exists a positive integer M such that

$$n \sum_{k=n+1}^{\infty} a_k > \frac{1}{4\lambda} - \varepsilon, \qquad n \geq M.$$

As $\lambda \to \mu[a]$, we have

$$n \sum_{k=n+1}^{\infty} a_k \geq \frac{1}{4\mu[a]} - \varepsilon, \qquad n \geq M,$$

so that

$$\Psi \leq \liminf_{n\to\infty} \frac{n \sum_{k=n+1}^{\infty} b_k}{\frac{1}{4\mu[a]} - \varepsilon} = \frac{1}{\frac{1}{4\mu[a]} - \varepsilon} \liminf_{n\to\infty} n \sum_{k=n+1}^{\infty} b_k.$$

Since this implies

$$\liminf_{n\to\infty} n \sum_{k=n+1}^{\infty} b_k > \frac{1}{4},$$

by Theorem 3.4, (1.1) is oscillatory. Q.E.D.

Theorem 4.3. *Let* $\{b_k\}_1^\infty$ *and* $\{a_k\}_1^\infty$ *be two nonnegative sequences each of which has infinitely many positive terms, and let* $\mu[b]$ $(0 < \mu[b] < \infty)$ *and* $\mu[a]$ $(0 < \mu[a] < \infty)$ *be the oscillation constants of* $\{b_k\}_1^\infty$ *and* $\{a_k\}_1^\infty$ *respectively. Then* $\Psi \leq \mu[a]/\mu[b]$, *where* Ψ *is defined by* (4.2), *and* (1.1) *is nonoscillatory if*

$$\tau = \limsup_{n\to\infty} \frac{\sum_{k=n+1}^{\infty} b_k}{\sum_{k=n+1}^{\infty} a_k} < \mu[a].$$

Proof. Note that by the definition of $\mu[b]$ the difference equation

$$\Delta^2 x_{k-1} + (\mu[b] - \varepsilon) b_k x_k = 0, \qquad k = 1, 2, 3, \cdots$$

is nonoscillatory for any $\varepsilon > 0$. Thus by Theorem 4.2,

$$(\mu[b] - \varepsilon)\Psi = \liminf_{n\to\infty} \frac{(\mu[b] - \varepsilon) \sum_{k=n+1}^{\infty} b_k}{\sum_{k=n+1}^{\infty} a_k} \leq \mu[a].$$

Since ε is arbitrary, $\Psi \leq \mu[a]/\mu[b]$ is clear.

Next, since

$$\frac{1}{\tau} = \liminf_{n\to\infty} \frac{\sum_{k=n+1}^{\infty} a_k}{\sum_{k=n+1}^{\infty} b_k}$$

by means of the first part of our Theorem, we have $1/\tau \leq \mu[b]/\mu[a]$, or equivalently, $\mu[a] \leq \mu[b]\tau$. If (1.1) is oscillatory, then $\mu[b] \leq 1$ by definition of $\mu[b]$, thus $\mu[a] \leq \tau$ as required. Q.E.D.

As an example, let $a_k = 1/k(k+1)$ for $k = 1, 2, \cdots$. Since (3.16) is nonoscillatory, the oscillation constant $\mu[a]$ of $\{a_k\}$ is greater than or equal to $1/4$. Since

$$\liminf_{n\to\infty} n \sum_{k=n+1}^{\infty} \frac{\lambda}{k(k+1)} = \lambda,$$

if $\lambda > 1/4$. then by Theorem 3.4,

$$\Delta^2 x_{k-1} + \frac{\lambda}{k(k+1)} x_k = 0$$

is oscillatory, thus μ is equal to $1/4$. As a consequence, if

$$\liminf_{n\to\infty} \frac{\sum_{k=n+1}^{\infty} b_k}{\sum_{k=n+1}^{\infty} \frac{1}{k(k+1)}} > \frac{1}{4},$$

then (1.1) is oscillatory, and if

$$\limsup_{n\to\infty} \frac{\sum_{k=n+1}^{\infty} b_k}{\sum_{k=n+1}^{\infty} \frac{1}{k(k+1)}} < \frac{1}{4},$$

then (1.1) is nonoscillatory. For instance, if $b_k = 3/4k(4k+1)$ for $k \geq 1$, then

$$\limsup_{n\to\infty} \frac{\sum_{k=n+1}^{\infty} \frac{3}{4k(4k+1)}}{\sum_{k=n+1}^{\infty} \frac{1}{k(k+1)}} \leq \limsup_{n\to\infty} \frac{\frac{3}{16}\sum_{k=n+1}^{\infty} \frac{1}{k(k-1)}}{\sum_{k=n+1}^{\infty} \frac{1}{k(k+1)}} = 3/16 < 1/4.$$

Thus

$$\Delta^2 x_{k-1} + \frac{3}{4k(4k+1)} x_k = 0, \qquad k \geq 1,$$

is nonoscillatory. Indeed, it has a nonoscillatory solution $x_0 = 1$, $x_k = 1 \cdot \frac{4}{1} \cdot \frac{8}{5} \cdots \frac{4k}{4k-3}$ for $k \geq 1$.

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