Asymptotic Stability Criteria for Nonlinear Volterra Integro-Differential Equations

By

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Dedicated to Professor Tosihusa Kimura on his 60th birthday

§1. Introduction

The purpose of this paper is to study the asymptotic behavior of solutions of a nonlinear Volterra integro-differential system of the form

\[ x'(t) = A(t)x(t) + \int_0^t G(t, s, x(s))ds \]

where \( A(t) \) is a continuous \( n \times n \) matrix on \([0, \infty)\) and \( G(t, s, x) \) is a continuous \( n \) vector on \( 0 \leq s \leq t < \infty \) and \( x \in \mathbb{R}^n \).

Equations of this type were studied by several authors in \([1-4, 6, 10-12]\). Most of them considered the equations of the form

\[ x'(t) = Ax(t) + \int_0^t C(t, s)x(s)ds \]  \hspace{1cm} (1.2)

or

\[ x'(t) = Ax(t) + \int_0^t D(t-s)x(s)ds \]  \hspace{1cm} (1.3)

where \( A \) is a constant \( n \times n \) matrix, \( C(t, s) \) is an \( n \times n \) matrix continuous for \( 0 \leq s \leq t < \infty \) and \( D(t) \) is an \( n \times n \) matrix continuous for \( t \geq 0 \). In case \( A \) is a stable matrix, there exists a symmetric positive definite matrix \( B \) such that

\[ A^TB + BA = -I \]  \hspace{1cm} (1.4)

and we can use the function \( V = x^TBx \) as a Liapunov function to investigate asymptotic behaviors of solutions of (1.2) (c.f. \([6, 12]\)). For the equation (1.3) there is another method, that is, we can use a nice resolvent \( Z(t) \) for (1.3) (c.f. \([2, 3, 10]\)). Burton \([2, 3, 4]\) constructed a number of Liapunov functional to study the asymptotic behavior of the solutions of the form (1.2) or (1.3).

In this paper we do not assume that \( A(t) \) is constant, and hence it is difficult to apply the above methods to (1.1).
For the scalar nonautonomous equation

\begin{equation}
    x'(t) = a(t)x(t) + \int_0^t c(t, s)x(s)ds,
\end{equation}

some results are shown in [2, 3, 4].

Few results on asymptotic stabilities are obtained when \( A(t) \) is an \( n \times n \) matrix valued function. Recent works for the stability and the boundedness of solutions of \( (1.1) \) have been given by Mahfoud [9] in which he used the Liapunov functionals, and by Lakshmikantham and Rao [8] using the comparison method.

We regard \( (1.1) \) as a perturbed equation of

\begin{equation}
    y' = A(t)y
\end{equation}

to obtain some new results on the asymptotic behavior of solutions of \( (1.1) \) using a fundamental matrix \( Y(t) \) for \( (1.6) \). The main tool in our analysis is the "variation of parameters" formula, namely, the solution of \( (1.1) \) with the initial function \( \phi \) on \([0, t_0]\) is written by

\[
    x(t; t_0, \phi) = Y(t)Y^{-1}(t_0)\phi(t_0) + \int_0^t Y(t)Y^{-1}(s) \int_0^s G(s, u, x(u))duds.
\]

It is well known that various stability properties for \( (1.6) \) are characterized by those of \( Y(t) \) [5, p. 54]. For example, the zero solution of \( (1.6) \) is uniformly asymptotically stable, if and only if there exist positive constants \( K \) and \( \lambda \) such that

\begin{equation}
    |Y(t)Y^{-1}(s)| \leq Ke^{-\lambda(t-s)} \quad \text{for} \quad t \geq s \geq 0.
\end{equation}

We notice that for the constant stable matrix \( A(t) = A \), \( (1.7) \) is equivalent to \( (1.4) \). Assuming \( (1.7) \) we will investigate conditions on \( G(t, s, x) \) under which the zero solution of \( (1.1) \) is uniformly asymptotically stable (See, Section 5).

Theorem 5.1 generalizes [3, Theorem 2] to \( n \)-dimensional nonautonomous case.

The remainder of this paper is organized as follows: Section 3, 4, 5 and 6 contain statements and proofs of the theorems for the uniform stability, asymptotic stability, uniform asymptotic stability and exponential asymptotic stability, respectively.

\section{Definitions}

Let \( \mathbb{R}^n \) denote the Euclidean \( n \)-space. For \( x \in \mathbb{R}^n \), let \( |x| \) be a suitable norm of \( x \). For an \( n \times n \) matrix \( A \), define the norm \( |A| \) of \( A \) by \( |A| = \sup_{|x| \leq 1} |Ax| \). Let \( \mathbb{R}^+ \) be the half line \( 0 \leq t < \infty \). For \( \phi \in C(\mathbb{R}^+) \) and \( t \in \mathbb{R}^+ \), define \( \|\phi\|_t = \max \{|\phi(s)|: 0 \leq s \leq t\} \).
Consider the nonlinear Volterra integro-differential equation

(V) \[ x'(t) = A(t)x(t) + \int_0^t G(t, s, x(s))ds \]

where \( A(t) \) is a continuous \( n \times n \) matrix on \([0, \infty)\) and \( G(t, s, x) \) is a continuous \( n \) vector on \( 0 \leq s \leq t < \infty \) and \( x \in \mathbb{R}^n \). The solution of (V) with initial values \((t_0, \phi)\) will be denoted by \( x(t; t_0, \phi) \), where \( t_0 \geq 0 \) and \( \phi: [0, t_0] \to \mathbb{R}^n \) is a continuous function.

We give the definitions of various kinds of stability. In Definitions 2.1–2.5 we assume \( G(t, s, 0) \equiv 0 \).

**Definition 2.1.** The zero solution of (V) is stable (S), if for every \( \varepsilon > 0 \) and any \( t_0 \geq 0 \) there exists \( \delta > 0 \) such that \( \|\phi\|_{t_0} < \delta \) and \( t \geq t_0 \) imply \( |x(t; t_0, \phi)| < \varepsilon \).

**Definition 2.2.** The zero solution of (V) is uniformly stable (US), if it is S and the above \( \delta \) is independent of \( t_0 \).

**Definition 2.3.** The zero solution of (V) is attractive (Att), if for any \( t_0 \geq 0 \) there exists \( \delta_0 \geq 0 \) such that \( \|\phi\|_{t_0} < \delta_0 \) implies \( |x(t; t_0, \phi)| \to 0 \) as \( t \to \infty \). The zero solution of (V) is asymptotically stable (AS), if it is S and Att. If, in addition, all solutions tend to zero, then the zero solution of (V) is globally asymptotically stable.

**Definition 2.4.** The zero solution of (V) is uniformly asymptotically stable (UAS), if it is US, the above \( \delta_0 \) in Definition 2.3 is independent of \( t_0 \), and for every \( \varepsilon > 0 \) there exists \( T > 0 \) such that \( \|\phi\|_{t_0} < \delta_0 \) and \( t \geq t_0 + T \) imply \( |x(t; t_0, \phi)| < \varepsilon \). If \( \delta_0 \) may be made arbitrary large, then the zero solution of (V) is globally uniformly asymptotically stable.

**Definition 2.5.** The zero solution of (V) is exponentially asymptotically stable (ExAS), if there exists \( \lambda > 0 \) and for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( t_0 \geq 0, \|\phi\|_{t_0} < \delta \) and \( t \geq t_0 \) imply \( |x(t; t_0, \phi)| < \varepsilon e^{-\lambda(t-t_0)} \). The zero solution of (V) is globally exponentially asymptotically stable, if there exists \( \lambda > 0 \) and for any \( \alpha > 0 \), there exists \( K(\alpha) > 0 \) such that \( \|\phi\|_{t_0} < \alpha \) and \( t \geq t_0 \) imply \( |x(t; t_0, \phi)| \leq K(\alpha)e^{-\lambda(t-t_0)} \|\phi\|_{t_0} \).

**Definition 2.6.** The solutions of (V) are uniformly bounded (UB), if for every \( \alpha > 0 \) there exists \( \beta(\alpha) > 0 \) such that \( t_0 \geq 0, \|\phi\|_{t_0} < \alpha \) and \( t \geq t_0 \) imply \( |x(t; t_0, \phi)| < \beta(\alpha) \).

§3. Uniform Stability

In this section, we assume the zero solution of the homogeneous linear equation
is $US$, so that there exists $K \geq 1$ such that
\begin{equation}
|Y(t)Y^{-1}(s)| \leq K \quad \text{for} \quad t \geq s \geq 0,
\end{equation}
where $Y(t)$ is a fundamental matrix of (E).

We investigate conditions on $G(t, s, x)$ under which the zero solution of (V) is $US$.

We make the following assumption:
\begin{equation}
|G(t, s, x)| \leq c(t, s)|x| \quad \text{where} \quad c(t, s) \quad \text{is continuous for} \quad t \geq s \geq 0 \quad \text{and} \quad |x| < H \quad \text{for some} \quad H > 0.
\end{equation}

**Theorem 3.1.** Suppose that the assumptions (3.1) and (H-1) hold and there exists a positive constant $M > 0$ such that
\begin{equation}
\int_0^\infty \int_0^t c(t, s)dsdt < M.
\end{equation}

Then the zero solution of (V) is uniformly stable.

**Proof.** For any $0 < \epsilon < H$ let $\delta(\epsilon) = \epsilon/Ke^{KM}$ and $\|\phi\|_{t_0} < \delta(\epsilon)$. Suppose that there exists $t_1 \geq t_0$ such that $|x(t_1)| = \epsilon$ and $|x(t)| < \epsilon$ on $[t_0, t_1)$. By the variation of parameters formula, we have
\begin{align*}
|x(t)| &\leq |Y(t)Y^{-1}(t_0)||\phi(t_0)| + \int_{t_0}^t |Y(t)Y^{-1}(s)| \int_0^s c(s, u)|x(u)|duds \\
&\leq K\delta(\epsilon) + K \int_{t_0}^t \int_0^s c(s, u)|x(u)|duds \quad \text{on} \quad [t_0, t_1].
\end{align*}

Define $r(t) = \sup_{0 \leq s \leq t} |x(s)|$ to obtain
\begin{equation*}
r(t) \leq K\delta(\epsilon) + K \int_{t_0}^t \int_0^s c(s, u)duds.
\end{equation*}

By Gronwall's inequality, we have
\begin{align*}
|x(t)| &\leq r(t) \leq K\delta(\epsilon) \exp\left\{K \int_{t_0}^t \int_0^s c(s, u)duds\right\} \\
&\leq Ke^{KM}\delta(\epsilon) < \epsilon \quad \text{on} \quad [t_0, t_1].
\end{align*}

Therefore $|x(t_1)| < \epsilon$, which is a contradiction. Thus the zero solution of (V) is $US$. The proof is now completed.

**Remark 3.1.** In Theorem 3.1, $A(t) \equiv 0$ is allowed. For the system (V) of convolution type, there are no kernels which satisfies the condition of Theorem
3.1 except for the identically zero one. But consider the equation

\[(3.3)\quad x'(t) = \int_0^t c e^{-(t-s)} x(s) ds,
\]

where \(c \geq 0\). The zero solution of (3.3) is uniformly stable if and only if \(c = 0\), i.e., the kernel is null. Thus the condition (3.2) is not too restrictive.

We give an example of integro-differential equation which satisfies the conditions of Theorem 3.1.

**Example 3.1.** Let \(A(t)\) be the \(2 \times 2\) zero matrix and

\[G(t, s, x) = e^{-t} \begin{pmatrix} -e^{-(t-s)} & -t \\ t & -e^{-(t-s)} \end{pmatrix} \omega(x),\]

where \(\omega(x) = ^T(x_1^3, x_2^4)\). Then the kernel \(G(t, s, x)\) satisfies (H-1) and (3.2) with \(c(t, s) = e^{-(2t-s)} + te^{-t}\). Therefore the zero solution of (V) is uniformly stable.

In place of the assumption (H-1) we make the following assumption:

\[(A-1)\quad |G(t, s, x)| \leq c(t, s)|x|
\]

Then we have

**Remark 3.2.** Suppose that the assumptions (3.1), (3.2) and (A-1) hold. Then the solutions of (V) are uniformly bounded.

The proof is quite similar to that of Theorem 3.1.

§4. Asymptotic Stability

In this section we study the asymptotic stability of the zero solution of (V).

We first assume that the zero solution of (E) is AS, exactly we assume that there exists \(L > 0\) such that

\[(4.1)\quad \int_0^t |Y(t) Y^{-1}(s)| ds \leq L \quad \text{for} \quad t \geq 0,
\]

and we investigate conditions on \(G(t, s, x)\) under which the zero solution of (V) is AS. We remark that under (4.1), we have

\[(4.2)\quad Y(t) \to 0 \quad \text{as} \quad t \to \infty.
\]

For the proof, see [5, p. 68].

**Theorem 4.1.** Suppose that the assumptions (H-1) and (4.1) hold and

\[(4.3)\quad \sup_{t \geq 0} \int_0^t c(t, s) ds < \frac{1}{L}.
\]
Furthermore suppose that

\[ \lim_{s \to \infty} \int_0^s c(s, u) \, du = 0 \quad \text{for all} \quad t \geq 0. \]

Then the zero solution of \((V)\) is asymptotically stable.

**Proof.** We first show the stability of the zero solution. From \((4.3)\) there exists a positive constant \(\gamma\) such that

\[ 0 < \gamma < \frac{1}{L} \quad \text{and} \quad \sup_{s \geq 0} \int_0^s c(s, u) \, du \leq \gamma. \]

From \((4.2)\) there exists a positive constant \(N\) such that

\[ |Y(t)| \leq N \quad \text{for all} \quad t \geq 0. \]

For any \(0 < \varepsilon < H\) and \(t_0 \geq 0\) let \(\delta = \delta(\varepsilon, t_0) < \min \{ (1 - \gamma L)\varepsilon/(N |Y^{-1}(t_0)|), \varepsilon \}.\)

Consider the solution of \((V)\) such that \(\|\phi\|_{t_0} < \delta\). Suppose that there exists \(t_1 > t_0\) such that \(|x(t_1)| = \varepsilon\) and \(|x(t)| < \varepsilon\) on \([t_0, t_1)\). For all \(t \in [t_0, t_1]\) we have

\[
|\phi(t)| \leq |Y(t)Y^{-1}(t_0)||\phi(t_0)| + \int_{t_0}^t |Y(t)Y^{-1}(s)| \int_0^s c(s, u)x(u)\, du \, ds
\]

\[
< N |Y^{-1}(t_0)| \delta + \varepsilon \int_{t_0}^t |Y(t)Y^{-1}(s)| \int_0^s c(s, u)\, du \, ds
\]

\[
< (1 - \gamma L)\varepsilon + \gamma L\varepsilon = \varepsilon.
\]

Therefore \(|x(t_1)| < \varepsilon\), which is a contradiction. Thus the zero solution of \((V)\) is stable.

Next we will show that the zero solutions of \((V)\) is \(\text{Att}\). From the stability let \(\varepsilon = 1\), then there exists \(\delta_0 = \delta(1, t_0) < 1\) such that \(t_0 \geq 0\) and \(\|\phi\|_{t_0} < \delta_0\) imply

\[ |x(t; t_0, \phi)| < \min (H, 1) \quad \text{for all} \quad t \geq 0. \]

Hereafter we consider the solutions such that \(\|\phi\|_{t_0} < \delta_0\). Among them suppose that there exist \(\phi(t)\) and \(x(t) = x(t; t_0, \phi)\) such that

\[ \lim_{t \to \infty} \sup |x(t)| = \mu > 0. \]

Since \(\gamma L < 1\) by \((4.5)\), there exists a constant \(\theta\) such that \(\gamma L < \theta < 1\). By \((4.8)\) there exists \(t_1 \geq t_0\) such that

\[ |x(u)| \leq \frac{\mu}{\theta} \quad \text{for all} \quad u \geq t_1. \]
By the use of (4.4), there exists $T > t_1$ such that

\[(4.10) \quad \int_0^{t_1} c(s, u)du < \frac{(\theta - \gamma L)\mu}{2\theta L} \quad \text{for} \quad s \geq T.\]

Then we have

\[
|x(t)| \leq |Y(t)||Y^{-1}(t_0)|\delta_0 + \int_{t_0}^{t} |Y(t)Y^{-1}(s)| \int_0^{s} c(s, u)|x(u)|duds \\
\leq |Y(t)||Y^{-1}(t_0)|\delta_0 + |Y(t)| \int_{t_0}^{T} |Y^{-1}(s)| \int_0^{s} c(s, u)|x(u)|duds \\
+ \int_{T}^{t} |Y(t)Y^{-1}(s)| \int_0^{t_1} c(s, u)|x(u)|duds \\
+ \int_{T}^{t} |Y(t)Y^{-1}(s)| \int_{t_1}^{s} c(s, u)|x(u)|duds.
\]

From (4.1), (4.7) and (4.10) we have

\[
\int_{T}^{t} |Y(t)Y^{-1}(s)| \int_0^{t_1} c(s, u)|x(u)|duds \leq \frac{(\theta - \gamma L)\mu}{2\theta}.
\]

From (4.1), (4.5) and (4.9) we have

\[
\int_{T}^{t} |Y(t)Y^{-1}(s)| \int_{t_1}^{s} c(s, u)|x(u)|duds \leq \frac{L\gamma\mu}{\theta}.
\]

Thus we have

\[
|x(t)| \leq |Y(t)||Y^{-1}(t_0)|\delta_0 + |Y(t)| \int_{t_0}^{T} |Y^{-1}(s)| \int_0^{s} c(s, u)|x(u)|duds + \frac{(\theta + \gamma L)\mu}{2\theta}.
\]

Since $Y(t) \to 0$ as $t \to \infty$ by (4.2), we have $\mu \leq (\theta + \gamma L)\mu/2\theta$ and thus $\mu < \mu$. This is impossible. Therefore the zero solution of (V) is Att. The proof is now completed.

We make the following assumption:

(H-2) \quad |G(t, s, x)| \leq d(t - s)|x| \quad \text{where} \quad d(t) \text{ is continuous for} \quad t \geq 0 \quad \text{and} \quad |x| < H.

Then we have

**Corollary 4.1.** Suppose that the assumptions (H-2) and (4.1) hold and

\[\int_0^{\infty} d(t)dt < 1/L. \quad \text{Then the zero solution of (V) is asymptotically stable.}\]

**Proof.** Since $\sup_{s \geq 0} \int_0^{s} d(s - u)du = \int_0^{\infty} d(t)dt < 1/L$, then for any $t \geq 0$ and
any $\varepsilon > 0$ there exists $T(t, \varepsilon) > t$ such that $s \geq T(t, \varepsilon)$ implies
\[
\int_0^t d(s-u)du = \int_{s-t}^s d(t)dt < \varepsilon.
\]
Thus by Theorem 4.1 the zero solution of (V) is AS. The proof is now completed.

**Corollary 4.2.** Suppose that the assumption (4.1) holds and $|G(t, s, x)| \leq f(t)g(s)|x|$ for $t \geq s \geq 0$ and $|x| < H$ where $f(t) \geq 0, g(s) \geq 0,$ and $\sup_{t \geq 0} f(t) \int_0^t g(s)ds < 1/L$ and $\int_0^\infty g(t)dt = \infty$. Then the zero solution of (V) is asymptotically stable.

**Proof.** Since $\int_0^s g(u)du$ is an increasing function which tends to $\infty$ and $\sup_{s \geq 0} f(s) \int_0^t g(u)du < \infty$, then $f(s) \to 0$ as $s \to \infty$. Therefore for any $t \geq 0$ and any $\varepsilon > 0$ there exists $T(t, \varepsilon) > t$ such that $s \geq T(t, \varepsilon)$ implies $f(s) \int_0^t g(u)du < \varepsilon$. Thus by Theorem 4.1 the zero solution of (V) is AS. The proof is now completed.

We present an example which satisfies the conditions of Theorem 4.1.

**Example 4.1.** Let $\alpha(t)$ be a real, continuously differentiable function, equal to 1 except in the intervals
\[
J_n = [n - 2^{-4n}, n + 2^{-4n}] \quad (n = 1, 2, \ldots);
\]
in $J_n$, $\alpha(t)$ lies between 1 and $2^{2n}$ and $\alpha(n) = 2^{2n}$. $\alpha(t)$ is due to Massera and Schäffer (cf. [5, p. 73]). Consider the scalar equation
\[
(4.11) \quad x'(t) = -\left(1 + \frac{\alpha'(t)}{\alpha(t)}\right)x(t) + \frac{3}{8} \int_0^t \frac{\sin t}{1 + (t+s)^2} x(s) \cos x(s)ds.
\]
Then we have
\[
\int_0^t |Y(t)Y^{-1}(s)|ds = \frac{1}{\alpha(t)} \int_0^t \alpha(s)e^{s-t}ds \leq \frac{5}{3} \quad \text{for} \quad t \geq 0
\]
and
\[
\sup_{t \geq 0} \frac{3}{8} \int_0^t \frac{ds}{1 + (t-s)^2} = \frac{3\pi}{16} < \frac{3}{5}.
\]
It is clear that (4.4) holds. Thus the zero solution of (4.11) is asymptotically stable by Theorem 4.1.
Remark 4.1. By the first half of the proof of Theorem 4.1 we see that the assumptions (4.1) and (4.3) imply the stability of the zero solution of (V). If the assumption (4.4) in Theorem 4.1 is omitted, the zero solution of (V) cannot be attractive. To see this, we show the following example.

Example 4.2. Consider the following scalar equation

\[(4.12)\quad x'(t) = -\alpha x(t) + \int_0^t g(s)x(s)ds .\]

where

\[\alpha > 0, \quad g(s) = \begin{cases} \gamma & (0 \leq s \leq 1) \\ 0 & (s > 1) \end{cases} \quad \text{and} \quad 0 < \gamma < \alpha.\]

It can be easily seen that the assumptions (4.1) and (4.3) hold, but (4.4) does not. Consider the solution \(x(t)\) such that \(t_0 \geq 1, \phi(t) \equiv x_0 \neq 0\) for \(t \in [0, t_0]\). Then for any \(t \geq t_0\), we have

\[x'(t) = -\alpha x(t) + \gamma x_0 .\]

Therefore it follows that

\[x(t) = x_0 e^{-\alpha(t-t_0)} + \frac{\gamma x_0}{\alpha} (1 - e^{-\alpha(t-t_0)}) \quad \rightarrow \frac{\gamma x_0}{\alpha} \neq 0 \quad \text{as} \quad t \rightarrow \infty .\]

Thus the zero solution of (4.12) is S but not Att.

Next we make the following assumption:

\[(4.13) \quad |G(t, s, x)| \leq c(t, s)\omega(x) \quad \text{where} \quad c(t, s) \text{ is continuous for} \ t \geq s \geq 0 \quad \text{and} \quad \omega(x) = o(|x|) \ (|x| \rightarrow 0) .\]

Theorem 4.2. Suppose that the assumptions (4.13) and (4.4) hold and

\[(4.14) \quad \sup_{t \geq 0} \int_0^t |Y(t)Y^{-1}(s)|ds < \infty , \]

\[(4.15) \quad \sup_{t \geq 0} \int_0^t c(t, s)ds < \infty .\]

Then the zero solution of (V) is asymptotically stable.

Proof. By (4.14) there exists a positive constant \(L\) such that \(\sup_{t \geq 0} \int_0^t |Y(t)Y^{-1}(s)|ds \leq L\). By (4.15) there exists a positive constant \(\varepsilon_0\) such
that \( \sup_{t \geq 0} \int_0^t \varepsilon_0 c(t, s) ds < 1/L \). Then by (4.13) there exists \( \delta(\varepsilon_0) > 0 \) such that 
\[ |x| \leq \delta(\varepsilon_0) \text{ implies } |G(t, s, x)| \leq \varepsilon_0 c(t, s)|x|. \] Hence we can use Theorem 4.1 to prove the zero solution of (V) is \( AS \). The proof is now completed.

Remark 4.2. In Theorem 4.1 if we replace the assumption (H-1) by (A-1), then the zero solution of (V) is globally asymptotically stable.

§5. Uniform Asymptotic Stability

In this section, we assume that the zero solution of (E) is \( UAS \), so that there exist \( K \geq 1 \) and \( \lambda > 0 \) such that
\[ |Y(t) Y^{-1}(s)| \leq Ke^{-\lambda(t-s)} \quad \text{for } t \geq s \geq 0, \]
and we assume the kernel \( G(t, s, x) \) of (V) satisfies
\[ |G(t, s, x)| \leq d(t-s)|x| \text{ where } d(t) \text{ is continuous for } t \geq 0 \text{ and } |x| < H \]
use of (H-2)
for some continuous function \( d(t) \) on \([0, \infty)\).

Theorem 5.1. Suppose that the assumptions (H-2) and (5.1) hold and
\[ \int_0^\infty d(s) ds < \frac{\lambda}{K}. \]
Then the zero solution of (V) is uniformly asymptotically stable.

Proof. The proof of uniform stability can be carried out in a similar way to that of Theorem 4.1. Therefore the proof is omitted. From the uniform stability there exists \( \delta_0 < 1 \) such that \( t_0 \geq 0 \) and \( \| \phi \|_{t_0} < \delta_0 \) imply \( |x(t; t_0, \phi)| < \min (H, 1) \) for all \( t \geq 0 \). We will show that \( \| \phi \|_{t_0} < \delta_0 \) implies \( x(t; t_0, \phi) \to 0 \) as \( t \to \infty \) independently of \( t_0 \). If this is false, there exist \( \varepsilon_0 > 0 \) and sequences \( \{ T_k \} \to \infty \) as \( k \to \infty \), \( \{ \tau_k \} \), \( \{ \phi_k \} \), \( \{ t_k \} \) and \( \{ x_k(t; \tau_k, \phi_k) \} \) such that
\[ \tau_k \geq 0, \quad \| \phi_k \|_{\tau_k} < \delta_0 \quad \text{and} \quad |x_k(t_k; \tau_k, \phi_k)| \geq \varepsilon_0 \quad \text{for} \quad t_k \geq \tau_k + T_k. \]
Let \( \theta = \frac{K}{\lambda} \int_0^\infty d(s) ds \), then \( \theta < 1 \). From (5.2) we also have \( \lim_{t \to \infty} e^{-\lambda t} \times \int_0^t e^{\lambda s} d(s) ds = 0 \), then there exists \( T > 0 \) such that
\[ \frac{K\delta_0}{1 - \theta} e^{-\lambda t} \leq \varepsilon_0 \quad \text{and} \]
\[ \frac{K}{\lambda(1 - \theta)} \left\{ \int_T^\infty d(s) ds + 2e^{-\lambda t} \int_0^t e^{\lambda s} d(s) ds \right\} < \frac{\varepsilon_0}{4} \quad \text{for all} \quad t \geq T. \]
Hereafter we may assume $T_k \geq kT$. We denote $x_k(t; \tau_k, \phi_k)$ by $x_k(t)$ and we define $\sup_{t-T \leq s \leq t} |x_k(s)|$ by $r_k(t)$.

For any $t \geq \tau_k$ we have by (5.1)

\begin{equation}
|x_k(t)| \leq K \delta_0 e^{-\lambda(t-\tau_k)} + K \int_{\tau_k}^t \int_0^s e^{-\lambda(t-s)} d(s-u) |x_k(u)| duds \\
\leq K \delta_0 e^{-\lambda(t-\tau_k)} + K \int_{\tau_k}^t \int_0^s e^{-\lambda(t-s)} d(s-u) duds \\
+ K \int_{\tau_k}^t \int_{\tau_k}^s e^{-\lambda(t-s)} d(s-u) |x_k(u)| duds.
\end{equation}

For the proof of Theorem 5.1 we show the following Claims 1–8.

**Claim 1.** For any $k \in \mathbb{N}$ and $t \geq \tau_k + T$, we have

$$
\int_{\tau_k}^t \int_0^s e^{-\lambda(t-s)} d(s-u) duds \leq \frac{1}{\lambda} \left\{ \int_{t-\tau_k}^\infty d(s) ds + \sup_{\tau_T \geq T} e^{-\lambda \tau} \int_0^\tau e^{\lambda s} d(s) ds \right\}.
$$

**Proof:** By repeating the interchange of the order of integration and the change of variables, we have

\begin{align*}
\int_{\tau_k}^t \int_0^s e^{-\lambda(t-s)} d(s-u) duds &= \int_{\tau_k}^t \int_0^s e^{-\lambda(u-s)} d(s-u) duds \\
&= \int_0^t \int_{\tau_k}^s e^{-\lambda(u-s)} d(s) duds \\
&= \int_{t-\tau_k}^t \int_{u-(t-\tau_k)}^u e^{-\lambda(u-s)} d(s) duds \\
&\leq \int_{t-\tau_k}^t \int_{u-(t-\tau_k)}^\infty e^{-\lambda(u-s)} d(s) duds \\
&= \int_{t-\tau_k}^t \int_{s+(t-\tau_k)}^\infty e^{-\lambda(u-s)} d(s) duds \\
&\leq \frac{1}{\lambda} \int_{t-\tau_k}^\infty d(s) ds + \frac{1}{\lambda} e^{-\lambda(t-\tau_k)} \int_0^{t-\tau_k} e^{\lambda s} d(s) ds \\
&\leq \frac{1}{\lambda} \left\{ \int_{t-\tau_k}^\infty d(s) ds + \sup_{\tau_T \geq T} e^{-\lambda \tau} \int_0^\tau e^{\lambda s} d(s) ds \right\}.
\end{align*}

Claim 1 is now proved.
Claim 2. For any $k \in \mathbb{N}$ and $t \geq \tau_k + T$, we have

$$K \int_{\tau_k}^{t} e^{-\lambda(t-s)} \int_{\tau_k}^{s} d(s - u)|x_k(u)|duds \leq \theta r_k(t) + \frac{K}{\lambda} \left\{ \int_{0}^{T} d(s)ds + e^{-\lambda T} \int_{0}^{T} e^{\lambda s}d(s)ds \right\}.$$

Proof. By interchanging the order of integration and the change of variables as in the proof of Claim 1, we have

$$K \int_{\tau_k}^{t} e^{-\lambda(t-s)} \int_{\tau_k}^{s} d(s - u)|x_k(u)|duds = K \int_{\tau_k}^{t} \int_{u}^{t} e^{-\lambda(t-s)}d(s)|x_k(u)|duds = K \int_{\tau_k}^{t} \int_{0}^{t-u} e^{-\lambda(u-s)}d(s)|x_k(t - u)|du \leq K r_k(t) \int_{0}^{T} \int_{0}^{u} e^{-\lambda(u-s)}d(s)duds + K \int_{T}^{t-\tau_k} \int_{s}^{T} e^{-\lambda(u-s)}d(s)duds \leq \theta r_k(t) + \frac{K}{\lambda} \left\{ \int_{0}^{T} d(s)ds + e^{-\lambda(T-s)}d(s)ds + \int_{T}^{t-\tau_k} d(s)ds \right\}.$$

Claim 2 is now proved.

From (5.5), Claims 1, 2, and (5.4) we have
Claim 3. For any $k \in \mathbb{N}$ and $t \geq \tau_k + T$, we have
\[
|x_k(t)| \leq K \delta_0 e^{-\lambda T} + \theta r_k(t) + \frac{K}{\lambda} \left\{ \int_{\tau_k}^{\infty} d(s) ds + 2 \sup_{r \geq T} e^{-\lambda r} \int_0^r e^{\lambda s} d(s) ds \right\} \\
\leq \theta r_k(t) + \frac{(1-\theta)e_0}{2}.
\]

Claim 4. For any $k \geq 2$ and $t \in [\tau_k + 2T, t_k]$, we have $r_k(t) \geq e_0$.

Proof. If the claim is false, there exist $k \geq 2$ and $\bar{t} \in [\tau_k + 2T, t_k]$ such that $r_k(\bar{t}) < e_0$. Then from (5.3), there exists $\tilde{t} \in [\bar{t}, t_k]$ such that $|x_k(\tilde{t})| = r_k(\tilde{t}) = e_0$. By Claim 3 we have
\[
r_k(\bar{t}) = |x_k(\bar{t})| \leq \theta r_k(\bar{t}) + \frac{(1-\theta)e_0}{2}.
\]
Therefore $r_k(\bar{t}) \leq e_0/2$, which is a contradiction. Claim 4 is now proved.

From Claims 3 and 4 we have

Claim 5. For any $k \geq 2$ and $t \in [\tau_k + 2T, t_k]$, we have $|x_k(t)| < r_k(t)$.

Proof. If the claim is false, there exist $k \geq 2$ and $\bar{t}_1, \bar{t}_2 \in [\tau_k + 2T, t_k]$ such that $\bar{t}_2 - T < \bar{t}_1 < \bar{t}_2$ and $r_k(\bar{t}_1) < r_k(\bar{t}_2)$. Then there exists $\bar{t}_3 \in [\bar{t}_1, \bar{t}_2]$ such that $r_k(\bar{t}_2) = \max_{\bar{t}_1 \leq s \leq \bar{t}_2} |x_k(s)| = |x_k(\bar{t}_3)|$. Hence $|x_k(\bar{t})| = r_k(\bar{t}_3)$, which contradicts to Claim 5. Claim 6 is now proved.

Claim 6. For any $k \geq 2$, $r_k(t)$ is a nonincreasing function on $[\tau_k + 2T, t_k]$.

Proof. If the claim is false, there exist $k \geq 2$ and $\bar{t}_1, \bar{t}_2 \in [\tau_k + 2T, t_k]$ such that $\bar{t}_2 - T < \bar{t}_1 < \bar{t}_2$ and $r_k(\bar{t}_1) < r_k(\bar{t}_2)$. Then there exists $\bar{t}_3 \in [\bar{t}_1, \bar{t}_2]$ such that $r_k(\bar{t}_2) = \max_{\bar{t}_1 \leq s \leq \bar{t}_2} |x_k(s)| = |x_k(\bar{t}_3)|$. Hence $|x_k(\bar{t})| = r_k(\bar{t}_3)$, which contradicts to Claim 5. Claim 6 is now proved.

Claim 7. For any $e > 0$, there exist $k_0 \geq 3$ and $s_{k_0} \in [\tau_{k_0} + 3T, t_{k_0}]$ such that $r_{k_0}(s_{k_0} - T) - r_{k_0}(s_{k_0}) < e$.

Proof. If the claim is false, there exists $e_1 > 0$ such that $r_k(s - T) - r_k(s) \geq e_1$ for any $k \geq 2$ and $s \in [\tau_k + 3T, t_k]$. Hence $r_k(\tau_k + jT) - r_k(\tau_k + (j + 1)T) \geq e_1$ for $2 \leq j \leq k - 1$. Therefore from the uniform stability and Claim 4 we have
\[
1 - e_0 > r_k(\tau_k + 3T) - r_k(\tau_k + kT) \\
= \sum_{j=3}^{k-1} \{r_k(\tau_k + jT) - r_k(\tau_k + (j + 1)T)\} \\
\geq (k - 3)e_1 \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty,
\]
which is a contradiction. Claim 7 is now proved.

By Claim 7 it follows that

Claim 8. There exist $k_0 \geq 3$ and $s_{k_0} \in [\tau_k + 3T, t_k]$ such that $r_{k_0}(s_{k_0} - T) - r_{k_0}(s_{k_0}) < (1 - \theta)e_0/2\theta$. 

Now we are ready to complete the proof. By Claim 8 we have
\[ r_{k_0}(s_{k_0} - s) - r_{k_0}(s_{k_0}) < \frac{(1 - \theta)\varepsilon_0}{2\theta} \text{ for any } s \in [0, T]. \]

By Claim 5 there exists \( \bar{s} \in (0, T] \) such that \( |x_{k_0}(s_{k_0} - \bar{s})| = r_{k_0}(s_{k_0}). \) By Claim 3 we have
\[ |x_{k_0}(s_{k_0} - \bar{s})| \leq \theta r_{k_0}(s_{k_0} - \bar{s}) + \frac{(1 - \theta)\varepsilon_0}{2}. \]

Since \( r_{k_0}(s_{k_0} - \bar{s}) < r_{k_0}(s_{k_0}) + (1 - \theta)\varepsilon_0/2\theta, \) we have
\[ r_{k_0}(s_{k_0}) = |x_{k_0}(s_{k_0} - \bar{s})| \leq \theta r_{k_0}(s_{k_0} - \bar{s}) + \frac{(1 - \theta)\varepsilon_0}{2} < \theta \left( r_{k_0}(s_{k_0}) + \frac{(1 - \theta)\varepsilon_0}{2\theta} \right) + \frac{(1 - \theta)\varepsilon_0}{2} = \theta r_{k_0}(s_{k_0}) + (1 - \theta)\varepsilon_0. \]

Therefore \( r_{k_0}(s_{k_0}) < \varepsilon_0, \) which is a contradiction. The proof is now completed.

**Theorem 5.2.** Suppose that the zero solution of (E) is UAS and \( G(t, s, x) \) satisfies
\[ |G(t, s, x)| \leq d(t-s)\omega(x) \]
where \( \int_0^\infty d(t)dt < \infty \) and \( \omega(x) = o(|x|) \) as \( |x| \to 0. \) Then the zero solution of (V) is uniformly asymptotically stable.

**Proof.** The proof of Theorem 5.2 can be carried out in a similar way to that of Theorem 4.2.

**Example 5.1.** Let \( a, c: [0, \infty) \to [0, \infty) \) and \( b: [0, \infty) \to \mathbb{R} \) be continuous functions and let
\[ A(t) = \begin{pmatrix} -a(t) & b(t) \\ 0 & -c(t) \end{pmatrix}. \]

We assume that there exists \( \dot{\lambda} > 0 \) such that
\[ a(t) \geq \dot{\lambda} \quad \text{and} \quad c(t) \geq \dot{\lambda} \quad \text{for all } t \geq 0, \]
and
\[ \sup_{t \geq 0} \int_t^{t+1} |b(s)|ds < \infty. \]
Then it is clear that the zero solution of (E) is \( UAS \). For an absolutely integrable continuous \( 2 \times 2 \) matrix \( D(t) \), consider the equation

\[(5.6) \quad x'(t) = A(t)x(t) + \int_0^t D(t - s)\omega(x(s))ds ,\]

where \( \omega(x) = x_1^2, x_2^2 \). Let \( d(t) = |D(t)| \), then the conditions of Theorem 5.2 are satisfied and hence the zero solution of (5.6) is uniformly asymptotically stable.

**Example 5.2.** Let \( a: [0, \infty) \to [0, \infty) \) and \( b: [0, \infty) \to R \) be continuous and let

\[ A(t) = \begin{pmatrix} -a(t) & -b(t) \\ b(t) & -a(t) \end{pmatrix} . \]

Then a fundamental matrix of (E) is given by

\[ Y(t) = r(t) \begin{pmatrix} -\cos \theta(t) & -\sin \theta(t) \\ -\sin \theta(t) & \cos \theta(t) \end{pmatrix} , \]

where \( r(t) = \exp \left( -\int_0^t a(s)ds \right) \) and \( \theta(t) = \int_0^t b(s)ds \). Therefore, letting \( |x| \) be the Euclidean norm of \( x \in R^2 \), we have

\[ |Y(t)Y^{-1}(s)| \leq \exp \left( -\int_s^t a(\tau)d\tau \right) \quad \text{for} \quad t \geq s \geq 0 . \]

Assume that there exists \( \lambda > 0 \) such that

\[ a(t) \geq \lambda \quad \text{for all} \quad t \geq 0 , \]

and let \( D(t) \) be an continuous \( 2 \times 2 \) matrix such that

\[ \int_0^\infty |D(t)|dt < \lambda , \]

Then Theorem 5.1 shows that the zero solution of

\[(5.7) \quad x'(t) = A(t)x(t) + \int_0^t D(t - s)x(s)ds \]

is uniformly asymptotically stable.

**Remark 5.1.** In Theorem 5.1 if we replace the assumption (H-2) by

\[(A-2) \quad |G(t, s, x)| \leq d(t - s)|x| \quad \text{for} \quad t \geq s \geq 0 \quad \text{and} \quad |x| \in R^n , \]

then the zero solution of (V) is globally uniformly asymptotically stable.
Remark 5.2. Consider a scalar equation

\[(5.8)\quad x'(t) = ax(t) + \int_0^t d(t-s)x(s)ds\]

in which \(a\) is constant and \(d(t)\) is continuous for \(0 \leq t \leq \infty\). In [3], Burton and Mahfoud showed that if \(a + \int_0^\infty |d(t)|dt < \infty\), then the zero solution of (5.8) is uniformly asymptotically stable. In this case, \(K\) and \(\lambda\) in our Theorem 5.1 are considered as \(K = 1\), \(\lambda = -a\). Thus \(a + \int_0^\infty |d(t)|dt < \infty\) satisfies the assumptions in Theorem 5.1. Therefore Theorem 5.1 generalizes Burton and Mahfoud's result to \(n\)-dimensional case.

§6. Exponential Asymptotic Stability

Finally, we investigate conditions on \(G(t, s, x)\) under which the zero solution of (V) is ExAS, assuming (5.1).

**Theorem 6.1.** Suppose that the assumptions (H-1) and (5.1) hold and there exists a positive constant \(\mu\) such that

\[(6.1)\quad \sup_{t \geq 0} \int_0^t e^{\mu(t-s)}c(t, s)ds < \frac{\lambda}{K} .\]

Then the zero solution of (V) is exponentially asymptotically stable.

**Proof.** For all \(t \geq t_0\) and \(\|\phi\|_{t_0} < H/K\), we have

\[(6.2)\quad |x(t)| \leq Ke^{-\lambda(t-t_0)}|\phi(t_0)| + K \int_{t_0}^t e^{-\lambda(t-s)} \int_0^s c(s, u)|x(u)|duds .\]

There exist positive constants \(\nu < \mu\) and \(\sigma\) such that \(\lambda = \nu + \sigma\) and \(\sup_{t \geq 0} \int_0^t e^{\nu(t-s)}c(t, s)ds < \sigma/K\). Multiply by \(e^{\nu t}\) both sides of (6.2) to obtain

\[e^{\nu t}|x(t)| \leq Ke^{\nu t_0}e^{-\sigma(t-t_0)}|\phi(t_0)| + Ke^{\nu t_0}e^{-\sigma(t-t_0)} \int_{t_0}^t e^{-\sigma(t-s)} \int_0^s c(s, u)|x(u)|duds .\]

\[= Ke^{\nu t_0}e^{-\sigma(t-t_0)}|\phi(t_0)| + K \int_{t_0}^t e^{-\sigma(t-s)} \int_0^s c(s, u)|x(u)|e^{\nu u}|x(u)|duds .\]
If we define \( \sup_{0 \leq s \leq t} e^{\sigma s} |x(s)| \) by \( r(t) \), it follows that
\[
e^{\sigma t} |x(t)| \leq K e^{\sigma t_0} e^{-\sigma (t-t_0)} |\phi(t_0)| + \sigma r(t) \int_{t_0}^{t} e^{-\sigma (s-t)} ds
\]
\[
\leq K e^{\sigma t_0} e^{-\sigma (t-t_0)} |\phi(t_0)| + \{1 - e^{-\sigma (t-t_0)}\} r(t).
\]

(I): In case \( e^{\sigma s} |x(s)| \leq e^{\sigma t} |x(t)| \) for any \( s \in [0, t] \), we see \( r(t) = e^{\sigma t} |x(t)| \).

Then from (6.3) we have
\[
 r(t) \leq K e^{\sigma t_0} e^{-\sigma (t-t_0)} |\phi(t_0)| + \{1 - e^{-\sigma (t-t_0)}\} r(t).
\]
Thus \( r(t) \leq K |\phi(t_0)| e^{\sigma t_0} \) for \( t \geq t_0 \). Then \( r(t) = e^{\sigma t} |x(t)| \) implies \( |x(t)| \leq K |\phi(t_0)| e^{-\sigma (t-t_0)} \) for \( t \geq t_0 \).

(II): In case there exists \( s \in [0, t] \) such that \( e^{\sigma s} |x(s)| > e^{\sigma t} |x(t)| \), we have the following two cases further:

(II)-(i): There exists \( t_1 \in [t_0, t) \) such that \( e^{\sigma t_1} |x(t_1)| = r(t) \). Then from (6.3) we have
\[
 r(t_1) = e^{\sigma t_1} |x(t_1)| \leq K |\phi(t_0)| e^{\sigma t_0} e^{-\sigma (t_1-t_0)} + \{1 - e^{-\sigma (t_1-t_0)}\} r(t_1),
\]
Thus \( r(t_1) \leq K |\phi(t_0)| e^{\sigma t_0} \) for \( t_1 \geq t_0 \). Then \( e^{\sigma t} |x(t)| < r(t_1) \) implies \( |x(t)| \leq K |\phi(t_0)| e^{-\sigma (t-t_0)} \) for \( t \geq t_0 \).

(II)-(ii): There exists \( t_2 \in [0, t_0] \) such that \( e^{\sigma t_2} |x(t_2)| = r(t) \). Then \( e^{\sigma t} |x(t)| < e^{\sigma t_2} |x(t_2)| < e^{\sigma t_0} \|\phi\|_{l_0} \), and we have \( |x(t)| \leq \|\phi\|_{l_0} e^{-\sigma (t-t_0)} \).

Thus from (I) and (II), the zero solution of (V) is ExAS. The proof is now completed.

**Example 6.1.** Consider the scalar equation
\[
x'(t) = -ax(t) + \int_{0}^{t} e^{-c(t-s)} x(s) ds,
\]
where \( a > 0 \) and \( c \in \mathbb{R} \). Then (H-1) and (5.1) hold, and (6.1) implies that there exists \( \mu > 0 \) such that
\[
\sup_{t_2 \geq 0} \int_{0}^{t} e^{-(c-\mu)(t-s)} ds < a.
\]
This is equivalent to that
\[
ac > 1 \quad \text{and} \quad c > 0.
\]
Therefore if (6.5) holds, then by Theorem 6.1 the zero solution of (6.4) is exponentially asymptotically stable.

On the other hand, we can solve (6.4) by Laplace transform in case \( t_0 = 0 \), and hence

\[
x(t, 0, x_0) = \frac{x_0}{\lambda_1 - \lambda_2} \left\{ (\lambda_1 - c)e^{-\lambda_1 t} + (c - \lambda_2)e^{-\lambda_2 t} \right\},
\]

where \(-\lambda_1, -\lambda_2\) are the real roots of \((\lambda + a)(\lambda + c) - 1 = 0\). Hence if the zero solution of (6.4) is asymptotically stable, then \(\lambda_1 > 0, \lambda_2 > 0\), which implies (6.5). Therefore for the equation (6.4), the condition (6.1) is a necessary and sufficient condition for the exponential asymptotic stability of the zero solution.

Remark 6.1. In Theorem 6.1 if we replace the assumption (H-1) by (A-1), the zero solution of (V) is globally exponentially asymptotically stable.

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