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Asymptotic Stability of an Integrodifferential System with a Nonintegrable Kernel

By

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1. Introduction

Consider the integrodifferential system

$$x' = Ax + \int_0^t B(t, s)x(s)ds\,, \tag{C}$$

where A is a constant $n \times n$ matrix, $B(t, s)$ is a continuous $n \times n$ matrix, $0 \le s \le t < \infty$, and $n \ge 1$.

Most of the known stability results for (C) require that $B(t, s)$ be integrable or absolutely integrable with respect to at least one argument on the interval $[0, \infty)$. Certain examples were given in [1] and [2] to show that some of the theory that require integrable kernels can be used to provide stability results for scalar equations with a special form of nonintegrable kernels. In this paper we consider the problem in general and discuss the stability of solutions of (C) where $B(t, s)$ is a nonintegrable kernel of the form $C(t, s) + D(t - s) + K$. We provide sufficient conditions so that all solutions of (C) approach zero as $t \to \infty$.

The following notation and terminology are used throughout this paper. For any $t_0 \ge 0$ and any continuous function ϕ: $[0, t_0] \to \boldsymbol{R}^n$, the function x: $[0, \infty) \to \boldsymbol{R}^n$ satisfying $x(t) = \phi(t)$ on $[0, t_0]$ will be denoted by $x(t, t_0, \phi)$. A solution of (C) is a function of the form $x(t, t_0, \phi)$ which satisfies (C) for all $t \ge t_0$. Under the stated conditions, (C) has a unique solution $x(t, t_0, \phi)$ or simply $x(t)$ if no confusion should arise. If D is a matrix or a vector, $|D|$ represents the sum of the absolute values of its elements. An $n \times n$ matrix is said to be stable if all its eigenvalues have negative real parts.

Stability definitions as well as existence and uniqueness results for (C) can be found in [1], [3], and [5].

2. Stability

Consider the system

$$x' = Ax + \int_0^t [C(t, s) + D(t - s) + K]x(s)ds\,, \tag{1}$$

where A and K are constant $n \times n$ matrices, $C(t, s)$ is a continuous $n \times n$ matrix, $0 \leq s \leq t < \infty$, and $D(t)$ is a continuously differentiable $n \times n$ matrix, $0 \leq t < \infty$, and $n \geq 1$.

By differentiating both sides of (1) with respect to t, we obtain

$$x'' = Ax' + [D(0) + K]x + \int_0^t D'(t-s)x(s)ds + \frac{d}{dt}\int_0^t C(t, s)x(s)ds . \tag{2}$$

If we let $x' = y$, then (2) can be written as a system of the form

$$\frac{d}{dt}\left[z - \int_0^t H(t, s)z(s)ds\right] = Lz + \int_0^t R(t-s)z(s)ds , \tag{3}$$

where $z = [x, y]^T$ is a $2n$-vector, and L, H, and R are $2n \times 2n$ matrices given by

$$L = \begin{bmatrix} O & I \\ D(0)+K & A \end{bmatrix}, \quad H(t, s) = \begin{bmatrix} O & O \\ C(t, s) & O \end{bmatrix}, \quad \text{and} \quad R(t) = \begin{bmatrix} O & O \\ D'(t) & O \end{bmatrix}; \tag{4}$$

I denotes the $n \times n$ identity matrix and O denotes the $n \times n$ zero matrix.

Stability results for (1) will be derived from the corresponding stability results of (3). The analysis below which leads to the main theorem of this paper concerns System (3).

Let $Z(t)$ be the principal matrix solution of the system

$$u' = Lu . \tag{5}$$

Let $Q(t, s)$ be a continuous $2n \times 2n$ matrix,

$$\Psi(t, s) = H(t, s) + \int_s^t Z(t-u)[LH(u, s) + R(u-s) - Q(u, s)]du , \tag{6}$$

and

$$U(t, z(\cdot)) = z(t) - \int_0^t \Psi(t, s)z(s)ds , \tag{7}$$

where $z(t)$ is a continuously differentiable $2n$-vector on $[t_0, \infty)$.

Lemma 1. *The function* $z(t) = z(t, t_0, \phi)$ *is a solution of* (3) *if and only if* $U(t, z(\cdot))$ *satisfies*

$$\frac{d}{dt}U(t, z(\cdot)) = LU(t, z(\cdot)) + \int_0^t Q(t, s)z(s)ds , \quad t \geq t_0 . \tag{8}$$

Proof. Differentiate $U(t, z(\cdot))$ with respect to t and make use of (6) to obtain

$$\begin{aligned}\frac{d}{dt}U(t, z(\cdot)) &= \frac{d}{dt}\left[z(t) - \int_0^t H(t, s)z(s)ds\right] \\ &\quad - \int_0^t [LH(t, s) + R(t-s) - Q(t, s)]z(s)ds \\ &\quad - L\int_0^t\int_s^t Z(t-u)[LH(u, s) + R(u-s) - Q(u, s)]z(s)duds \\ &= \frac{d}{dt}\left[z(t) - \int_0^t H(t, s)z(s)ds\right] - \int_0^t R(t-s)z(s)ds - L\int_0^t \Psi(t, s)z(s)ds \\ &\quad + \int_0^t Q(t, s)z(s)ds \\ &= \frac{d}{dt}\left[z(t) - \int_0^t H(t, s)z(s)ds\right] - \int_0^t R(t-s)z(s)ds - Lz(t) + LU(t, z(\cdot)) \\ &\quad + \int_0^t Q(t, s)z(s)ds\,.\end{aligned}$$

Thus, $z(t)$ is a solution of (3) if and only if $U(t, z(\cdot))$ satisfies (8). This completes the proof.

Lemma 2. *If every solution $z(t)$ of* (3) *is in $L^1[0, \infty)$, then every solution $x(t)$ of* (1) *tends to zero as $t \to \infty$. Furthermore, if $z^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1, \ldots, m$, then $x^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1, \ldots, m+1$.*

Proof. Suppose that solutions of (3) are in $L^1[0, \infty)$ and let $x(t, t_0, \phi_1)$ be a solution of (1). Then $x'(t, t_0, \phi_1)$ satisfies (2) for all $t \geq t_0$. Let ϕ_2: $[0, t_0] \to \boldsymbol{R}^n$ be a continuous function such that $\phi_2(t_0) = x'(t_0, t_0, \phi_1)$. Define $y(t, t_0, \phi_2)$ as follows:

$$\begin{aligned} y(t, t_0, \phi_2) &= x'(t, t_0, \phi_1) \quad \text{for} \quad t \geq t_0 \\ &= \phi_2(t) \qquad\quad\;\; \text{for} \quad 0 \leq t \leq t_0\,. \end{aligned}$$

Let $\phi = [\phi_1, \phi_2]^T$. Then the function $z(t, t_0, \phi) = [x(t, t_0, \phi_1), y(t, t_0, \phi_2)]^T$ is a solution of (3). Thus, $z(t, t_0, \phi)$ is in $L^1[0, \infty)$ and so are $x(t, t_0, \phi_1)$ and $x'(t, t_0, \phi_1)$. Hence, $x(t, t_0, \phi_1) \to 0$ as $t \to \infty$. The rest is obvious and the proof is complete.

If L is a stable matrix, then there is a unique positive definite symmetric matrix E such that

$$L^TE + EL = -I\,. \tag{9}$$

If α^2 and β^2 denote the smallest and largest eigenvalues of E respectively, then for any $y \in \boldsymbol{R}^{2n}$,

$$\alpha^2 |y|^2 \le y^T E y \le \beta^2 |y|^2 . \tag{10}$$

We assume throughout this paper that L is a stable matrix, E is the positive definite symmetric matrix satisfying (9), and α^2 and β^2 are the smallest and largest eigenvalues of E, respectively. Let

$$\lambda(t, s) = |\Psi(t, s)| + (2\beta/\alpha)|EQ(t, s)| , \tag{11}$$

$U = U(t, x(\cdot))$, and consider the Lyapunov functional

$$W(t, z(\cdot)) = [U^T E U]^{1/2} + \frac{1}{2\beta}\int_0^t \int_t^\infty \lambda(u, s)|z(s)| du ds . \tag{12}$$

By (8) and (9), the derivative $W'(t, z(\cdot))$ of $W(t, z(\cdot))$ with respect to t along the solution $z(t) = z(t, t_0, \phi)$ of (3) satisfies

$$\begin{aligned} W'(t, z(\cdot)) = (1/2)[U^T E U]^{-1/2}\left[-U^T U + 2U^T E \int_0^t Q(t, s) z(s) ds \right] \\ + \frac{1}{2\beta}\int_t^\infty \lambda(u, t) du |z(t)| - \frac{1}{2\beta}\int_0^t \lambda(t, s)|z(s)| ds . \end{aligned}$$

By (10), (6), and (7),

$$\begin{aligned} W'(t, z(\cdot)) &\le -\frac{1}{2\beta}|U| + \frac{1}{\alpha}\int_0^t |EQ(t, s)||z(s)| ds + \frac{1}{2\beta}\int_t^\infty \lambda(u, t) du |z(t)| \\ &\quad - \frac{1}{2\beta}\int_0^t \lambda(t, s)|z(s)| ds \\ &\le -\frac{1}{2\beta}|z(t)| + \frac{1}{2\beta}\int_0^t |\Psi(t, s)||z(s)| ds \\ &\quad + \frac{1}{2\beta}\int_0^t [(2\beta/\alpha)|EQ(t, s)| - \lambda(t, s)]|z(s)| ds \\ &\quad + \frac{1}{2\beta}\int_t^\infty \lambda(u, t) du |z(t)| . \end{aligned}$$

By (11),

$$W'(t, z(\cdot)) \le -\frac{1}{2\beta}\left[1 - \int_t^\infty \lambda(u, t) du \right]|z(t)| .$$

If $\int_t^\infty \lambda(u, t) du \le r < 1$ for some positive constant r, then $W'(t, z(\cdot)) \le -\gamma|z(t)|$

for some positive constant γ and all $t \geq t_0$. Integration from t_0 to t yields

$$0 \leq W(t, z(\cdot)) \leq W(t_0, \phi) - \gamma \int_{t_0}^{t} |z(s)| ds . \tag{13}$$

The following theorem is the main result of this paper.

Theorem 1. *Suppose L is a stable matrix and there is a positive constant r such that*

$$\int_{t}^{\infty} [|\Psi(u, t)| + (2\beta/\alpha)|EQ(u, t)|] du \leq r < 1 . \tag{14}$$

Then the following statements hold:

(i) *Every solution* $z(t) = z(t, t_0, \phi)$ *of* (3) *is in* $L^1[0, \infty)$.

(ii) *If* $|\Psi(t, s)| \leq h(t - s)$ *and* $|Q(t, s)| \leq q(t - s)$, *where* $h, q: [0, \infty) \to [0, \infty)$ *are continuous, in* $L^1[0, \infty)$, *and* $h(t) \to 0$ *as* $t \to \infty$, *then* $z(t) \to 0$ *as* $t \to \infty$.

(iii) *If* (ii) *is satisfied and* $|\partial\Psi(t, s)/\partial t| \leq k(t - s)$ *for some continuous function* $k: [0, \infty) \to [0, \infty)$ *with* $k(t) \to 0$ *as* $t \to \infty$, *then* $z^{(i)}(t) \to 0$ *as* $t \to \infty$, $i = 0, 1$.

Proof. (i) That $z(t)$ is in $L^1[0, \infty)$ follows immediately from (13).

(ii) From (7), we have

$$|U(t, z(\cdot))| \leq |z(t)| + \int_0^t h(t - s)|z(s)| ds .$$

Since $h(t)$ and $z(t)$ are in $L^1[0, \infty)$, then the convolution in the above inequality is in $L^1[0, \infty)$ (Cf. [4, p. 379]), and hence $U(t, z(\cdot))$ is in $L^1[0, \infty)$. Now, by (8),

$$\left| \frac{d}{dt} U(t, z(\cdot)) \right| \leq |L||U(t, z(\cdot))| + \int_0^t q(t - s)|z(s)| ds , \tag{15}$$

and the integral is the convolution of two L^1-functions. It follows that the convolution in (15) is in $L^1[0, \infty)$ and hence $U'(t, z(\cdot))$ is in $L^1[0, \infty)$. Thus, $U(t, z(\cdot)) \to 0$ as $t \to \infty$. By (7),

$$|z(t)| \leq |U(t, z(\cdot))| + \int_0^t h(t - s)|z(s)| ds .$$

Since $z(t)$ is in $L^1[0, \infty)$ and $h(t) \to 0$ as $t \to \infty$, then the convolution above tends to zero and hence $z(t) \to 0$ as $t \to \infty$.

(iii) Since $z(t) \to 0$ as $t \to \infty$ and $q(t)$ is in $L^1[0, \infty)$, then the convolution in (15) tends to zero and hence $U'(t, z(\cdot)) \to 0$ as $t \to \infty$. By differentiating both sides of (7) with respect to t, we have

$$U'(t, z(\cdot)) = z'(t) - \Psi(t, t)z(t) - \int_0^t \frac{\partial \Psi(t, s)}{\partial t} z(s) ds$$

and hence

$$|z'(t)| \leq |U'(t, z(\cdot))| + h(0)|z(t)| + \int_0^t k(t-s)|z(s)|ds \,.$$

Since $k(t) \to 0$ and $z(t)$ is in $L^1[0, \infty)$, then the convolution in the inequality above tends to zero and hence $z'(t) \to 0$ as $t \to \infty$. This completes the proof.

In order to understand Theorem 1 and its implication regarding the stability of solutions of (1), one needs to look closely at Condition (14) and its effect on the matrices A, C, D, and K. Since $Q(t, s)$ is an arbitrary matrix, many values can be assigned to $Q(t, s)$. However, only two values are of special interest; namely, $Q(t, s) \equiv 0$ and $Q(t, s) = LH(t, s) + R(t-s)$. Thus, we let

$$Q(t, s) = \begin{bmatrix} Q_{11}(t, s) & O \\ Q_{21}(t, s) & O \end{bmatrix},$$

where $Q_{11}(t, s)$ and $Q_{21}(t, s)$ are continuous $n \times n$ matrices. Let $Z_{ij}(t)$ and E_{ij}, $i = 1, 2$, be the $n \times n$ block submatrices of $Z(t)$ and E respectively; i.e.,

$$Z(t) = \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

Then by (4),

$$LH(v, t) + R(v-t) - Q(v, t) = \begin{bmatrix} C(v, t) - Q_{11}(v, t) & O \\ AC(v, t) + D'(v-t) - Q_{21}(v, t) & O \end{bmatrix}$$

and

$$EQ(u, t) = \begin{bmatrix} E_{11}Q_{11}(u, t) + E_{12}Q_{21}(u, t) & O \\ E_{21}Q_{11}(u, t) + E_{22}Q_{21}(u, t) & O \end{bmatrix}.$$

Let

$$Z_i^*(t) = Z_{i1}(t) + Z_{i2}(t)A\,,$$

$$E_i^* = E_{i1} + E_{i2}A\,, \quad i = 1, 2, \quad \text{and}$$

$$\bar{D}(v, t) = D'(v-t) - Q_{21}(v, t)\,.$$

Then by (6),

$\Psi(u, t)$

$$= \begin{bmatrix} \int_t^u [Z_1^*(u-v)C(v, t) + Z_{12}(u-v)\bar{D}(v, t) - Z_{11}(u-v)Q_{11}(v, t)]dv & O \\ C(u, t) + \int_t^u [Z_2^*(u-v)C(v, t) + Z_{22}(u-v)\bar{D}(v, t) - Z_{21}(u-v)Q_{11}(v, t) & O \end{bmatrix},$$

and Condition (14) reduces to

$$\sum_{i=1}^{2} \int_t^{\infty} \Big| (i-1)C(u,t) + \int_t^u [Z_i^*(u-v)C(v,t) + Z_{i2}(u-v)\bar{D}(v-t) - Z_{i1}(u-v)Q_{11}(v,t)]dv \Big| du + \frac{2\beta}{\alpha} \sum_{i=1}^{2} \int_t^{\infty} |E_{i1}Q_{11}(u,t) + E_{i2}Q_{21}(u,t)|du \le r < 1 .$$

Theorem 2. *Suppose L is stable and there is a positive constant r such that*

$$\sum_{i=1}^{2} \int_t^{\infty} \Big| (i-1)C(u,t) + \int_t^u [Z_i^*(u-v)C(v,t) + Z_{i2}(u-v)D'(v-t)]dv \Big| du \le r < 1 . \tag{16}$$

Then the following statements hold:

(i) *Every solution* $x(t) = x(t, t_0, \phi)$ *of* (1) *tends to zero as* $t \to \infty$.

(ii) *If* $|C(t,s)| \le \gamma(t-s)$ *for some continuous function* $\gamma\colon [0,\infty) \to [0,\infty)$ *and if* $\gamma(t)$ *and* $D'(t)$ *are in* $L^1[0,\infty)$ *and* $\gamma(t) \to 0$ *as* $t \to \infty$, *then* $x^{(i)}(t) \to 0$ *as* $t \to \infty$, $i = 0, 1$.

(iii) *If* (ii) *is satisfied and* $|\partial C(t,s)/\partial t| \le \gamma_1(t-s)$ *for some continuous function* $\gamma_1\colon [0,\infty) \to [0,\infty)$ *with* $\gamma_1(t) \to 0$ *as* $t \to \infty$ *and if* $D'(t) \to 0$, *then* $x^{(i)}(t) \to 0$ *as* $t \to \infty$, $i = 0, 1, 2$.

Proof. Let $Q_{11}(t,s) = Q_{21}(t,s) \equiv 0$. Then $Q(t,s) \equiv 0$ and hence Condition (14) reduces to $\int_t^{\infty} |\Psi(u,t)|du \le r < 1$ which is precisely (16).

(i) This part follows at once from Theorem 1 and Lemma 2.

(ii) From (6), we have

$$|\Psi(t,s)| \le \gamma(t-s) + \int_s^t |Z(t-u)|[|L|\gamma(u-s) + |D'(u-s)|]du .$$

Let

$$h(t) = \gamma(t) + \int_0^t |Z(t-u)|[|L|\gamma(u) + |D'(u)|]du .$$

Then $|\Psi(t,s)| \le h(t-s)$. As L is stable, then $Z(t)$ is in $L^1[0,\infty)$ and hence the convolution above is in $L^1[0,\infty)$. Thus, $h(t)$ is in $L^1[0,\infty)$. Also, $Z(t)$ tending to zero implies that the convolution tends to zero and hence $h(t) \to 0$ as $t \to \infty$. By Theorem 1 and Lemma 2, this part is proved.

(iii) By differentiating both sides of (6) with respect to t, we have

$$\partial\Psi(t, s)/\partial t = \partial H(t, s)/\partial t + R(t - s) + L\Psi(t, s),$$

and hence

$$|\partial\Psi(t, s)/\partial t| \leq \gamma_1(t - s) + |D'(t - s)| + |L|h(t - s).$$

Let $k(t) = \gamma_1(t) + |D'(t)| + |L|h(t)$. Then $|\partial\Psi(t, s)/\partial t| \leq k(t - s)$ with $k(t) \to 0$ as $t \to \infty$. By Theorem 1 and Lemma 2, the proof is complete.

Theorem 3. *Suppose L is stable and there is a positive constant r such that*

$$(17) \quad \sum_{i=1}^{2} \int_t^{\infty} \left[(i - 1)|C(u, t)| + \frac{2\beta}{\alpha}|E_i^* C(u, t) + E_{i2}D'(u - t)| \right] du \leq r \leq 1,$$

Then the following statements hold:

(i) *Every solution $x(t) = x(t, t_0, \phi)$ of* (1) *tends to zero as $t \to \infty$.*

(ii) *If $|C(t, s)| \leq \gamma(t - s)$ for some continuous function $\gamma: [0, \infty) \to [0, \infty)$ and if $\gamma(t)$ and $D'(t)$ are in $L^1[0, \infty)$ and $\gamma(t) \to 0$ as $t \to \infty$, then $x^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1$.*

(iii) *If* (ii) *is satisfied and $|\partial C(t, s)/\partial t| \leq \gamma_1(t - s)$ for some continuous function $\gamma_1: [0, \infty) \to [0, \infty)$ with $\gamma_1(t) \to 0$ as $t \to \infty$, then $x^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1, 2$.*

Proof. Let $Q_{11}(t, s) = C(t, s)$ and $Q_{21}(t, s) = D'(t - s) + AC(t, s)$. Then $LH(t, s) + R(t - s) - Q(t, s) \equiv 0$, and hence $\Psi(t, s) = H(t, s)$. Thus, Condition (14) reduces to $\int_t^{\infty} [|C(u, t)| + (2\beta/\alpha)|EQ(u, t)|]du \leq r < 1$ which is precisely (17).

(i) This part follows at once from Theorem 1 and Lemma 2.

(ii) Since $|\Psi(t, s)| = |C(t, s)| \leq \gamma(t - s)$ and since $|Q(t, s)| \leq |L||H(t, s)| + |R(t - s)| \leq |L|\gamma(t - s) + |D'(t - s)|$, then, by letting $q(t) = |L|\gamma(t) + |D'(t)|$, we have $q(t)$ in $L^1[0, \infty)$ and $|Q(t, s)| \leq q(t - s)$. Thus, (ii) of Theorem 1 is satisfied, and this part of the proof follows from Theorem 1 and Lemma 2.

(iii) Since $\Psi(t, s) = H(t, s)$, then $|\partial\Psi(t, s)/\partial t| \leq \gamma_1(t - s)$. Thus (iii) of Theorem 1 is satisfied and this completes the proof.

Remark 1. Condition (17) of Theorem 3 is often easier to verify than Condition (16) of Theorem 2. However, (16) may be a lot weaker than (17); see [6] for an illustration. On the other hand, (iii) of Theorem 3 is stronger than (iii) of Theorem 2 in that the requirement $D'(t) \to 0$ as $t \to \infty$ is relaxed.

Remark 2. If $C(t, s) = C(t - s)$, the L^1-conditions in (ii) and (iii) of Theorems 2 and 3 can be relaxed. To see this, one selects the arbitrary matrix Q to be a convolution matrix. Since C is a convolution matrix, then $H(t, s) =$

$H(t-s)$, and hence by (6), $\Psi(t,s) = \Psi(t-s)$ with

$$\Psi(t) = H(t) + \int_0^t Z(t-u)[LH(u) + R(u) - Q(u)]du \tag{18}$$

and

$$\Psi'(t) = H'(t) + R(t) - Q(t) + L\Psi(t)\,. \tag{19}$$

In this case, Condition (14) is reduced to

$$\int_0^\infty |\Psi(t)|dt + \frac{2\beta}{\alpha}\int_0^\infty |EQ(t)|dt < 1 \tag{20}$$

or

$$\begin{aligned}\sum_{i=1}^{2}\int_0^\infty \Bigg|(i-1)C(t) \\ &+ \int_0^t [Z_i^*(t-s)C(s) + Z_{i2}(t-s)\bar{D}(s) - Z_{i1}(t-s)Q_{11}(s)]ds\Bigg|dt \\ &+ \frac{2\beta}{\alpha}\sum_{i=1}^{2}\int_0^\infty |E_{i1}Q_{11}(t) + E_{i2}Q_{21}(t)|dt < 1\,.\end{aligned} \tag{21}$$

It follows from (20) that $\Psi(t)$ and $EQ(t)$ are in $L^1[0,\infty)$. Since E is invertible, then $Q(t)$ is in $L^1[0,\infty)$. We may now choose $h(t) = |\Psi(t)|$ and $q(t) = |Q(t)|$ in Theorem 1. Since L is stable, then $Z(t)$ is in $L^1[0,\infty)$. If $C(t)\to 0$ and $R(t) - Q(t)\to 0$ as $t\to\infty$, then $LH(t) + R(t) - Q(t)\to 0$ and hence, the convolution in (18) tends to zero as $t\to\infty$. Thus, $\Psi(t)\to 0$ as $t\to\infty$ and hence, all the conditions in (ii) of Theorem 1 are satisfied. If, in addition, $C'(t)\to 0$ as $t\to\infty$, then by (19), $\Psi'(t)\to 0$ as $t\to\infty$ and, by letting $k(t) = |\Psi'(t)|$, all the conditions in (iii) of Theorem 1 are satisfied.

In summary, if L is stable, $C(t,s) = C(t-s)$, and (21) is satisfied, then by Theorem 1 and Lemma 2, the following statements hold:

(a) Every solution $x(t) = x(t, t_0, \phi)$ of (1) tends to zero as $t\to\infty$.

(b) If $C(t)\to 0$ and $R(t) - Q(t)\to 0$ as $t\to\infty$, then $x^{(i)}(t)\to 0$ as $t\to\infty$, $i = 0, 1$.

(c) If $C^{(i)}(t)\to 0$, $i = 0, 1$, and $R(t) - Q(t)\to 0$ as $t\to\infty$, then $x^{(i)}(t)\to 0$ as $t\to\infty$, $i = 0, 1, 2$.

By choosing $Q(t)\equiv 0$, (a)–(c) yield a result analogous to Theorem 2 without the assumption that $C(t)$ and $D'(t)$ are in $L^1[0,\infty)$. If $Q(t) = LH(t) + R(t)$, the result derived from (a)–(c) is analogous to Theorem 3 without the assumption that $D'(t)$ is in $L^1[0,\infty)$. The assumption that $C(t)$ is in $L^1[0,\infty)$ is then implied by Condition (21). As a consequence, interesting stability criteria are obtained for the convolution system

$$x' = Ax + \int_0^t [B(t-s) + K]x(s)ds\,, \tag{22}$$

where A and K are constant $n \times n$ matrices and $B(t)$ is a continuous $n \times n$ matrix for $0 \leq t < \infty$.

First, we let $C(t, s) = B(t-s)$ and $D(t) \equiv 0$ in (1). If $Q_{11}(t) = Q_{21}(t) \equiv 0$, then $R(t) - Q(t) \equiv 0$ in (b) and (c), and Condition (21) is reduced to

$$\sum_{i=1}^{2} \int_0^\infty \left|(i-1)B(t) + \int_0^t Z_i^*(t-s)B(s)ds\right| dt < 1\,. \tag{23}$$

If $Q_{11}(t) = B(t)$ and $Q_{21}(t) = AB(t)$, then $Q(t) - LH(t) \equiv 0$, Condition (21) is reduced to

$$\sum_{i=1}^{2} \int_0^\infty \left[(i-1)|B(t)| + \frac{2\beta}{\alpha}|E_i^* B(t)|\right] dt < 1\,, \tag{24}$$

and $B(t) \to 0$ implies $Q(t) \to 0$. Thus, (a)–(c) yield the following result:

Theorem 4. *If* $L = \begin{bmatrix} O & I \\ K & A \end{bmatrix}$ *is stable and either* (23) *or* (24) *is satisfied, then the following statements hold*:

(i) *Every solution* $x(t) = x(t, t_0, \phi)$ *of* (22) *tends to zero as* $t \to \infty$.

(ii) *If* $B(t) \to 0$ *as* $t \to \infty$, *then* $x^{(i)}(t) \to 0$ *as* $t \to \infty$, $i = 0, 1$.

(iii) *If* $B^{(i)}(t) \to 0$, $i = 0, 1$, *then* $x^{(i)}(t) \to 0$ *as* $t \to \infty$, $i = 0, 1, 2$.

Now, let $C(t, s) \equiv 0$ and $D(t) = B(t)$ in (1). If $Q_{11}(t) = Q_{21}(t) \equiv 0$, then $\bar{D}(t) = B'(t)$, and Conditions (21) is reduced to

$$\sum_{i=1}^{2} \int_0^\infty \left|\int_0^t Z_{i2}(t-s)B'(s)ds\right| dt < 1. \tag{25}$$

The conditions in (b) and (c) are now the same, and therefore we have

Theorem 5. *If* $L = \begin{bmatrix} O & I \\ B(0) + K & A \end{bmatrix}$ *is stable and* (25) *is satisfied, then the following statements hold*:

(i) *Every solution* $x(t) = x(t, t_0, \phi)$ *of* (22) *tends to zero as* $t \to \infty$.

(ii) *If* $B'(t) \to 0$ *as* $t \to \infty$, *then* $x^{(i)}(t) \to 0$ *as* $t \to \infty$, $i = 0, 1, 2$.

If $Q_{11}(t) \equiv 0$ and $Q_{21}(t) = B'(t)$, then $R(t) - Q(t) \equiv 0$, the conditions in (b) and (c) are satisfied, and (21) is reduced to

$$\frac{2\beta}{\alpha} \sum_{i=1}^{2} \int_0^\infty |E_{i2}B'(t)|dt < 1\,. \tag{26}$$

We now have

Theorem 6. *If* $L = \begin{bmatrix} O & I \\ B(0) + K & A \end{bmatrix}$ *is stable and* (26) *is satisfied, then every solution* $x(t) = x(t, t_0, \phi)$ *of* (22) *has the property that* $x^{(i)}(t) \to 0$ *as* $t \to \infty$, $i =$ 0, 1, 2.

We illustrate below an application of Theorems 4 and 6 to the scalar equation

$$x' = Ax + \int_0^t [B(t-s) + K]x(s)ds\,, \tag{27}$$

where A and K are real numbers and $B(t)$ is a continuous real valued function defined on $[0, \infty)$.

Let F denote either K or $B(0) + K$, and suppose that $L = \begin{bmatrix} O & 1 \\ F & A \end{bmatrix}$ is a stable matrix. A simple calculation shows that the positive definite symmetric matrix E satisfying (9) is given by

$$E = \begin{bmatrix} \dfrac{A}{2F} + \dfrac{F}{2A} - \dfrac{1}{2A} & -\dfrac{1}{2F} \\[2ex] -\dfrac{1}{2F} & -\dfrac{1}{2A} + \dfrac{1}{2FA} \end{bmatrix}.$$

Furthermore,

$$E_1^* = E_{11} + E_{12}A = (F-1)/(2A)\,,$$

$$E_2^* = E_{21} + E_{22}A = -1/2\,,$$

and the conditions (24) and (26) reduce respectively to

$$\int_0^\infty |B(t)|dt < \frac{A}{A + \beta(A + K - 1)/\alpha}\,, \tag{28}$$

and

$$\int_0^\infty |B'(t)|dt < \frac{\alpha}{\beta}\frac{[B(0) + K]A}{1 - [B(0) + K + A]}\,. \tag{29}$$

The following corollaries are immediate consequences of Theorems 4 and 6 respectively.

Corollary 1. *If* $A < 0$, $K < 0$, *and* (28) *is satisfied, then the conclusions of Theorem* 4 *hold.*

Corollary 2. *If $A < 0$, $B(0) + K < 0$, and (29) is satisfied, then the conclusions of Theorem 6 hold.*

Example. Consider the equation

$$x' = -x + \int_0^t [a(t-s+1)^{-p} - b]x(s)ds\,, \tag{30}$$

where $a > 0$, $b > 0$, and $p > 0$.

Here, $A = -1$, $B(0) = a$, $K = -b$, and the eigenvalues of E are solutions of the equation $\mu^2 - q\mu + q/2 = 0$, where $q = 1 - (1/F) - (F/2)$ and F denotes either K or $B(0) + K$. Thus,

$$\alpha^2/\beta^2 = (q - \sqrt{q^2 - 2q})/(q + \sqrt{q^2 - 2q}), \quad \text{and hence}\,,$$

$$\begin{aligned}\alpha/\beta &= 2\sqrt{-F}/(2-F)[\sqrt{1 + 2(F-1)/(F-2)^2} + \sqrt{1 + 2(3F-1)/(F-2)^2}\,] \\ &\geq \sqrt{-F}/(2-F)\,.\end{aligned}$$

If $p > 1$, then $\int_0^\infty |B(t)|dt = a/(p-1)$, and for $F = K = -b$, we have $\alpha/\beta \geq \sqrt{b}/(2+b)$. Since

$$A/[A + \beta(A + K - 1)/\alpha] \geq \sqrt{b}/[\sqrt{b} + (b+2)^2]\,,$$

then (28) is satisfied if $a < \sqrt{b}(p-1)/[\sqrt{b} + (b+2)^2]$.

If $p \leq 1$, then the integral in (28) does not exist and hence, Corollary 1 does not apply. However, $\int_0^\infty |B'(t)|dt = a$, and for $F = B(0) + K = a - b < 0$, we have

$$\alpha F A/\beta[1 - (F + A)] \geq (b-a)^{3/2}/(2 + b - a)^2.$$

Thus, (29) is satisfied if $a < b$ and $a < (b-a)^{3/2}/(2+b-a)^2$ which holds for sufficiently small a. In summary:

(i) If $p > 1$ and $a < \sqrt{b}(p-1)/[\sqrt{b} + (b+2)^2]$, then, by Corollary 1, $x^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1, 2$.

(ii) If $0 < p \leq 1$, $a < b$, and a is sufficiently small, then, by Corollary 2, $x^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1, 2$.

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