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On a Functional Equation on C*-Algebras

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1. Introduction

Watatani [7] has proved that if α and β are $*$-automorphisms of the algebra $B(H)$ of all bounded linear operators on a Hilbert space H satisfying the equation

$$\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x) \tag{1}$$

for all x in $B(H)$, then they commute. The purpose of this note is to consider this problem in a more general setting of C^*-algebras and provide a simpler proof without using Kadison's technical result about extreme points as in [7] and get Watatani's result as an easy corollary to our result (Theorem 2.2).

We may mention in brief that equation (1) has played a fundamental role in the geometric interpretation of the Tomita-Takesaki theory [1] and also in the generalization of the Tomita-Takesaki theory for Jordan algebras [2]. Also in recent years this equation has been solved (in various forms) by central projections. For instance it has been shown in [5] that if α and β are commuting $*$-automorphisms of a von Neumann algebra M satisfying equation (1) then there exists a central projection p in M such that $\alpha=\beta$ on Mp and $\alpha=\beta^{-1}$ on $M(1-p)$. For more details about this type of decomposition we refer to [5] and [6].

2. Results

Our main result depends on the following lemma of [4] which has been proved for von Neumann algebras. Essentially the same argument can work for C^*-algebras as well. We include it here for the sake of completeness and immediate reference.

Lemma 2.1 ([4]). *Let α be a $*$-automorphism of a C^*-algebra M. If x in M satisfies that $\alpha(x)+\alpha^{-1}(x)=2x$, then $\alpha(x)=x$.*

Proof. By assumption, we have $\alpha^2(x)-2\alpha(x)+x=0$. This implies that $(\alpha-1)(\alpha-1)(x)=0$. Put $(\alpha-1)(x)=y$. Then $\alpha(y)=y$ or one gets $\alpha^n(y)=y$ for all positive integers n. But $\alpha(x)=x+y$, so $\alpha^n(x)=x+ny$. This means

$$n\|y\| = \|\alpha^n(x)-x\| \leq \|\alpha^n(x)\| + \|x\| = 2\|x\|.$$

Therefore, $n\|y\| \leq 2\|x\|$ for all n. This shows that $\|y\| \leq 2\|x\|/n$ and as $n \to \infty$, we get that $y=0$. This shows that $y=0$ or $\alpha(x)=x$.

We now prove the commutativity result which partially answers the problem proposed in [5].

Theorem 2.2. *Let α and β be $*$-automorphisms of a C^*-algebra M satisfying $\alpha(x)+\alpha^{-1}(x)=\beta(x)+\beta^{-1}(x)$ for all x in M. Suppose α (or β) is inner then α and β commute.*

Proof. Suppose α is inner. Then $\alpha(x)=uxu^{-1}$, $(x \in M)$ for an invertible u in M. It follows that

$$\beta(x) + \beta^{-1}(x) = uxu^{-1} + u^{-1}xu.$$

In particular when $x=u$, we get $\beta(u)+\beta^{-1}(u)=2u$. By Lemma 2.1, $\beta(u)=u$. Therefore, for any x in M,

$$\begin{aligned}(\alpha\beta)(x) = \alpha(\beta(x)) &= u\beta(x)u^{-1} = \beta(u)\beta(x)\beta(u^{-1}) \\ &= \beta(uxu^{-1}) = (\beta\alpha)(x).\end{aligned}$$

This shows that α and β commute.

Remark that for $M=B(H)$, we obtain Watatani's result as an immediate corollary to the above theorem.

Since every $*$-automorphism of a type I von Neumann algebra leaving the center pointwise fixed is inner (Sakai [3, Corollary 2.9.32]), therefore we have the following

Corollary 2.3. *Let M be a type I von Neumann algebra and α, β be $*$-automorphisms satisfying $\alpha(x)+\alpha^{-1}(x)=\beta(x)+\beta^{-1}(x)$ for all x in M. If α (or β) leaves the center pointwise fixed, then α and β commute.*

References

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