Funkcialaj Ekvacioj, 34 (1991) 103–115

Continuous Dependence on Obstacles in Variational Inequalities

By

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§ 1. Introduction

We shall study elliptic variational inequalities of the type,

(P) $$u \in K(\psi) = \{v \in H_0^1(\Omega) \colon v \geq \psi \text{ a.e. in } \Omega\} \colon$$
$$\langle Au, v - u \rangle \geq \langle f, v - u \rangle \qquad {}^{\forall}v \in K(\psi)$$

where A is an elliptic operator and $\langle \cdot, \cdot \rangle$ is the dual bracket between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

The aim of this paper is to present some results on the continuous dependence on the obstacle ψ in (P). To state our problem more precisely, we consider a sequence of obstacles $(\psi_n)_{n \geq 1}$ converging to ψ in suitable sense as $n \to \infty$, and for each $n \geq 1$ we consider variational inequalities:

(P_n) $$u_n \in K(\psi_n) = \{v \in H_0^1(\Omega) \colon v \geq \psi_n \text{ a.e. in } \Omega\} \colon$$
$$\langle Au_n, v - u_n \rangle \geq \langle f, v - u_n \rangle \qquad {}^{\forall}v \in K(\psi_n) \, .$$

Our purpose here is to study sufficient conditions on (ψ_n) which imply that the sequence of the solution (u_n) converges to u in some sense. This kind of problems are sometimes called "varying obstacle problems."

We have already known some results on these problems. In 1969, U. Mosco [4] introduced a very useful concept for stability of variational inequalities. First we recall the definition of Mosco sense convergence. Let K_n be a closed convex set in $H^1(\Omega)$. A sequence of convex sets (K_n) converges to K in the sense of Mosco, if following two hypotheses hold.

(1) For all $v \in K$ there exists $v_n \in K_n$ such that

$$v_n \to v \qquad \text{strongly in } H^1(\Omega) \, .$$

(2) For any subsequence $v_n \in K_n$ such that

$$v_n \to v \qquad \text{weakly in } H^1(\Omega), \text{ we can deduce } v \in K \, .$$

Applying Mosco's theorem in [4] to obstacle problems, we find that (u_n) converges to u on condition that $(K(\psi_n))$ converges to $K(\psi)$ in the sense of

Mosco. After that, H. Attouch and C. Picard showed that

$$K(\psi_n) \to K(\psi) \quad \text{and} \quad K(-\psi_n) \to K(-\psi) \quad \text{in the sense of Mosco}$$

if and only if

$$\psi_n \to \psi \quad \text{in } L_c^2(\Omega),$$

where $L_c^2(\Omega)$ is $L^2(\Omega)$–capacity. To see more detail, we refer to H. Attouch and C. Picard [1]. Consequently, we find that (u_n) converges to u in $H^1(\Omega)$ provided that (ψ_n) converges to ψ in $L_c^2(\Omega)$. Also, by easy computation we can show that (u_n) converges to u in $L^\infty(\Omega)$ when (ψ_n) converges to ψ in $L^\infty(\Omega)$.

On the other hand it has been known that the solution (u_n) does not converge to u with highly oscillating obstacles or fakir's carpets. We will see a simple example in Section 3. Furthermore readers may find some other examples in D. Cioranescue and F. Murat [3], and H. Attouch and C. Picard [2].

In this paper we give new results on these problems. To mention our results, we offer some definitions of obstacles. We say that a sequence of obstacles $(\psi_n)_{n\geq 1}$ approaches to ψ from below (resp. above) a.e. in Ω if following two hypotheses holds.

(i) $$\psi_n \to \psi \quad \text{a.e. in } \Omega \quad \text{as } n \to \infty,$$

and for all $n \geq 1$,

(ii) $$\psi_n \leq \psi \quad (\text{resp. } \psi_n \geq \psi) \quad \text{a.e. in } \Omega.$$

Also, we say that $(\psi_n)_{n\geq 1}$ approaches to ψ from below in $L^2(\Omega)$ when $(\psi_n)_{n\geq 1}$ converges to ψ in $L^2(\Omega)$ and $\psi_n \leq \psi$ a.e. in Ω. We shall deal with three different cases; obstacles approaching from below, concave obstacles approaching from above and convex obstacles approaching from above.

This paper is organized as follows: In Section 2 we deal with obstacles approaching from below. We show that (u_n) converges to u in $H^1(\Omega)$ provided that (ψ_n) approaches to ψ from below a.e. in Ω. (See Theorem 2.2.) In Section 3, we deal with obstacles approaching from above. We need to assume some additional hypotheses for obstacles since at the end of Section 3 we may see a counterexample such that (u_n) does not converge to u in any sense. It was proved in Theorem 3.1 that (u_n) converges to u in $H_0^1(\Omega)$ when (ψ_n) approaches to ψ from above in $L^2(\Omega)$ and ψ_n is convex. Further, we show in Theorem 3.3 that (u_n) converges to u in $L^2(\Omega)$ provided that (ψ_n) approaches to ψ from above in $L^2(\Omega)$ and ψ_n is concave. In Section 4 we will deal with singular perturbation problems in variational inequalities when the obstacle (ψ_n) converges to ψ in some sense.

Throughout this paper we use the following notation:

$\Omega \subset \boldsymbol{R}^n$; a bounded domain with its smooth boundary $\partial\Omega$.

$H^1(\Omega) = W^{1,2}(\Omega) = \{u \in L^2(\Omega);\ \nabla u \in [L^2(\Omega)]^n\}$, where ∇u is generalized gradient of u.

$H_0^1(\Omega) = H^1$-closure of $C_0^\infty(\Omega)$.

$Q(\Omega)$ = the vector space of equivalent classes for quasi-continuous functions.

$L_c^2(\Omega) = \{\psi \in Q(\Omega);\ \tilde{v} \geq |\psi|$ quasi everywhere in Ω for some $v \in H_0^1(\Omega)\}$, where $\tilde{v}$ is quasi-continuous representative of v.
(We see the definition of quasi-continuous functions and quasi everywhere in Rodrigues [5] Section 5:8.)

$(\cdot, \cdot)$; the inner product of $L^2(\Omega)$.

$\langle \cdot, \cdot \rangle$; the dual bracket between $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$.

§ 2. Obstacles approaching from below

We consider variational inequalities with obstacles approaching from below in this section. We assume some hypotheses on obstacles, ψ and ψ_n, and the mapping A:

(H.1) $\psi,\ \psi_n \in L_c^2(\Omega)$,

(H.2) $A: H_0^1(\Omega) \to H^{-1}(\Omega)$ is Lipschitz and coercive, i.e. for all v and w, we have

$$\|Au - Aw\|_{H^{-1}(\Omega)} \leq M\|v - w\|_{H_0^1(\Omega)}$$

and

$$\langle Av - Aw, v - w \rangle \geq \alpha\|v - w\|_{H_0^1(\Omega)}^2$$

where M and α are positive constants,

and

(H.3) the given function f belongs to $H^{-1}(\Omega)$.

Then we can consider the following variational inequalities.

$$(\mathrm{P}_n) \qquad u_n \in K(\psi_n) = \{v \in H_0^1(\Omega) \colon v \geq \psi_n \text{ a.e. in } \Omega\}\colon$$

$$\langle Au_n, v - u_n \rangle \geq \langle f, v - u_n \rangle \qquad {}^\forall v \in K(\psi_n)$$

and their limit;

$$(\mathrm{P}) \qquad u \in K(\psi) = \{v \in H_0^1(\Omega) \colon v \geq \psi \text{ a.e. in } \Omega\}\colon$$

$$\langle Au, v - u \rangle \geq \langle f, v - u \rangle \qquad {}^\forall v \in K(\psi)\,.$$

By Lions–Stampacchia's standard theorem for variational inequalities, we have unique solutions of (P_n) and (P), u_n and u under the hypotheses (H.1)–(H.3). (See Rodrigues [5].) First we give the following a-priori estimate.

Lemma 2.1. (A-priori Estimate) *Suppose there exists* $v_0 \in \bigcap_{n\geq 1} K(\psi_n)$ *and* (ψ_n) *converges to* ψ *a.e. in* Ω. *We have the estimate for* u_n;

$$\|u_n - v_0\|^2_{H_0^1(\Omega)} \leq \frac{1}{\alpha}(\|f\|_{H^{-1}(\Omega)} + \|Av_0\|_{H^{-1}(\Omega)}) . \tag{2.1}$$

Moreover we may take a subsequence of (u_n), *which we denote by* (u_n) *again, satisfying*

$$u_n \to u^* \quad \textit{weakly in } H_0^1(\Omega) , \tag{2.2}$$

$$u_n \to u^* \quad \textit{strongly in } L^2(\Omega) , \tag{2.3}$$

$$u_n \to u^* \quad \textit{a.e. in } \Omega , \tag{2.4}$$

for some $u^* \in K(\psi)$.

Proof of Lemma 2.1. We take $v = v_0$ in (P_n) to have

$$\langle Au_n, v_0 - u_n \rangle \geq \langle f, v_0 - u_n \rangle . \tag{2.5}$$

By coercivity of the mapping A, we find

$$\alpha\|u_n - v_0\|^2 \leq \langle Au_n - Av_0, u_n - v_0 \rangle . \tag{2.6}$$

We use (2.5) and (2.6) to show

$$\alpha\|u_n - v_0\|^2 \leq \langle f - Av_0, u_n - v_0 \rangle . \tag{2.7}$$

Since A is Lipschitz and $f \in H^{-1}(\Omega)$, we deduce

$$\|u_n - v_0\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}(\|f\|_{H^{-1}(\Omega)} + \|Av_0\|_{H^{-1}(\Omega)}) .$$

Then we can take a subsequence which satisfies

$$u_n \to u^* \quad \text{weakly in } H_0^1(\Omega) , \tag{2.8}$$

$$u_n \to u^* \quad \text{strongly in } L^2(\Omega) , \tag{2.9}$$

$$u_n \to u^* \quad \text{a.e. in } \Omega , \tag{2.10}$$

for some $u^* \in H_0^1(\Omega)$.

Now we show $u^* \in K(\psi)$. Since $u_n \in K(\psi_n)$, we have

$$u_n \geq \psi_n \quad \text{a.e. in } \Omega . \tag{2.11}$$

Letting $n \to \infty$ in (2.11), we find by (2.10) and by the assumption of this lemma,

$$u^* \geq \psi \quad \text{a.e. in } \Omega . \tag{2.12}$$

Then we deduce $u^* \in K(\psi)$. □

Theorem 2.2. (Obstacles Approaching from Below) *Under the hypotheses* (H.1), (H.2) *and* (H.3), *if the sequence of obstacles* $(\psi_n)_{n\geq 1}$ *approaches to* ψ *from below a.e. in* Ω, *then the solution converges in* $H_0^1(\Omega)$;

$$u_n \to u \qquad \textit{strongly in } H_0^1(\Omega)\,. \tag{2.13}$$

Proof of Theorem 2.2. Since $(\psi_n)_{n\geq 1}$ approaches to ψ from below, we have for all $n \geq 1$,

$$K(\psi) \subset K(\psi_n)\,. \tag{2.14}$$

Also, since $\psi \in L_c^2(\Omega)$, $K(\psi)$ is not empty. (See Rodrigues [5] P 181.) So there exists $v_0 \in \bigcap_{n\geq 1} K(\psi_n)$. Using Lemma 2.1, we find

$$u_n \to u^* \qquad \text{weakly in } H_0^1(\Omega)\,, \tag{2.15}$$

for some $u^* \in K(\psi)$.

Now, we show $u^* = u$. By Minty's Lemma, (See Rodrigues [5] p 99.)

$$\langle Av, v - u_n\rangle \geq \langle f, v - u_n\rangle \qquad {}^\forall v \in K(\psi_n)\,.$$

Using (2.14), we can take $K(\psi)$ instead of $K(\psi_n)$.

$$\langle Av, v - u_n\rangle \geq \langle f, v - u_n\rangle \qquad {}^\forall v \in K(\psi)\,.$$

Then we pass to the limit $n \to \infty$, according to the convergence (2.15).

$$u^* \in K(\psi)\colon \langle Av, v - u^*\rangle \geq \langle f, v - u^*\rangle \qquad {}^\forall v \in K(\psi)\,.$$

By using Minty's Lemma again, we have

$$u^* \in K(\psi)\colon \langle Au^*, v - u^*\rangle \geq \langle f, v - u^*\rangle \qquad {}^\forall v \in K(\psi)\,.$$

By the uniqueness of the problem (P), we find

$$u^* = u\,.$$

Consequently we found

$$u_n \to u \qquad \text{weakly in } H_0^1(\Omega)\,.$$

To end this proof, we show strong convergence. Put $v_0 = u$ in (2.7) to have

$$\alpha\|u_n - u\|^2_{H_0^1(\Omega)} \leq \langle f - Au, u_n - u\rangle\,. \tag{2.16}$$

Let $n \to \infty$ in (2.16), then we find

$$u_n \to u \qquad \text{strongly in } H_0^1(\Omega)\,. \quad \square$$

Remark 2.4. In this paper we consider Dirichlet boundary value problems with zero boundary data for simplicity. However it is very easy to extend our results to the general Dirichlet boundary value problems. In those cases we need a compatibility condition for the obstacles such that

$$\psi_n - \tilde{g} \in L_c^2(\Omega), \tag{2.17}$$

where $\tilde{g}$ is the usual extension of $g \in H^{1/2}(\partial\Omega)$ which is boundary data of Dirichlet problem. Then, instead of (2.13), we have

$$u_n \to u \qquad \text{strongly in } H^1(\Omega).$$

§3. Obstacles approaching from above

We shall consider obstacles approaching from above in $L^2(\Omega)$ and give a counterexample. First we offer a case that the solution converges in $H_0^1(\Omega)$ when the convex obstacle converges in $L^2(\Omega)$.

Theorem 3.1. (Obstacles Approaching from Above 1) *Let obstacles ψ_n belong to $H_0^1(\Omega)$ and satisfy a hypothesis,*

$$-\Delta\psi_n \le 0 \qquad \text{in } H^{-1}(\Omega). \tag{3.1}$$

If $(\psi_n)_{n\ge1}$ approaches to ψ from above in $L^2(\Omega)$, then the solution (u_n) converges to u strongly in $H_0^1(\Omega)$;

$$u_n \to u \qquad \textit{strongly in } H_0^1(\Omega).$$

An important step in the proof of Theorem 3.1 is the following

Lemma 3.2. *For all $w \in K(\psi)$ we have an approximation of w denoted by (w_n), which satisfies*

$$w_n \in K(\psi_n) \qquad \textit{for all } n \ge 1,$$

and

$$w_n \to w \qquad \textit{strongly in } H_0^1(\Omega).$$

Proof of Lemma 3.2. By (3.1) and the usual maximum principle for $-\Delta$, we find that

$$\psi_n \le 0 \qquad \text{a.e. in } \Omega.$$

Hence we have

$$0 \in K(\psi_n) \qquad \text{for all } n \ge 1.$$

Then we can deduce an a-priori estimate by the same technique as in Lemma 2.1.

$$\|u_n\|_{H_0^1(\Omega)} \le C, \tag{3.2}$$

where C is a positive constant independent of n.

Define w_n as the approximation for each $w \in K(\psi)$ such that

$$w_n \in H_0^1(\Omega):\ -\frac{1}{n}\Delta w_n + w_n = w \vee \psi_n\,, \tag{3.3}$$

where $w \vee \psi_n = \max(w, \psi_n)$.

In fact, w_n converges to w in $H_0^1(\Omega)$ by the theory of singular perturbation problems;

$$w_n \to w \vee \psi = w \qquad \text{strongly in } H_0^1(\Omega)\,. \tag{3.4}$$

(See Rodrigues [5] Section 4:9.)

Now, we show that $w_n \in K(\psi_n)$. Multiplying (3.3) by $(\psi_n - w_n)^+$ and integrating by parts, we find

$$\frac{1}{n}\int_\Omega \nabla w_n \nabla(\psi_n - w_n)^+ + \int_\Omega w_n(\psi_n - w_n)^+ = \int_\Omega w \vee \psi_n(\psi_n - w_n)^+\,.$$

Adding $\int_\Omega \left(-\frac{1}{n}\Delta\psi_n + \psi_n\right)(\psi_n - w_n)^+$ to both sides, we have

$$\frac{1}{n}\int_\Omega |\nabla(\psi_n - w_n)^+|^2 + \int_\Omega |(\psi_n - w_n)^+|^2$$
$$= \int_\Omega \left(-\frac{1}{n}\Delta\psi_n + \psi_n - w \vee \psi_n\right)(\psi_n - w_n)^+\,.$$

By the assumption (3.1) we deduce

$$-\frac{1}{n}\Delta\psi_n + \psi_n - w \vee \psi_n = -\frac{1}{n}\Delta\psi_n - (w - \psi_n)^+ \le 0\,.$$

Hence we have

$$\frac{1}{n}\int_\Omega |\nabla(\psi_n - w_n)^+|^2 + \int_\Omega |(\psi_n - w_n)^+|^2 \le 0\,.$$

Then we find that

$$w_n \ge \psi_n \qquad \text{a.e. in } \Omega\,.$$

Consequently we have an approximation (w_n) for each $w \in K(\psi)$, i.e.

$$w_n \in K(\psi_n) \qquad \text{such that}$$
$$w_n \to w \qquad \text{strongly in } H_0^1(\Omega)\,. \quad \square$$

Then we proceed the proof of Theorem 3.1.

Proof of Theorem 3.1. Using Minty's Lemma on (P_n), we find

$$\langle Aw, w - u_n \rangle \geq \langle f, w - u_n \rangle \qquad {}^{\forall} w \in K(\psi_n) . \tag{3.5}$$

By Lemma 3.2 we have an approximation (v_n) of $v \in K(\psi)$. Taking $w = v_n$ in (3.5), we find

$$\langle Av_n, v_n - u_n \rangle \geq \langle f, v_n - u_n \rangle . \tag{3.6}$$

Letting $n \to \infty$ in (3.6), we find

$$\langle Av, v - u^* \rangle \geq \langle f, v - u^* \rangle \qquad {}^{\forall} v \in K(\psi) ,$$

where u^* is used in Lemma 2.1.

On the other hand we find $u^* \in K(\psi)$ by the same technique as we used in Section 2. Then, using Minty's lemma, we have

$$u^* \in K(\psi): \langle Au^*, v - u^* \rangle \geq \langle f, v - u^* \rangle \qquad {}^{\forall} v \in K(\psi) ,$$

where u^* is used in Lamma 3.1.

By the uniqueness of solution of (P), we find

$$u^* = u .$$

Hence we find

$$u_n \to u \qquad \text{weakly in } H_0^1(\Omega) .$$

We will proceed strong convergence. From Lemma 3.2 we can take an approximation of u, denoted by $(\tilde{u}_n)$. Since A is the coercive mapping, we have

$$\alpha \| u_n - \tilde{u}_n \|_{H_0^1(\Omega)}^2 \leq \langle Au_n - A\tilde{u}_n, u_n - \tilde{u}_n \rangle .$$

Taking into account of (P_n), we have

$$\alpha \| u_n - \tilde{u}_n \|_{H_0^1(\Omega)}^2 \leq \langle f - A\tilde{u}_n, u_n - \tilde{u}_n \rangle .$$

Let $n \to \infty$, and we easily find that

$$u_n \to u \qquad \text{strongly in } H_0^1(\Omega) . \quad \square$$

Remark 3.3. The assumption (3.1) implies that

$$\psi_n \leq 0 \qquad \text{a.e. in } \Omega ,$$

and ψ_n is convex.

Now we will deal with another case that the solution converges in $L^2(\Omega)$ when the concave obstacle converges in $L^2(\Omega)$. We need to recall the definition of strictly T-monotone to use the comparison theorem.

Definition: Let V be a Hilbert lattice, for example $L^2(\Omega)$ or $H_0^1(\Omega)$. One says that an operator A is strictly T-monotone in V, if for all v and $w \in V$ with $(v-w)^+ \neq 0$, we have

$$\langle Av - Aw, (v-w)^+ \rangle > 0 . \tag{3.7}$$

Theorem 3.4. (Obstacles Approaching from Above 2) *We assume* (H.1), (H.3) *and the following hypotheses*:

(H.2′) *A: $H_0^1(\Omega) \to H^{-1}(\Omega)$ is linear bounded, coercive and strictly T-monotone operator*,

and

(H.4) *There exists $g_n \in H^{-1}(\Omega)$ such that*

$$g_n \geq A\psi - A\psi_n \qquad \text{in } H^{-1}(\Omega) , \tag{3.8}$$

$$g_n \to 0 \qquad \text{strongly in } H^{-1}(\Omega) . \tag{3.9}$$

If the sequence of obstacles $(\psi_n)_{n \geq 1}$ approaches to ψ from above in $L^2(\Omega)$, then we have

$$u_n \to u \qquad \text{strongly in } L^2(\Omega) .$$

Remark 3.5. By the standard theory of elliptic problems, we find that there exists $w_n \in H_0^1(\Omega)$ such that

$$w_n \geq \psi - \psi_n \qquad \text{a.e. in } \Omega ,$$

and

$$w_n \to 0 \qquad \text{strongly in } H_0^1(\Omega)$$

under the assumption (H.4).

Moreover if $g_n \leq 0$ in $H^{-1}(\Omega)$, then we find that

$$\psi_n \geq \psi \qquad \text{a.e. in } \Omega . \tag{3.10}$$

Hence we need not to assume the hypotheses (3.10) in this case.

If $A = -\Delta$ and $\psi = 0$, then (3.9) implies that ψ_n is concave.

Proof of Theorem 3.4. Define w_n as a unique solution of auxiliary variational inequality below,

$$w_n \in K(\psi) \colon \langle Aw_n, v - w_n \rangle \geq \langle f + g_n, v - w_n \rangle \qquad \forall v \in K(\psi) . \tag{3.11}$$

We can use the classical theorem for stability of variational inequalities to find

$$w_n \to u \qquad \text{strongly in } H_0^1(\Omega) .$$

On the other hand since A is linear, we have

$$\langle A(u_n - \psi_n + \psi), v - u_n\rangle \geq \langle f - A\psi_n + A\psi, v - u_n\rangle \qquad {}^{\forall}v \in K(\psi_n)\,.$$

Put $\tilde{u}_n = u_n - \psi_n + \psi \in K(\psi)$, and we have

(3.12) $\quad \tilde{u}_n \in K(\psi)\colon \langle A\tilde{u}_n, v - \tilde{u}_n\rangle \geq \langle f - A\psi_n + A\psi, v - \tilde{u}_n\rangle \qquad {}^{\forall}v \in K(\psi).$

Comparing (3.11) with (3.12), we find

$$w_n \geq \tilde{u}_n \qquad \text{a.e. in } \Omega\,.$$

Here we use (H.4) and the standard comparison theorem for variational inequalities in aid of the strictly T-monotonicity of the operator A. Now we find

$$w_n - \psi + \psi_n \geq u_n \qquad \text{a.e. in } \Omega\,. \tag{3.13}$$

Since $\psi_n \geq \psi$ a.e. in Ω, we can use the same theorem which we used above. Then, we find easily

$$u_n \geq u \qquad \text{a.e. in } \Omega\,. \tag{3.14}$$

By (3.13) and (3.14), we have

$$w_n - \psi + \psi_n \geq u_n \geq u \qquad \text{a.e. in } \Omega\,. \tag{3.15}$$

Letting $n \to \infty$ in (3.15), we easily find

$$u_n \to u \qquad \text{strongly in } L^2(\Omega)\,. \quad \square$$

Moreover, if we assume the "super-obstacles" which satisfy the hypotheses in Theorem 3.4, we obtain the same result in the case that obstacles do not satisfy them.

Corollary 3.6. *Let* $\tilde{\psi}_n$ *be the super-obstacle of* ψ_n, *i.e.*,

$$\tilde{\psi}_n \geq \psi_n \qquad \textit{a.e. in } \Omega\,.$$

If $\tilde{\psi}_n$ *satisfies* (H.1), (H.2′), (H.3) *and* (H.4), *and* $(\tilde{\psi}_n)_{n\geq 1}$ *approaches to* ψ *from above in* $L^2(\Omega)$, *then we have*

$$u_n \to u \qquad \textit{strongly in } L^2(\Omega)\,. \quad \square$$

Here we consider a counterexample that (u_n) does not converges to u when (ψ_n) approaches to ψ from above.

Example 3.7. Let $\Omega = (-1, 1)$ and

$$A = -\frac{d^2}{dx^2} \tag{3.16}$$

(3.17) $$f \equiv 0 \quad \text{in } \Omega,$$

(3.18) $$\psi_n = \begin{cases} 1 & \text{in } \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases}$$

(3.19) $$\psi = 0 \quad \text{a.e. in } \Omega.$$

Then we have the following simple variational inequalities,

($\bar{P}_n$) $$u_n \in K(\psi_n): \int_\Omega \frac{d}{dx}u_n \frac{d}{dx}(v - u_n) \geq 0 \qquad {}^\forall v \in K(\psi_n).$$

and

($\bar{P}$) $$u \in K(\psi): \int_\Omega \frac{d}{dx}u \frac{d}{dx}(v - u) \geq 0 \qquad {}^\forall v \in K(\psi) = K(0).$$

It is easy to show that the solution of ($\bar{P}$) is zero;

$$u \equiv 0.$$

Also, we easily calculate the solution of ($\bar{P}_n$), since $H_0^1(-1, 1)$ is included in $C_0^{0,1/2}(-1, 1)$.

(3.20) $$u_n(x) = \begin{cases} \frac{n}{n-1}x + \frac{n}{n-1} & x \in \left[-1, -\frac{1}{n}\right] \\ 1 & x \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ -\frac{n}{n-1}x + \frac{n}{n-1} & x \in \left[\frac{1}{n}, 1\right]. \end{cases}$$

Letting $n \to \infty$ in (3.20), we find that (u_n) converges to u^* uniformly, where

$$u^*(x) = \begin{cases} x + 1 & \text{in } [-1, 0] \\ -x + 1 & \text{in } [0, 1]. \end{cases}$$

Hence we have an example that solution does not converge to the solution of ($\bar{P}$) on condition that the obstacle is not concave.

§ 4. Singular perturbation problems in variational inequalities

We also obtained nearly the same results of singular perturbation problems in variational inequalities when the obstacle converges in some sense. Here we state our results without proof.

(A.1) $B: L^2(\Omega) \to L^2(\Omega)$ is a Lipshitz and coercive mapping,

(A.2) $f \in L^2(\Omega)$,

and we assume (H.1) and (H.2). Then we have unique solutions of following variational inequalities and we can consider singular perturbation problems. (See Rodrigues [5] to find some results of singular perturbation problems.)

(P'_n) $u_n \in K(\psi_n) = \{v \in H_0^1(\Omega)\colon v \geq \psi_n \text{ a.e. in } \Omega\}$:

$$\left\langle \frac{1}{n} A u_n, v - u_n \right\rangle + (B u_n, v - u_n) \geq (f, v - u_n) \qquad {}^\forall v \in K(\psi_n)\,,$$

and their limit problem,

(P′) $u \in \overline{K(\psi)} = \{v \in L^2(\Omega)\colon v \geq \psi \text{ a.e. in } \Omega\}$:

$$(Bu, v - u) \geq (f, v - u) \qquad {}^\forall v \in \overline{K(\psi)}\,.$$

Here we denote $\overline{K(\psi)}$ is closure of $K(\psi)$ in $L^2(\Omega)$.

Theorem 4.1. *Under the assumption* (A.1), (A.2), (H.1) *and* (H.2), *if the sequence of obstacles* $(\psi_n)_{n \geq 1}$ *approaches to* ψ *from below a.e. in* Ω, *then we have*

$$u_n \to u \qquad \text{strongly in } L^2(\Omega)\,. \quad \square$$

Theorem 4.2. *We assume that A and B are strictly T-monotone operators. Let the sequence of obstacles* $(\psi_n)_{n \geq 1}$ *be monotone approaching to* ψ *from above in* $L^2(\Omega)$, *i.e.,*

$$\psi_n \geq \psi_m \qquad \text{a.e. in } \Omega \qquad \text{for all } n \leq m\,,$$

and

$$\psi_n \to \psi \qquad \text{strongly in } L^2(\Omega)$$

and (A.1), (A.2), (H.1) *and* (H.2) *be satisfied. We have*

$$u_n \to u \qquad \text{strongly in } L^2(\Omega)\,. \quad \square$$

Theorem 4.3. *Let A and B be strictly T-monotone and linear operators. We assume* (A.1), (A.2), (H.1), (H.2) *and there exists* (g_n) *which is a sequence in* $H^{-1}(\Omega)$ *such that*

$$g_n \geq \frac{1}{n}(A\psi - A\psi_n) + B\psi - B\psi_n \qquad \text{in } H^{-1}(\Omega)\,,$$

and

$$g_n \to 0 \qquad \text{in } H^{-1}(\Omega)\,.$$

If the sequence of obstacles $(\psi_n)_{n\geq 1}$ *approaches to* ψ *from above in* $L^2(\Omega)$, *then we have*

$$u_n \to u \qquad \textit{strongly in } L^2(\Omega)\,. \quad \square$$

We can obtain all of these results for singular perturbation problems by the same method as we used in previous sections and by the theory of singular perturbation.

Acknowledgement. I wish to express my appreciation to Professor M. Tsutsumi for his many valuable suggestions. I am grateful to Mr. S. Koike for his constructive criticism of the work.

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(Ricevita la 31-an de marto, 1989)
(Reviziita la 2-an de novembro, 1989)