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One Parameter Semi-Groups of Operators of Schatten Class C_p

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I. Introduction

Let H be a Hilbert space. A one parameter family $T(t)$, $0 \leq t < \infty$, is called a semi-group of operators if:

(i) $T(0) = I$, the identity operator of H

(ii) $T(s+t) = T(s)T(t)$ for every $t, s \geq 0$.

The semi group $T(t)$ is called C_0-semi group if $\lim_{t \to 0} T(t)x = x$ for all $x \in H$. The infinitesimal generator of the semigroup $T(t)$ is by definition the linear operator A defined by

$$D(A) = \left\{ x \in H : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} \qquad \text{for every} \quad x \in D(A).$$

It is well known that if $T(t)$ is a C_0-semigroup, then A is a densely defined closed operator, [1]. The compactness of $T(t)$, $0 < t < \infty$, was discussed in Pazy [2].

The object of this paper is to discuss when $T(t)$, $0 < t < \infty$ is in the Schatten class C_p, $0 < p < \infty$. For semigroups on Hilbert spaces, the problem of being in C_p is more interesting than of being compact. This is due to the fact that for a C_0-semigroup $T(t)$, $0 < t < \infty$, $\{\|T(t)\|, 0 < t \leq a\}$ is bounded in H for every finite a. But if $T(t) \in C_p$, $0 < t < \infty$, then $\|T(t)\|_p$ need not to be bounded in any interval $(0, a)$ for any finite a. For the basic theory of semigroups we refer to Hille and Phillips [1].

II. When $T(t) \in C_p$

For a Hilbert space H, let $L(H)$ be the space of all bounded linear operator on H. For $1 \leq p < \infty$, set:

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$$C_p = \left\{ T \in L(H) \colon \sup \sum_{n=1}^{\infty} |\langle Te_n, f_n \rangle|^p < \infty \right\},$$

where the supremum is taken over all orthonormal bases (e_n) and (f_n) of H. For $T \in C_p$, one defines $\|T\|_p = \sup (\sum |\langle Te_n, f_n \rangle|^p)^{1/p}$. This defines a norm on C_p. With this norm, C_p is a two sided Banach ideal in $L(H)$. For more on C_p, we refer to Pietsch [4].

The following lemma (whose proof is known in the literiture) will be used oftenly throughout the paper. We give a proof of the lemma for completeness.

Lemma 2.1. *Let $T_n \in C_p$ such that* $\sup_n \|T_n\|_p < \infty$. *If $T_n \to T\ (n \to \infty)$ in the operator norm, then $T \in C_p$.*

Proof. Since $T_n \in C_p$, each T_n is compact. Hence

$$T_n = \sum_{k=1}^{\infty} \sigma_{nk} e_{nk} \otimes f_{nk},$$

where $\sum_{k=1}^{\infty} |\sigma_{nk}|^p \le \lambda < \infty$ for all n, (e_{nk}), (f_{nk}) are orthonormal sequences for each n. Since $\|T_n - T\| \to 0\ (n \to \infty)$, it follows that T is compact. Let $T = \sum_{k=1}^{\infty} \sigma_k e_k \otimes f_k$. Using Theorem 1.20 of [6], we get $\sigma_{nk} \to \sigma_k\ (n \to \infty)$ for all k. Since

$$\sum_{k=1}^{r} |\sigma_k|^p = \sum_{k=1}^{r} \lim_n |\sigma_{nk}|^p = \lim_n \sum_{k=1}^{r} |\sigma_{nk}|^p \le \lambda$$

is true for every r, it follows that $\|T\|_p = (\sum_{k=1}^{\infty} |\sigma_k|^p)^{1/p} \le \lambda$. This ends the proof.

Lemma 2.2. *Let $(T(t))$ be a C_0-semigroup in $L(H)$. If for some $t_0 > 0$, $T(t_0) \in C_p$, then $T(t) \in C_p$ for all $t > t_0$. Further there exists M and a in $(0, \infty)$ such that $\|T(t)\|_p \le \|T(t_0)\|_p M e^{a(t-t_0)}$.*

Proof. From the semigroup property, we have $T(t) = T(t_0)T(t - t_0)$. Since C_p is a two sided ideal, it follows that $T(t) \in C_p$. Further the Banach ideal property of C_p gives $\|T(t)\|_p \le \|T(t_0)\|_p \|T(t - t_0)\|$. Since $(T(t))$ is a C_0-semigroup, then there exists an M and a in $(0, \infty)$ such that $\|T(s)\| \le Me^{as}$, [3]. This gives the result.

Definition 2.3. Let $(T(t))$ be a C_0-semigroup in $L(H)$. We say $(T(t))$ is of type p if:

(i) $T(t) \in C_p$ for all $t > 0$

(ii) There exists an $\varepsilon > 0$ and an $\alpha > 0$ such that $\|T(t)\|_p \le \alpha$ for all $t \in (0, \varepsilon)$.

Let $(T(t))$ be a C_0-semigroup of operators with generator A. Let $\lambda \in \rho(A)$ such that $\mathrm{Re}(\lambda) > a$, where $\|T(t)\| \le Me^{at}$. We define a family of operators

$(R_t(\lambda, A))$, where $R_t(\lambda, A)x = \int_t^\infty e^{-\lambda s}T(s)xds$. We say $(R_t(\lambda, A))$ is of type p if

(i) $R_t(\lambda, A) \in C_p$ for all t and all $\lambda \in \rho(A)$, Re $(\lambda) > a$.

(ii) There exists $\beta > 0$ such that $\|\lambda R_t(\lambda, A)\|_p \leq \beta$ for all $t \in (0, \infty)$ and $\lambda \in \rho(A)$, Re $(\lambda) > a_1 > a$.

Now we prove.

Theorem 2.4. *Let $(T(t))$ be a C_0-semigroup with generator A. Then the following are equivalent:*

(i) *$(T(t))$ is of type p*

(ii) *$(R_t(\lambda, A))$ is of type p and $(T(t))$ is uniformly continuous on* $(0, \infty)$.

Proof. (i) → (ii). Since $T(t) \in C_p$, it follows that $T(t)$ is compact for all $t \in (0, \infty)$. Hence $T(t)$ is uniformly continuous on $(0, \infty)$, [2]. Consequently $R(\lambda, A) = \int_0^\infty e^{-\lambda s}T(s)ds = \lim_{t\to 0} R_t(\lambda, A)$, where the limit is the uniform limit.

Now,

$$R_t(\lambda, A) = \int_t^\infty e^{-\lambda s}T(s)dx$$

$$= T(t)\int_t^\infty e^{-\lambda s}T(s-t)ds\,.$$

Since $T(t) \in C_p$, it follows that $R_t(\lambda, A) \in C_p$. Further

$$\|R_t(\lambda, A)\|_p \leq \|T(t)\|_p \int_0^\infty e^{-\lambda s}Me^{a(s-t)}ds$$

$$\leq b\|T(t)\|_p\,,$$

where b is a constant that is independent of t and λ. Hence $R_t(\lambda, A) \in C_p$ for all λ and $t > 0$.

Now, consider:

$$\|\lambda R_t(\lambda, A)\|_p = |\lambda| \left\| \int_t^\infty e^{-\lambda s}T(s)ds \right\|_p$$

$$\leq |\lambda|\, \|T(t)\|_p \left| \int_t^\infty e^{-\lambda s}Me^{a(s-t)}ds \right|$$

$$\leq \|T(t)\|_p \xi \frac{|\lambda|}{|a-\lambda|}\,.$$

Consequently, if $t \in (0, \delta)$, $\delta \leq \varepsilon$, we get $\|\lambda R_t(\lambda, A)\|_p \leq \beta$.

Conversely. (ii) → (i). Since $(T(t))$ is uniformly continuous, it follows that

$R_t(\lambda, A) \to R(\lambda, A) = \int_0^\infty e^{-\lambda s}T(s)ds$ uniformly. By the assumption, $\|R_t(\lambda, A)\|_p \leq \beta$. It follows from convergence theorems in C_p, [5] that $R(\lambda, A) \in C_p$. Further

$$\|\lambda R(\lambda, A)\|_p \leq \varliminf_t \|\lambda R_t(\lambda, A)\|_p \leq \beta .$$

Further; it follows from [3; the proof of Theorem 3.3] that

$$\lambda R(\lambda, A)T(t) \underset{\lambda}{\rightarrow} T(t) \text{ uniformly,}$$

and

$$\begin{aligned}\|\lambda R(\lambda, A)T(t)\|_p &\leq \|T(t)\| \, \|\lambda R(\lambda, A)\|_p \\ &\leq \beta\|T(t)\| .\end{aligned}$$

For $t \in (0, \varepsilon]$, $\|T(t)\| \leq \eta$ for some η. Thus $\lambda R(\lambda, A)T(t)$ is uniformly bounded in C_p. Consequently, [5], $T(t) \in C_p$ for all $t \in (0, \varepsilon]$. It follows from the semi-group property that $T(t) \in C_p$ for all $t > 0$. Further:

$$\begin{aligned}\|T(t)\|_p &\leq \varliminf_\lambda \|\lambda R(\lambda, A)T(t)\|_p \\ &\leq \beta\|T(t)\| \\ &\leq \beta\eta\end{aligned}$$

for $t \in (0, \varepsilon]$. This ends the proof.

Remarks. (i) If $(T(t))$ is of type p, then the resolvent operator $R(\lambda, A) \in C_p$. To see that:

$$R_t(\lambda, A) \in C_p \quad \text{and} \quad \|R_t(\lambda, A)\|_p \leq \beta .$$

Further $R_t(\lambda, A) \to R(\lambda, A)$ $(t \to 0)$ uniformly. Hence, [5], $R(\lambda, A) \in C_p$.

(ii) There exists a C_0-semigroup of operators $(T(t))$ such that $T(t) \in C_p$ for all $t \in (0, \infty)$, but $\|T(t)\|_p \to \infty$ as $t \to 0$ as the following example shows:

Example 2.5. Let A be a positive compact operator which is not of finite rank and $\|A\| \leq 1$. So $A = \sum_{n=1}^\infty \lambda_n e_n \otimes e_n$, for some $0 < \lambda_n < 1$ and decreasing, and (e_n) is some orthonormal basis. Define a one parameter family of operators as follows:

$$T(t) = \sum_{n=1}^\infty \lambda_n^t e_n \otimes e_n .$$

It is easily seen that $(T(t))$ is a C_0-semigroup of operators on H. Choose $(\lambda_n) \in \bigcap_{p>0} l^p$, where l^p is the space of p-summable sequences. Then $T(t) \in C_p$ for all

p and all t. Now, $\|T(t)\|_p = (\sum_{n=1}^{\infty} \lambda_n^{tp})^{1/p}$. Further $\|T(t)\|_p \leq \|T(s)\|_p$ for $t > s$. The Monotone Convergence Theorem implies that $\|T(t)\|_p \to \infty$ as $t \to 0$.

Another main result of this section:

Theorem 2.6. *Let $T(t)$ be C_0-semigroup of operators in $L(H)$ with generator A. If $w \in (0, \infty)$ such that $\|T(t)\| \leq e^{-wt}$, then the following are equivalent.*

(i) *$T(t) \in C_p$ for $t \in (0, \infty)$ and $\|T(1/n)\|_p \leq \gamma$ for all $n \geq n_0$, for some n_0.*

(ii) *$R(\lambda, A) \in C_p$ and $\|R(\lambda, A)\|_p \leq \gamma/(\lambda + w)$ for some $\gamma > 0$ and all $\lambda > 0$.*

Proof. (i) $\to$ (ii). Set $R_n(\lambda, A)x = \int_{1/n}^{\infty} e^{-\lambda s} T(s) x ds$. Since $T(t) \in C_p$ and $\|T(t)\|_p \leq \gamma$, for all t in some neighborhood of zero, then

$$R_n(\lambda, A)x = T\left(\frac{1}{n}\right) \int_{1/n}^{\infty} e^{-\lambda s} T\left(s - \frac{1}{n}\right) x ds$$

is an element of C_p and

$$\begin{aligned}\|R_n(\lambda, A)\|_p &\leq \left\| T\left(\frac{1}{n}\right) \right\|_p \frac{1}{\lambda + w} \\ &\leq \frac{\gamma}{\lambda + w}\end{aligned}$$

for large values of n. But $R_n(\lambda, A)x \to R(\lambda, A)x$ for all $x \in H$. Consequently Lemma 2.1, implies that $R(\lambda, A) \in C_p$ and $\|R(\lambda, A)\|_p \leq \gamma/(\lambda + w)$.

(ii) $\to$ (i) By the expansion formula of $T(t)$, [1, p. 352] we have $T(t)x = \lim_{\lambda \to \infty} e^{-\lambda t} \sum_{n=0}^{\infty} (\lambda^n t^n)/(n!) [\lambda R(\lambda, A)]^n x$, for $\lambda > -w$, where w is as given in the assumption. Then

$$\|T(t)\|_p \leq \lim_{\lambda \to \infty} e^{-\lambda t} \sum_{0}^{\infty} \frac{\lambda^n t^n}{n!} \lambda^n \|R(\lambda, A)\|^{n-1} \|R(\lambda, A)\|_p.$$

But $\|R(\lambda, A)\| \leq \int_0^{\infty} e^{-\lambda s} \|T(s)\| ds \leq \frac{1}{\lambda + w}$. Hence

$$\begin{aligned}\|T(t)\|_p &\leq \lim_{\lambda \to \infty} e^{-\lambda t} \sum_{0}^{\infty} \frac{\lambda^n t^n}{n!} \frac{\lambda^n}{(\lambda + w)^{n-1}} \frac{\gamma}{(\lambda + w)} \\ &\leq \lim_{\lambda \to \infty} e^{-\lambda t} \gamma \sup_n \frac{\lambda^n}{(\lambda + w)^n} \sum_{0}^{\infty} \frac{\lambda^n t^n}{n!} \\ &\leq \gamma \lim_{\lambda \to \infty} \sup_n \frac{\lambda^n}{(\lambda + w)^n}.\end{aligned}$$

Since $\lambda > 0$ and $w > 0$, then $\sup \lambda^n/(\lambda + w)^n = \lambda/(\lambda + w)$. Hence $\|T(t)\|_p \leq \gamma$. Consequently $T(t) \in C_p$. This ends the proof.

III. Further Results

In this section we prove:

Theorem 3.1. *Let* $(T(t))$ *be a* C_0*-semigroup. If* $T(t)$ *is self adjoint in* C_p *for all* $t > 0$, and *some* $p_0 > 0$, *then* $T(t) \in C_{p_0}$ *for all* $t > 0$ and all $p > 0$.

Proof. Since $T(t)$ is self adjoint, then for each $t > 0$ there exist a positive decreasing sequence $(\lambda_{n,t})$ and an orthonormal sequence $(e_{n,t})$ such that

$$T(t) = \sum_{n=1}^{\infty} \lambda_{n,t} e_{n,t} \otimes e_{n,t} \, .$$

With no loss of generality, we assume $T(t) \in C_1$ for all $t > 0$. For simplicity, we show that $T(1) \in C_p$ for all p.

If $p > 1$, then $C_1 \subseteq C_p$ and there is nothing to prove. Assume $p < 1$. Thus $p = 1/\varepsilon$ for some $\varepsilon > 1$. Choose a positive integer n such that $n > \varepsilon$. Assume

$$T\left(\frac{1}{n}\right) = \sum_{k=1}^{\infty} \xi_k e_k \otimes e_k \, ,$$

(e_k) is orthonormal basis and (ξ_k) is a positive decreasing sequence in l^1. Now:

$$T(1) = T^n\left(\frac{1}{n}\right) = \sum_{k=1}^{\infty} \xi_k^n e_k \otimes e_k \, .$$

Since $(\xi_k) \in l^1$, it follows that $(\xi_k^n) \in l^{1/n} \subseteq l^p$. Thus $T(1) \in l^p$. In a similar way we can show $T(t) \in l^p$ for $t > 0$. This ends the proof.

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