Funkcialaj Ekvacioj, 32 (1989) 479–481

On a Functional Equation over Hilbert Spaces

By

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1. Introduction

During the last few years a lot of work has been done on the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, where α and β are $*$-automorphisms of a von Neumann algebra M, say. To mention briefly, this operator equation arose for modular operators in the new proof of the Tomita-Takesaki theory [10]. Later on, this equation has been studied for arbitrary automorphisms as well as for one-parameter groups of automorphisms of von Neumann algebras. For more details on this operator equation and related works, we refer to [1, 3, 7, 8, 9, 12]. This equation has played an important role in the geometric interpretation of the Tomita-Takesaki theory [4] and in the generalization of the Tomita-Takesaki theory for Jordan algebras [5]. It has been proved in [1] (see also [7]) that if α, β are commuting $*$-automorphisms of a von Neumann algebra M such that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ then M can be decomposed by a (central) projection p in M such that $\alpha = \beta$ on Mp and $\alpha = \beta^{-1}$ on $M(1-p)$. A non-commutative version of this result (in the case of one-parameter groups of automorphisms) has been proved in [8] with its proof depending on Arveson's theory of spectral subspaces [2, 11]. More precisely, it has been shown that if $\{\alpha_t : t \in \boldsymbol{R}\}$ and $\{\beta_t : t \in \boldsymbol{R}\}$ are strongly continuous one-parameter groups of $*$-automorphisms of a von Neumann algebra M such that $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in \boldsymbol{R}$, then there exists a (central) projection p in M such that $\alpha_t = \beta_t$ on Mp, $\alpha_t = \beta_{-t}$ on $M(1-p)$.

In this note, we consider this operator equation for a more general situation of unitaries on a Hilbert space. We obtain a decomposition of a Hilbert space analogous to the decomposition theorem of [8], as mentioned above. We prove that if $\{u_t : t \in \boldsymbol{R}\}$ and $\{v_t : t \in \boldsymbol{R}\}$ are commuting one-parameter groups of unitary operators on a Hilbert space H such that $u_t + u_{-t} = v_t + v_{-t}$ for all $t \in \boldsymbol{R}$, then H can be decomposed by a projection p on H such that $u_t = v_t$ on pH and $u_t = v_{-t}$ on $(1-p)H$.

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2. Decomposition of a Hilbert space

Notation. Let T be a linear operator on a Hilbert space H into H. We denote by $N(T)$ the null space of T and by $R(T)$ the range space of T.

Our first decomposition result in the following

Proposition 2.1. *Let u and v be unitary operators on a Hilbert space H such that $u + v^{-1}u^{-1}v = v + v^{-1}$. Then there exists a projection p on H such that $u = v$ on pH and $u = v^{-1}$ on $(1 - p)H$.*

Proof. Consider the normal operator $(u^{-1}v - 1)$. Then by [6, p. 332], $N(u^{-1}v - 1) \oplus \overline{R(u^{-1}v - 1)} = H$. Since

$$(u - v^{-1})(u^{-1}v - 1) = (v + v^{-1}) - (u + v^{-1}u^{-1}v) = 0\,,$$

it follows that $R(u^{-1}v - 1) \subseteq N(u - v^{-1})$ and hence $\overline{R(u^{-1}v - 1)} \subseteq N(u - v^{-1})$. If p is the (orthogonal) projection associated with $N(u^{-1}v - 1)$, we get that $u = v$ on pH and $u = v^{-1}$ on $(1 - p)H$ and this completes the proof.

Remark 2.2. In case u and v commute then it follows (from the above proposition) that $u + u^{-1} = v + v^{-1}$ and $u = v$ on pH, $u = v^{-1}$ on $(1 - p)H$ for a projection p on H. The commutativity of u and v implies that pH and $(1 - p)H$ remain invariant under u and v and hence p commutes with u and v.

We now come to our main result about one-parameter groups of unitary operators.

Theorem 2.3. *Let $\{u_t : t \in \boldsymbol{R}\}$ and $\{v_t : t \in \boldsymbol{R}\}$ be two commuting one-parameter groups of unitary operators on a Hilbert space H such that $u_t + u_{-t} = v_t + v_{-t}$ for all $t \in \boldsymbol{R}$. Then there is a projection p on H such that $u_t = v_t$ on pH, $u_t = v_{-t}$ on $(1 - p)H$ and p commutes with u_t and v_t for all $t \in \boldsymbol{R}$.*

Proof. Let p_n, $n \in \boldsymbol{N}$, be the projection such that $u_{2^{-n}} = v_{2^{-n}}$ on p_nH and $u_{2^{-n}} = v_{2^{-n}}^{-1}$ on $(1 - p_n)H$ (by Remark 2.2). We first show that $\{p_n\}$ is a decreasing sequence. Now $u_{2^{-(n+1)}} = v_{2^{-(n+1)}}$ on $p_{n+1}H$. Since $p_{n+1}H$ is invariant under $u_{2^{-(n+1)}}$ and $v_{2^{-(n+1)}}$, therefore $u_{2^{-(n+1)}}^2 = v_{2^{-(n+1)}}^2$ on $p_{n+1}H$. This means that $u_{2^{-n}} = v_{2^{-n}}$ on $p_{n+1}H$. It follows that $p_{n+1}H \subset p_nH$ and consequently $\{p_n\}$ is a decreasing sequence. Put $p = \lim_{n\to\infty} p_n$, in the strong operator topology (see for example, [13, p. 84]). It is easy to see that pH is also invariant under $u_{2^{-n}}$ and $v_{2^{-n}}$ and hence for any $\xi \in pH \subseteq p_nH$, we have $u_{k2^{-n}}(\xi) = v_{k2^{-n}}(\xi)$ for any $k \in \boldsymbol{Z}$. The density of the set $\{k2^{-n} : k \in \boldsymbol{Z}, n \in \boldsymbol{N}\}$ in $\boldsymbol{R}$ and the continuity of $t \to u_t$ imply that $u_t(\xi) = v_t(\xi)$, $\xi \in pH$. Similarly, $u_t = v_{-t}$ on $(1 - p)H$. Since p_n commutes with $u_{2^{-n}}$ and $v_{2^{-n}}$, then applying the density argument (as above) we get that p commutes with u_t and v_t for all $t \in \boldsymbol{R}$. This completes the proof.

Acknowledgements

The authors are grateful to Professor Y. Watatani for suggesting the problem. They would also like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

References

[1] Awami, M. and Thaheem, A. B., A short proof of a decomposition theorem of a von Neumann algebra, Proc. Amer. Math. Soc., **92** (1984), 81–82.

[2] Arveson, W., On groups of automorphisms of operator algebras, J. Funct. Anal., **15** (1974), 217–243.

[3] Ciorănescu, I. and Zsidó, L., Analytic generator for one-parameter groups, Tôhoku Math. J., **28** (1976), 327–362.

[4] Haagerup, U. and Skau, C. F., Geometric interpretation of the Tomita-Takesaki theory II, Math. Scand., **48** (1981), 241–252.

[5] Haagerup, U. and Hanche-Olsen, H., Tomita-Takesaki theory for Jordan algebras, J. Operator Theory, **11** (1984), 343–364.

[6] Taylor, A. E., *Introduction to Functional Analysis*, John Wiley & Sons, London, 1958.

[7] Thaheem, A. B., Decomposition of a von Neumann algebra, Rend. Sem. Mat. Univ. Padova, **65** (1981), 1–7.

[8] Thaheem, A. B., Van Daele, A. and Vanheeswijck, L., A result on two one-parameter groups of automorphisms, Math. Scand., **51** (1982), 261–274.

[9] Thaheem, A. B., A functional equation on C^*-algebras, Funkcial. Ekvac., **31** (1988), 411–413.

[10] Van Daele, A., A new approach to the Tomita-Takesaki theory for generalized Hilbert algebras, J. Funct. Anal., **15** (1974), 387–393.

[11] Van Daele, A., Arveson's theory of spectral subspaces, Nieuw Arch. Wisk., **27** (1979), 215–237.

[12] Watatani, Y., On commuting automorphisms, Amer. Math. Monthly, **88** (1981), 449.

[13] Weidmann, J., *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York, 1980.

nuna adreso:
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(Ricevita la 12-an de marto, 1988)
(Reviziita la 12-an de julio, 1988)