Nonlinear Schrödinger Equations in  
Weighted Sobolev Spaces  

By  
Nakao Hayashi*, Kuniaki Nakamitsu** and Masayoshi Tsutsumi*  
(*Waseda University and **Tokyo Denki University, Japan)  

1. Introduction  
As can be seen from the explicit formula for its propagator, the free  
Schrödinger equation on $\mathbb{R}^n$ shows a strong smoothing property, in case the  
initial data belongs to the domain in $L^2$ of the multiplication operator $(1+|x|^2)^{m/2}$,  
for some positive integer $m$. Indeed, any data in that domain can yield a solution  
which instantly becomes $m$-times differentiable locally in $L^2$. The explicit formula  
for the free propagator also shows that, if, in addition, the data is $m$-times  
differentiable in $L^2$, the solution remains well localized, i.e., remains in the domain  
of $(1+|x|^2)^{m/2}$ under time evolution. It has been shown that the equation with a  

We are concerned with the case where the perturbation term is nonlinear.  
It has been observed that the Schrödinger equation with a regular power non-  
linearity has the analogous smoothing [7] [8] and localizing [14] properties.  
Special results of this type have been appeared also in some earlier works [2]  
[3] [13] in connection with the scattering theory for nonlinear Schrödinger  
equations.  

We prove here those smoothing and localizing effects for a fairly general  
class of nonlinearities, including nonlocal ones. The main results, described  
precisely in Section 2, imply that any nonlinear Schrödinger equation of the  
usual type has those properties, provided the nonlinear term is suitably regular  
in certain weighted Sobolev spaces. The proofs of the main results will be given  
by Section 3. In Section 4 we apply the results to some typical equations arising  
in physics.  

2. The main results  
We begin with some notation. Let $k$ and $m$ be nonnegative integers, and let  
$1 \leq p \leq \infty$. We denote by $W_{m}^{k,p}(R^n)$ the complex Banach space with norm  
$$
\|\phi\|_{W_{m}^{k,p}} = \left\{ \sum_{|\alpha| \leq k} \|\partial^\alpha \phi\|_p^p + \sum_{|\alpha| \leq m} \|x^\alpha \phi\|_p^p \right\}^{1/p},
$$
where $\| \cdot \|_p$ is the norm in $L^p=L^p(\mathbb{R}^n)$. $L^p_m$ and $H^k_m$ denote the spaces $W^{0,p}_m$ and $W^{k,2}_m$, respectively. We will regard $H^k_m$ as a Hilbert space equipped with its obvious inner product $(\cdot, \cdot)_{H^k_m}$. $W^{k,p}$ and $H^{k} = W^{k,2}$ denote the usual Sobolev spaces. Thus $W^{k,p}_0=W^{k,p}$ and $H^{k}_0=H^{k}$. The conjugate exponent to $p$ will be written $p'$, so $1/p+1/p'=1$. Positive constants will be denoted by the same letter $c$.

We shall consider the Cauchy problem for nonlinear Schrödinger equations on $\mathbb{R}^n$, $n \geq 1$, of the form

$$i \frac{\partial u}{\partial t} = -\Delta u + F(u)$$

with initial condition $u|_{t=0} = u_0 \in H^k_m$. Here $\Delta$ is the Laplace operator on $\mathbb{R}^n$, and $F$ denotes a possibly nonlinear map. This problem can be conveniently rewritten in the form of the integral equation

$$(2.1) \quad u(t) = S(t)u_0 - i \int_0^t S(t-s)F(u(s))ds,$$

where $S(t) = \exp(it\Delta)$. The well-known explicit formula for $S(t)$ implies that $S(t)H^k_m \subset H^k_m$ for all $t \in \mathbb{R}$ when $k \geq m$, and $S(t)H^k_m \subset H^k_m \cap H^0_{loc}$ for all $t \in \mathbb{R}\setminus\{0\}$ when $k < m$. This means that the free Schrödinger equation with $H^k_m$-data shows a localizing property in case $k \geq m$ and a smoothing property in case $k < m$. Accordingly, we distinguish those two cases here.

We now state the main results.

**Theorem 1.** Let $k$ and $m$ be nonnegative integers with $k \geq m$. Let $\lambda \geq 1$ and $2 \leq p < \infty$ with $1/2 - 2/n(\lambda + 1) < 1/p$. Let $V$ be a Banach space in which $H^k_m$ is embedded continuously. Let $F$ be a mapping from $W^{k,p}_m \cap H^k_m$ to $W^{k,p'}_m$ such that

$$(2.2) \quad \|F(\phi) - F(\psi)\|_{W^{k,p'}_m} \leq c\{1 + \|\phi\|_{W^{k,p}_m} + \|\psi\|_{W^{k,p}_m}\}^{\lambda - 1} \|\phi - \psi\|_{W^{k,p}_m},$$

$$(2.3) \quad \|F(\phi)\|_{W^{k,p'}_m} \leq c\{1 + \|\phi\|_{V}\}^{\lambda - 1} \|\phi\|_{W^{k,p}_m}$$

for all $\phi, \psi \in W^{k,p}_m \cap H^k_m$. Then for any $u_0 \in H^k_m$, there is a $T > 0$, and a unique solution $u$ of (2.1) on $[0, T)$ with

$$u \in C([0, T); H^k_m) \cap L^{4p/n(p-2)}(0, T; W^{k,p}_m) \equiv X_{(0,T)}.$$

We may assume either $T = \infty$ or $\lim_{t \uparrow T} \|u(t)\|_V = \infty$. Furthermore, if $u_{0j} \in H^k_m$, $j = 1, 2, \ldots$, and $u_{0j} \to u_0$ in $H^k_m$ as $j \to \infty$, then for any $T'$ with $0 < T' < T$, we may assume that $u_j \in X_{(0,T_j)}$ with $T' < T_j$, $u_j$ being the solution for $u_{0j}$, for sufficiently large $j$, and $u_j \to u$ in $X_{(0,T')}$.

**Theorem 2.** Let $k$ and $m$ be nonnegative integers with $k < m$. Let $\lambda \geq 1$
and \(2 \leq p < \infty\) with \(1/2 - 2/n(\lambda + 1) < 1/p\). Let \(1 \leq q_i \leq \infty\), \(i = 1, 2\), with \(H^k\) embedded continuously in \(L^{q_1} \cap L^{q_2}\). Let \(F\) be a mapping such that \(F(W^{j_1,p}_j \cap H^k_j) \subset W^{j_2,p}_j\) for \(j = k, m\), \(F(W^{m,p}_m \cap H^m) \subset W^{m,p}_m\), and \(F(\phi) = g(|\phi|)\phi\) for some \(g\) and for all \(\phi \in W^{m,p}_m \cap H^m\). Assume that

\[
\tag{2.4} \|F(\phi) - F(\psi)\|_{W^{j_1,p}_j} \leq c \{1 + \|\phi\|_{W^{j_2,p}_j} + \|\psi\|_{W^{j_2,p}_j}\}^{\lambda - 1} \|\phi - \psi\|_{W^{j_1,p}_j}^\lambda
\]

\[
\tag{2.5} \|F(\phi)\|_{W^{j_1,p}_j} \leq c \{1 + \|\phi\|_{q_1} + \|\phi\|_{q_2}\}^{\lambda - 1} \|\phi\|_{W^{j_1,p}_j}^\lambda
\]

for \(j = k, m\) and for all \(\phi, \psi \in W^{j_1,p}_j \cap H^k_j\), and that

\[
\tag{2.6} \sum_{|a|=m} \|\partial^a F(\phi)\|_{p'} \leq c \{1 + \|\phi\|_{q_1} + \|\phi\|_{q_2}\}^{\lambda - 1} \sum_{|a|=m} \|\partial^a \phi\|_p
\]

for all \(\phi \in W^{m,p}_m \cap H^m\). Then for any \(u_0 \in H^k\), there is a \(T > 0\), and a unique solution \(u\) of (2.1) on \([0, T)\) with

\[
u \in C([0, T); H^k) \cap L^{4p/(p-2)}(0, T; W^{k,p}_k)
\]

and

\[
x^a(\cdot)u(\cdot) \in C([0, T); L^2)
\]

whenever \(|x| = m\), where \(x^a(t)\) denotes the operator defined by

\[
\tag{2.7} x^a(t) \phi = \begin{cases} 
\exp \left( i \frac{|x|^2}{4t} \right) (2it\partial)^a \exp \left( - i \frac{|x|^2}{4t} \right) \phi & \text{for } t \neq 0, \\
x^a \phi & \text{for } t = 0,
\end{cases}
\]

in the distribution sense. Thus, in particular, \(u(t)\) is in \(H^k_{10}\) for all \(t \in (0, T)\). We may assume either \(T = \infty\) or \(\lim_{t \to T} \sum_{i=1,2} \|u(t)\|_{q_i} = \infty\).

The proofs of these results will be given in the next section. It will be seen from the proofs that, in case \(p = 2\), one may replace \(\{ \cdot \}^{\lambda - 1}\) in (2.2)–(2.6) by any positive nondecreasing function \(f(\cdot)\).

3. Proofs of Theorems 1 and 2

The proofs of Theorems 1 and 2 depend on several properties of \(S(t) = \exp (itD)\). It is well known (see, e.g., [1]) that for any \(p\) with \(2 \leq p \leq \infty\) and for any \(t \in R \setminus \{0\}\), \(S(t)\) is a bounded operator as the one from \(L^p(R^n)\) to \(L^p(R^n)\), with

\[
\tag{3.1} \|S(t)\phi\|_p \leq (4\pi|t|)^{-\delta(p)}\|\phi\|_{p'},
\]

where \(\delta(p) = n/2 - n/p\), for all \(\phi \in L^p\), and the mapping \(t \to S(t)\) is strongly con-
Nakao Hayashi, Kuniaki Nakamitsu and Masayoshi Tsutsumi

continuous. In case $p=2$, the $S(t)$ form a $C_0$-group of unitary operators. Furthermore, if $2 \leq p, q \leq \infty$ with $0 \leq \delta(p) < 1$ and $2/q = \delta(p)$, we have

$$\|S(\cdot)\phi; L^q(R; L^p)\| \leq c\|\phi\|_2$$

for all $\phi \in L^2$ [4] (the result given there, stated only for $n \geq 3$, naturally extends to $n \geq 1$).

The following facts will be useful.

**Lemma 1.** Let $2 \leq p, q \leq \infty$ with $0 \leq \delta(p) < 1$ and $2/q = \delta(p)$, and let $T > 0$. Then, with

$$I(t, u) \equiv \int_0^t S(t-s)u(s)ds,$$

we have

$$\|I(\cdot, u); L^q(0, T; L^p)\| \leq c\|u; L^{q'}(0, T; L^{p'})\|$$

for all $u \in L^{p'}(0, T; L^{q'})$, and

$$\|I(t, u)\|_2 \leq c\|u; L^{q'}(0, T; L^{p'})\|$$

for all $u \in L^{q'}(0, T; L^{p'})$ and all $t \in [0, T]$. Furthermore, for each $u \in L^{q'}(0, T; L^{p'})$, $I(\cdot, u) \in C([0, T]; L^2)$.

**Proof.** For $p=2$, (3.3) follows from the unitarity of $S(t)$ in $L^2$. In case $2 < p \leq \infty$, we use (3.1) and the Hardy-Littlewood-Sobolev inequality (see, e.g., [9]). Then

$$\|I(\cdot, u); L^p(0, T; L^q)\| \leq \left\{ \int_0^T \|S(t-s)u(s)\|_p^q dt \right\}^{1/q} \leq c\left\{ \int_0^T dt \left[ \int_0^t (t-s)^{-\delta(p)}\|u(s)\|_p dt \right]^{q} \right\}^{1/q} \leq c\left\{ \int_0^T \|u(t)\|_{p'}^{q'} dt \right\}^{1/q'},$$

since $1 + 1/q = \delta(p) + 1/q$, and $0 < \delta(p) < 1$. To prove (3.4), by density and duality we need only show that, for all $\phi \in \mathcal{S}(R^n)$, and for all $t \in [0, T]$,

$$|\langle \phi, I(t, u) \rangle| \leq c\|\phi\|_2\|u; L^{q'}(0, T; L^{p'})\|,$$

where $\langle \cdot, \cdot \rangle$ represents the duality. Now the above calculation also implies $I(t, u) \in L^p$ for a.e. $t \in [0, T]$. For those $t$, we have
\[ |\langle \phi, I(t, u) \rangle | = \left| \int_0^t \langle S(t-s)\phi, u(s) \rangle \, ds \right| \]
\[ \leq \int_0^t \| S(t-s)\phi \|_p \| u(s) \|_p \, ds \]
\[ \leq \| S(\cdot)\phi \|_{L^p(\mathbb{R}; L^p)} \| u \|_{L^p(0, T; L^p')} \]

and hence (3.4) by (3.2). It remains to show that \( I(\cdot, u) \in C([0, T]; L^2) \).
For that purpose, we choose \( u_j \in C_0([0, T] \times \mathbb{R}^n), j = 1, 2, \ldots \), so that \( u_j \to u \)
in \( L^p(0, T; L^p) \) as \( j \to \infty \). Then \( I(\cdot, u_j) \in C([0, T]; L^2) \) because of the
\( C_0 \)-group property of \( S(t) \) on \( L^2 \), and \( I(\cdot, u_j) \to I(\cdot, u) \) in \( L^\infty([0, T]; L^2) \)
as \( j \to \infty \) by (3.4). Thus \( I(\cdot, u) \in C([0, T]; L^2) \).

We shall derive the corresponding properties of \( S(t) \) in the spaces \( W_{m,p}^{k,p} \).
We remark that for any nonnegative integer \( m \) and for any \( p \) with \( 1 < p < \infty \),
one has
\[ \sum_{|x| \leq m} \| x^\alpha \phi \|_p \leq c \| \phi \|_{W_{m,p}^{k,p}} \]
for all \( \phi \in W_{m,p}^{k,p} \). See [16] for a proof. In particular, for those \( m \) and \( p \),
\[ (x - 2it\partial)^s \phi \|_p \leq c(1 + |t|)^m \| \phi \|_{W_{m,p}^{k,p}} \]
whenever \( |x| \leq m, t \in \mathbb{R} \) and \( \phi \in W_{m,p}^{k,p} \). This inequality enable us to prove the
following fact.

**Lemma 2.** Let \( m \) be any nonnegative integer, and let \( p \) be any number with
\( 2 \leq p < \infty \). Then for any \( t \in \mathbb{R} \), \( S(t) \) is a bounded operator as the one from
\( W_{m,p'}^{k,p'} \) to \( W_{m,p}^{k,p} \). Furthermore, for any \( t, s \in \mathbb{R} \) with \( t \neq s \) and for any \( \phi \in W_{m,p'}^{k,p'} \),
we have
\[ (x + 2it\partial)^s S(t-s)\phi = S(t-s)(x + 2is\partial)^s \phi \]
whenever \( |x| \leq m \). For \( p = 2 \), \( S(t) \) is bounded for all \( t \in \mathbb{R} \), and the above equality
holds for all \( t, s \in \mathbb{R} \).

**Proof.** Let \( \phi \in \mathcal{S} \). For such a \( \phi \), we have (3.6) in \( \mathcal{S} \) for all \( t, s \in \mathbb{R} \) and all
\( x \), by using the explicit formula for \( S(t) \) as well as the identity property of \( S(0) \).
In particular,
\[ x^\alpha S(t)\phi = S(t)(x - 2it\partial)^s \phi \]
for all \( t \) and \( x \). We now fix a \( t \in \mathbb{R} \), and assume \( t \neq 0 \) in case \( 2 < p < \infty \). The above
equality and (3.1) or unitarity of \( S(t) \) in \( L^2 \) imply
\[ \| x^\alpha S(t)\phi \|_p = \| S(t)(x - 2it\partial)^s \phi \|_p \]
\[ \leq c \| (x - 2it\partial)^s \phi \|_p \]
for all \( \alpha \). Using (3.5), we obtain
\[
\|S(t)\phi\|_{L_{m}^{q}} \leq c\|\phi\|_{W_{m}^{m,p'}}.
\]
On the other hand, (3.1) or unitarity of \( S(t) \) in \( L^{2} \) also implies that \( S(t) \) is bounded from \( W_{m}^{m,p'} \) to \( W_{m}^{m,p} \), since \( S(t) \) and \( \delta^{a} \) commute on \( W_{m}^{m,p} \) whenever \( |\alpha| \leq m \). Therefore we have
\[
\|S(t)\phi\|_{W_{m}^{m,p'}} \leq c\|\phi\|_{W_{m}^{m,p}}.
\]
Since \( \mathcal{S} \) is dense in \( W_{m}^{m,p'} \), we conclude that \( S(t) \) is bounded from \( W_{m}^{m,p'} \) to \( W_{m}^{m,p} \). This implies that (3.6) holds not only for \( \phi \in \mathcal{S} \) but for any \( \phi \in W_{m}^{m,p'} \) under the indicated assumption on \( t \) and \( s \).

We now give some properties of \( S(t) \) in the spaces \( W_{m}^{k,p} \).

**Lemma 3.** Let \( k \) and \( m \) be given nonnegative integers with \( k \geq m \). Then for any \( p \) with \( 2 \leq p < \infty \) and for any \( t \in \mathbb{R} \setminus \{0\} \), \( S(t) \) is a bounded operator as the one from \( W_{m}^{k,p} \) to \( W_{m}^{k,p} \) with
\[
\|S(t)\phi\|_{W_{m}^{k,p}} \leq c(1+|t|^{m})|t|^{-\delta(p)}\|\phi\|_{W_{m}^{k,p'}}
\]
for all \( \phi \in W_{m}^{k,p'} \), and the mapping \( t \mapsto S(t) \) is strongly continuous. In case \( p = 2 \), the same assertion holds for any \( t \in \mathbb{R} \), and the \( S(t) \) form a \( C_{0} \)-group. Furthermore, for any \( p \) and \( q \) satisfying \( 2 \leq p < \infty \), \( 2 \leq q \leq \infty \), \( 0 \leq \delta(p) < 1 \) and \( 2/q = \delta(p) \), and with
\[
I(t, u) \equiv \int_{0}^{t} S(t-s)u(s)ds,
\]
the following results hold:

\[
\|S(\cdot)\phi; L^{q}(0, T; W_{m}^{k,p})\| \leq c(1+T)^{m}\|\phi\|_{H_{m}^{k}}
\]
for all \( \phi \in H_{m}^{k} \);

\[
\|I(\cdot, u); L^{q}(0, T; W_{m}^{k,p})\| \leq c(1+T)^{m}\|u\|_{L^{q}(0, T; W_{m}^{k,p'})}
\]
for all \( u \in L^{q}(0, T; W_{m}^{k,p'}) \); and

\[
\|I(t, u)\|_{H_{m}^{k}} \leq c(1+T)^{m}\|u\|_{L^{q}(0, T; W_{m}^{k,p'})}
\]
for all \( u \in L^{q}(0, T; W_{m}^{k,p'}) \) and all \( t \in [0, T] \). Moreover, for each \( u \in L^{q}(0, T; W_{m}^{k,p'}) \), we have \( I(\cdot, u) \in C([0, T]; H_{m}^{k}) \).

**Proof.** In view of (3.6) and (3.5), all the assertions are immediate from the previously indicated properties of \( S(t) \) in the scale of \( L^{p} \)-spaces. Indeed, let
\( \phi \in W_{m}^{k,p'}, k \geq m, 2 \leq p < \infty \), and let \( t \in \mathbb{R} \). Assume \( t \neq 0 \) if \( p \neq 2 \). Then (3.6) implies

\[ x^a S(t)\phi = S(t)(x - 2it\partial)^a \phi \]

in \( L^p \) whenever \( |x| \leq m \). Of course, we also have

\[ \partial^a S(t)\phi = S(t)\partial^a \phi \]

in \( L^p \) whenever \( |x| \leq k \). Recalling the definition of the \( W_{m}^{k,p} \)-norm and using the above two inequalities and (3.5), we have (3.7), (3.8), (3.9) and (3.10) from (3.1), (3.2) (with the integral over \( \mathbb{R} \) on the left hand side replaced by that on \([0, T]\)), (3.3) and (3.4), respectively. The strong continuity assertion below (3.7) can be verified in a similar way by using the strong continuity in the scale of \( L^p \)-spaces of \( S(t) \), mentioned below (3.1), and noting that

\[ x^a S(s)\phi - x^a S(t)\phi = [S(s) - S(t)](x - 2is\partial)^a \phi \\
+ S(t)[(x - 2is\partial)^a \phi - (x - 2it\partial)^a \phi] \]

whenever \( \phi \in W_{m}^{k,p'}, |x| \leq m \) and \( t, s \in \mathbb{R} \setminus \{0\} \) (\( t, s \in \mathbb{R} \) if \( p = 2 \)). That the \( S(t) \) form a group on \( H_{m}^{k} \) if \( k \geq m \) is obvious since each \( S(t) \) leaves such an \( H_{m}^{k} \) invariant. To show the last continuity assertion for \( I(\cdot, u) \), we choose \( u_j \in C_{0}([0, T]; H_{m}^{k}) \), \( j = 1, 2, \ldots \), so that \( u_j \to u \) in \( L^q(0, T; L^p) \) as \( j \to \infty \). Then \( I(\cdot, u_j) \in \mathbb{C}([0, T]; H_{m}^{k}) \) and \( I(\cdot, u_j) \to I(\cdot, u) \) in \( L^\infty([0, T]; H_{m}^{k}) \) by the \( C_{0} \)-group property of the \( S(t) \) on \( H_{m}^{k} \) and \( I(\cdot, u_j) \to I(\cdot, u) \) in \( L^\infty([0, T]; H_{m}^{k}) \) by (3.10). Thus we have \( I(\cdot, u) \in \mathbb{C}([0, T]; H_{m}^{k}) \).

We are now in position to prove Theorems 1 and 2.

**Proof of Theorem 1.** Let \( k, m, \lambda, p, V \) and \( F \) be as in the hypotheses of the theorem, and let \( q = 2/\delta(p) \). The condition \( 1/2 - 2/n(\lambda + 1) < 1/p \) implies \( (\lambda + 1)/q < 1 \). Let \( T > 0 \). We write \( Y = \mathbb{C}([0, T]; H_{m}^{k}), Z = L^q(0, T; W_{m}^{k,p'}) \) and \( X = Y \cap Z \). For the norm in \( X \) we will choose \( \|u\|_X = \max\{\|u\|_Y, \|u\|_Z\} \). Let \( u_0 \in H_{m}^{k} \) and set

\[ Gu(t) \equiv S(t)u_0 - i \int_{0}^{t} S(t-s)F(u(s))ds. \]

We will write as before

\[ I(t, w) = \int_{0}^{t} S(t-s)w(s)ds. \]

We remark that (3.9), (3.10) and the last assertion of Lemma 3 imply that if \( w \in L^q(0, T; W_{m}^{k,p'}) \), then \( I(\cdot, w) \in X \), with

\[ \|I(\cdot, w)\|_X \leq c\|w\|_X \]

and

\[ \|I(\cdot, w)\|_X \leq c\|w\|_L^q(0, T; W_{m}^{k,p'}) \].
We construct a solution of $u(t) = Gu(t)$, i.e., of (2.1) in $X$ for some $T > 0$ by using the method of contraction mapping. Let $u, v \in X$. We recall that $(\lambda + 1)/q < 1$. Then (3.11) and (2.2) give

$$
\| Gu - Gv \|_X
\leq \| I(\cdot, F(u) - F(v)) \|_X
\leq c(1 + T)^{m} \| F(u) - F(v) \|_{L^{q}(0, T; W^{k,p}_{m})}
\leq c(1 + T)^{m} T^{1-(\lambda + 1)/q} \| F(u) - F(v) \|_{L^{q}(0, T; W^{k,p}_{m})}
\leq c(1 + T)^{m} T^{1-(\lambda + 1)/q} \{ 1 + \| u \|_Z + \| v \|_Z \}^{\lambda - 1} \| u - v \|_Z .
$$

Thus we obtain

$$
\| Gu - Gv \|_X
\leq c(1 + T)^{m} T^{1-(\lambda + 1)/q} \{ 1 + \| u \|_Z + \| v \|_Z \}^{\lambda - 1} \| u - v \|_X .
$$

We next estimate $Gu$. From (3.7) with $p = 2$ and (3.8) we have

$$
\| S(\cdot) u_0 \|_X \leq a(1 + T)^{m} \| u_0 \|_{H^k_m}
$$

for some $a > 0$. On the other hand, (3.11), (2.2) and $F(0) = 0$, which follows from (2.3), imply

$$
\| I(\cdot, F(u)) \|_X \leq c(1 + T)^{m} \| F(u) \|_{L^{q}(0, T; W^{k,p}_{m})}
\leq c(1 + T)^{m} T^{1-(\lambda + 1)/q} \{ 1 + \| u \|_Z \}^{\lambda - 1} \| u \|_Z .
$$

Hence we obtain

$$
\| Gu \|_X \leq a(1 + T)^{m} \| u_0 \|_{H^k_m}
+ c(1 + T)^{m} T^{1-(\lambda + 1)/q} \{ 1 + \| u \|_Z \}^{\lambda - 1} \| u \|_Z .
$$

In particular,

$$
\| Gu \|_X \leq a(1 + T)^{m} \| u_0 \|_{H^k_m}
+ c(1 + T)^{m} T^{1-(\lambda + 1)/q} \{ 1 + \| u \|_Z \}^{\lambda - 1} \| u \|_X .
$$

Now let $K$ be any number such that $K \geq \| u_0 \|_{H^k_m}$. We set, with a fixed $b > 0$,

$$
B_K = \{ u \in X; \| u \|_X \leq (a + b)K + b \} .
$$

From (3.12) and (3.14) it follows that $G$ is a contraction from $B_K$ to $B_K$ for a sufficiently small $T > 0$. Thus the equation $u(t) = Gu(t)$ has a unique solution $u \in B_K$ on $[0, T]$ for such a $T$, which we shall denote by $T_K$. Since for each $K > 0, T_K$ can be chosen uniformly for $u_0$ in the ball in $H^k_m$ of radius $K$ and center 0,
we conclude, by a standard argument, that the solution \( u \) extends uniquely to some larger interval \( [0, T_{\text{max}}) \) so that

\[
    u \in C([0, T_{\text{max}}); H^k) \cap L^q(0, T_{\text{max}}; W^{k,p}).
\]

with either \( T_{\text{max}} = \infty \) or \( \lim_{t \uparrow T_{\text{max}}} \|u(t)\|_{H^k} = \infty \).

We claim that either \( T_{\text{max}} = \infty \) or \( \lim_{t \uparrow T_{\text{max}}} \|u(t)\|_{V} = \infty \). To this end, suppose that \( T_{\text{max}} < \infty \) and \( \|u(t)\|_{V} < c \) whenever \( 0 < t < T_{\text{max}} \). We need show that \( \|u(t)\|_{H^k} < c \) on the same interval, again for some constant \( c \). In case \( p = 2 \), the proof is simple. Indeed, let \( 0 < t < T_{\text{max}} \). From the equation, (3.7) and (2.3), we have

\[
    \|u(t)\|_{H^k} \leq \|S(t)u_0\|_{H^k} + \int_0^t \|S(t-s)F(u(s))\|_{H^k} ds
\]

\[
    \leq c(1 + t^m)\|u_0\|_{H^k} + c(1 + t^m) \int_0^t \|F(u(s))\|_{H^k} ds
\]

\[
    \leq c(1 + t^m)\|u_0\|_{H^k} + c(1 + t^m) \int_0^t [1 + \|u(s)\|_{V}]^{\lambda-1} \|u(s)\|_{H^k} ds.
\]

Thus

\[
    \|u(t)\|_{H^k} \leq c(1 + T_{\text{max}}^m)\|u_0\|_{H^k} + c(1 + T_{\text{max}}^m) \int_0^t \|u(s)\|_{H^k} ds.
\]

Gronwall’s inequality then implies that \( \|u(t)\|_{H^k} < c \) whenever \( 0 < t < T_{\text{max}} \), as desired. We now assume \( 2 < p < \infty \), in which case \( 2 < q < \infty \). To prove that \( \|u(t)\|_{H^k} < c \) whenever \( 0 < t < T_{\text{max}} \), we need only show that \( u \in L^q(0, T_{\text{max}}; W^{k,p}_m) \), in view of (3.13) and the equation \( u = Gu \). Let \( 0 < T < T_{\text{max}} \). The equation and (3.8) give

\[
    \|u; L^q(0, T; W^{k,p}_m)\|
\]

\[
    \leq \|S(\cdot)u_0; L^q(0, T; W^{k,p}_m)\| + \|I(\cdot, F(u)); L^q(0, T; W^{k,p}_m)\|
\]

\[
    \leq c(1 + T^m)\|u_0\|_{H^k} + \|I(\cdot, F(u)); L^q(0, T; W^{k,p}_m)\|.
\]

From (3.7) and (2.3) we have

\[
    \|I(\cdot, F(u)); L^q(0, T; W^{k,p}_m)\|
\]

\[
    \leq c(1 + T^m) \left[ \int_0^T dt \left\{ \int_0^t (t-s)^{-\delta(p)} [1 + \|u(s)\|_{V}]^{\lambda-1} \|u(s)\|_{W^{k,p}_m} ds \right\}^{q-1/q} \right]^{1/q}
\]

\[
    \leq c(1 + T^m) \left[ \int_0^T dt \left\{ \int_0^t (t-s)^{-\delta(p)} \|u(s)\|_{W^{k,p}_m} ds \right\}^{q-1/q} \right]^{1/q}.
\]

It follows that

\[
    \|u; L^q(0, T; W^{k,p}_m)\| \leq c(1 + T_{\text{max}}^m)\|u_0\|_{H^k} + c(1 + T_{\text{max}}^m)J,
\]

(3.15)
where

\[ J = \left[ \int_0^T dt \left\{ \int_0^t (t-s)^{-\delta(p)} \| u(s) \|_{W_{m}^{k,p}} ds \right\}^{q/3} \right]^{2q/3}. \]

Now

\[ J \leq c \{ J_1 + J_2 + J_3 \} \]

with

\[ J_1 = \left[ \int_0^\epsilon dt \left\{ \int_0^t (t-s)^{-\delta} \| u(s) \|_{W_{m}^{k,p}} ds \right\}^{q/3} \right]^{2q/3}, \]

\[ J_2 = \left[ \int_\epsilon^T dt \left\{ \int_0^t (t-s)^{-\delta(p)} \| u(s) \|_{W_{m}^{k,p}} ds \right\}^{q/3} \right]^{2q/3}, \]

and

\[ J_3 = \left[ \int_\epsilon^T dt \left\{ \int_{t-\epsilon}^T (t-s)^{-\delta(p)} \| u(s) \|_{W_{m}^{k,p}} ds \right\}^{q/3} \right]^{2q/3}, \]

where \( 0 < \epsilon < T \). For \( J_1 \) we have

\[ J_1^{q/3} \leq \int_0^\epsilon dt \left\{ \int_0^t (t-s)^{-\delta(p)} ds \right\}^{q/3} \left\{ \int_0^t (t-s)^{-\delta(p)} \| u(s) \|_{W_{m}^{k,p}}^{3/2} ds \right\}^{2q/3}, \]

\[ \leq c \epsilon^{(1-\delta(p))q/3} \int_0^\epsilon dt \left\{ \int_0^t (t-s)^{-\delta(p)} \| u(s) \|_{W_{m}^{k,p}}^{3/2} ds \right\}^{2q/3}, \]

\[ \leq c \epsilon^{(q-2)/3} \int_0^T dt \left\{ \int_0^t (t-s)^{-\delta(p)} \| u(s) \|_{W_{m}^{k,p}}^{3/2} ds \right\}^{2q/3}, \]

since \( 2/q = \delta(p) \). The Hardy-Littlewood-Sobolev inequality implies

\[ \left[ \int_0^T dt \left\{ \int_0^t (t-s)^{-\delta(p)} \| u(s) \|_{W_{m}^{k,p}}^{3/2} ds \right\}^{2q/3} \right]^{3/2q}, \]

\[ \leq \left\{ \int_0^T \| u(s) \|_{W_{m}^{k,p}}^{3q/(2q-1)} ds \right\}^{(2q-1)/2q}, \]

since \( 3/2q = (2q-1)/2q + \delta(p) - 1 \). We also have \( (2q-1)/3 > 1 \). Thus

\[ J_1^{q/3} \leq c \epsilon^{(q-2)/3} \left\{ \int_0^T \| u(s) \|_{W_{m}^{k,p}}^{3q/(2q-1)} ds \right\}^{(2q-1)/3}, \]

\[ \leq c \epsilon^{(q-2)/3} T^{2(q-2)/3} \int_0^T \| u(s) \|_{W_{m}^{k,p}}^{q} ds. \]

We estimate \( J_3 \) in a completely analogous way. We have
Nonlinear Schrödinger Equations

\[ J_3^q \leq \int_\epsilon^T dt \left\{ \int_{t-\epsilon}^t (t-s)^{-\delta(p)} ds \right\}^{q/3} \left\{ \int_{t-\epsilon}^t (t-s)^{-\delta(p)} \| u(s) \|_{W_{m}^{k,p}}^{3/2} ds \right\}^{2q/3} \leq c \epsilon^{(q-2)/3} \int_0^T dt \left\{ \int_0^{t-\epsilon} (t-s)^{-\delta(p)} \| u(s) \|_{W_{m}^{k,p}}^{3/2} ds \right\}^{2q/3} \leq c \epsilon^{(q-2)/3} T^{2(q-2)/3} \int_0^T \| u(s) \|_{W_{m}^{k,p}}^{q} ds. \]

For \( J_2 \) we have

\[ J_2^q \leq \int_\epsilon^T dt \left\{ \int_0^{t-\epsilon} (t-s)^{-\delta(p)q'/q} ds \right\}^{q/q'} \int_0^t \| u(s) \|_{W_{m}^{k,p}}^{q} ds \leq c T^{(q-3)/q} \int_0^T \| u(s) \|_{W_{m}^{k,p}}^{q} ds. \]

From (3.15) and the above estimates, we obtain

\[ \| u; L^q(0, T; W_{m}^{k,p}) \| \leq c (1 + T_{\max})^m \| u_0 \|_{H_{m}^k} + c (1 + T_{\max})^m T_{\max}^{(q-3)/q} \left\{ \int_0^T \| u; L^q(0, t; W_{m}^{k,p}) \|^{q} dt \right\}^{1/q} \]

\[ + c \| u; L^q(0, T; W_{m}^{k,p}) \|. \]

Since this holds for any \( \epsilon > 0 \) with \( \epsilon < T \), it follows that

\[ \| u; L^q(0, T; W_{m}^{k,p}) \| \leq c (1 + T_{\max})^m \| u_0 \|_{H_{m}^k} \]

\[ + c (1 + T_{\max})^m T_{\max}^{(q-3)/q} \int_0^T \| u; L^q(0, t; W_{m}^{k,p}) \|^{q} dt, \]

whenever \( 0 \leq T < T_{\max} \). Thus by Gronwall's inequality we have

\[ \| u; L^q(0, T; W_{m}^{k,p}) \| < c \]

whenever \( 0 < T < T_{\max} \). Thus \( u \in L^q(0, T_{\max}; W_{m}^{k,p}) \), as desired.

Now let \( u_{0j} \in H_{m}^k \), \( j = 1, 2, \ldots \), and let \( u_j \in X_{(0, T_j)} \) be the solution of \( u_j(t) = G_j u_j(t) \) for each \( j \), where \( T_j = T_{j,\max} \) and \( G_j \) denotes \( G \) with \( u_0 \) replaced by \( u_{0j} \). Assume that \( u_{0j} \rightarrow u_0 \) in \( H_{m}^k \) as \( j \rightarrow \infty \). We shall consider the convergence of \( u_j \).
to the solution \( u \in X_{[0,T_{\text{max}}]} \) of \( u(t) = Gu(t) \). As was shown previously, for each \( k > 0 \), there is a \( T_{K} > 0 \), and a ball \( B_{K} \) in \( X_{[0,T_{K}]} \) of radius \( ck + c \) and center 0, such that if \( u_{0} \) is in the ball in \( H_{m}^{k} \) of radius \( K \) and center 0, then the mapping \( G \) is a contraction from \( B_{K} \) to \( B_{K} \). The solution \( u \) is in \( B_{K} \) on \([0, T_{K}] \) by its construction. Assume that \( u_{0} \) is in such a ball in \( H_{m}^{k} \) of radius \( K \). Since \( u_{0j} \rightarrow u_{0} \) in \( H_{m}^{k} \), it follows that \( u_{j} \in B_{2K} \) for sufficiently large \( j \). We also have, with \( X = X_{[0,T_{2K}]} \),

\[
\|u - u_{j}\|_{X} = \|Gu - G_{j}u_{j}\|_{X} \\
\leq \|Gu - Gu_{j}\|_{X} + \|Gu_{j} - G_{j}u_{j}\|_{X} \\
\leq \|Gu - Gu_{j}\|_{X} + \|S(\cdot)(u_{0} - u_{0j})\|_{X}.
\]

Using (3.7) with \( p = 2 \), (3.8) and the fact that \( G \) is a contraction from \( B_{2K} \) to itself, we obtain

\[
\|u - u_{j}\|_{X} \leq c(1 + T_{2K})^{m}\|u_{0} - u_{0j}\|_{H_{k}^{k}}.
\]

Thus \( u_{j} \rightarrow u \) in \( X_{[0,T_{2K}]} \). A repeated application of this result leads to the conclusion, in a finite number of steps, that for any \( T' \) with \( 0 < T' < T_{\text{max}} \), \( u_{j} \in X_{[0,T']} \) for sufficiently large \( j \), and \( u_{j} \rightarrow u \) in \( X_{[0,T']} \). This completes the proof of Theorem 1.

**Proof of Theorem 2.** Let \( k, m, \lambda, p, q_{1}, q_{2} \) and \( F \) be as in the hypotheses of the theorem, and let \( q = 2/\delta(p) \), as before. We remark that \( F \) satisfies the conditions of Theorem 1 with \( (k, m) = (k, k), (m, m) \) and \( V = L^{q_{1}} \cap L^{q_{2}} \). Let \( u_{0} \in H_{k}^{k} \). Since, in particular, \( u_{0} \in H_{k}^{k} \), Theorem 1 implies that there is a \( T > 0 \), and a unique solution \( u \) of (2.1) on \([0, T)\) with

\[
u \in C([0, T); H_{k}^{k}) \cap L^{q}(0, T; W_{k}^{q_{1}, p}) \equiv X_{[0,T)}^{k}\]

and either \( T = \infty \) or \( \lim_{t \uparrow T} \sum_{i=1,2} \|u(t)\|_{q_{i}} = \infty \). Thus we need only show that, with \( x^{a}(t) \) defined in (2.7),

\[
x^{a}(\cdot)u(\cdot) \in C([0, T); L^{2})
\]

whenever \( |x| = m \). To this end, we choose \( u_{0j} \in H_{m}^{m}, j = 1, 2, \ldots \), so that \( u_{0j} \rightarrow u_{0} \) in \( H_{k}^{k} \) as \( j \rightarrow \infty \). Again by Theorem 1, there is for each \( j \) a \( T_{j} > 0 \), and a unique solution \( u_{j} \) on \([0, T_{j})\) of (2.1) with \( u_{0} \) replaced by \( u_{0j} \), such that \( u \in X_{[0,T_{j})}^{m} \) with either \( T_{j} = \infty \) or \( \lim_{t \uparrow T_{j}} \sum_{i=1,2} \|u_{j}(t)\|_{q_{i}} = \infty \). We remark that since \( u_{j} \in X_{[0,T_{j})}^{m} \), an explicit computation and (3.5) give

(3.17)

\[
x^{a}(t)u_{j}(t) = (x + 2it\partial)^{a}u_{j}(t)
\]

and

\[
x^{a}(\cdot)u_{j}(\cdot) \in C([0, T_{j}); L^{2}) \cap L^{q}(0, T_{j}; L^{p})
\]
whenever \(|x| \leq m\).

We prove the indicated property of \(x^a(t)u(t)\) by an approximating procedure using \(x^a(t)u_j(t)\). Let \(T'\) be any number with \(0 < T' < T\). By the last assertion of Theorem 1, we may assume, by choosing a subsequence if necessary, that \(u_j \in X_{[0, T')}^k\) for all \(j\), and \(u_j \to u\) in \(X_{[0, T')}^k\). Since each \(u_j\) is then bounded in \(H_{m}^k\) on \([0, T']\), it is bounded in \(L^q_i \cap L^{q_2}\) on that interval, by the assumption that \(H^k\) is embedded continuously in \(L^q_i \cap L^{q_2}\). Thus we have \(T_j > T'\) for every \(j\). Now let \(0 \leq t \leq T'\). We first derive an estimate for \(x(t)F(u_j(t))\). By assumption, \(F\) maps \(H_{m}^k \cap W_{m}^{m,p}\) and \(H^m \cap W_{m}^{m,p}\) into \(W_{m}^{m,p'}\) and \(W_{m}^{m,p'}\), respectively. Furthermore, we may write \(F(u_j(t)) = g(|u_j(t)|)u_j(t)\) with some mapping \(g(|\cdot|)\) defined on \(H^m \cap W_{m}^{m,p}\). We also notice that \(\exp(-it|x|^2/4t) \in H^m \cap W_{m}^{m,p}\), since \(\phi \in H_{m}^k \cap W_{m}^{m,p}\) implies \(x^a \partial^\phi \psi \in L^2 \cap L^p\) whenever \(|x| + |\beta| \leq m\), as mentioned above (3.5). Then for any multi-index \(\alpha\) with \(|\alpha| \leq m\) we have

\[
x^a(t)F(u_j(t)) = \exp\left(i \cdot \frac{|x|^2}{4t}\right)(2it)^\alpha \left\{ g\left(|u_j(t)|\right) \exp\left(-i \cdot \frac{|x|^2}{4t}\right) u_j(t) \right\}
= \exp\left(i \cdot \frac{|x|^2}{4t}\right)(2it)^\alpha \left\{ g\left(\exp\left(-i \cdot \frac{|x|^2}{4t}\right) u_j(t)\right) \right\} \exp\left(-i \cdot \frac{|x|^2}{4t}\right) u_j(t)
= \exp\left(i \cdot \frac{|x|^2}{4t}\right)(2it)^\alpha F\left(\exp\left(-i \cdot \frac{|x|^2}{4t}\right) u_j(t)\right).
\]

This result together with (2.6) gives

\[
\sum_{|\alpha| = m} \|x^a(t)F(u_j(t))\|_{p'}
= (2it)^m \sum_{|\alpha| = m} \|\partial^\alpha F\left(\exp\left(-i \cdot \frac{|x|^2}{4t}\right) u_j(t)\right)\|_{p'}
\leq c(2it)^m \left\{1 + \sum_{i=1,2} \left\|\exp\left(-i \cdot \frac{|x|^2}{4t}\right) u_j(t)\right\|_{q_i}\right\} \lambda^{1-}
\times \sum_{|\alpha| = m} \|\partial^\alpha \left(\exp\left(-i \cdot \frac{|x|^2}{4t}\right) u_j(t)\right)\|_{p'}
= c\left(1 + \sum_{i=1,2} \|u_j(t)\|_{q_i}\right)^{\lambda-1} \sum_{|\alpha| = m} \|x^a(t)u_j(t)\|_p.
\]

Since the \(u_j\) are uniformly bounded in \(C([0, T'); L^{q_i} \cap L^{q_2})\), we obtain

\[
(3.18) \quad \sum_{|\alpha| = m} \|x^a(t)F(u_j(t))\|_{p'} \leq c \sum_{|\alpha| = m} \|x^a(t)u_j(t)\|_p.
\]

We shall derive a bound on \(x^a(t)u_j(t)\) with the help of this inequality. We first consider the case \(p = 2\). The equation, (3.17), (3.6) and the unitarity of \(S(t)\) in \(L^2\) give
\[
\sum_{|\alpha|=m} \| x^\alpha(t)u_j(t) \|_2 \\
\leq \sum_{|\alpha|=m} \| x^\alpha(t)S(t)u_{0j} \|_2 + \int_0^t \sum_{|\alpha|=m} \| x^\alpha(t)S(t-s)F(u_j(s)) \|_2 \, ds \\
\leq \sum_{|\alpha|=m} \| S(t)x^\alpha u_{0j} \|_2 + \int_0^t \sum_{|\alpha|=m} \| S(t-s)x^\alpha(s)F(u_j(s)) \|_2 \, ds \\
\leq \sum_{|\alpha|=m} \| x^\alpha u_{0j} \|_2 + \int_0^t \sum_{|\alpha|=m} \| x^\alpha(s)F(u_j(s)) \|_2 \, ds.
\]

Using (3.18) we obtain
\[
\sum_{|\alpha|=m} \| x^\alpha(t)u_j(t) \|_2 \\
\leq \sum_{|\alpha|=m} \| x^\alpha u_{0j} \|_2 + c \int_0^t \sum_{|\alpha|=m} \| x^\alpha(s)u_j(s) \|_2 \, ds.
\]

Gronwall's inequality then implies
\[(3.19) \quad \sum_{|\alpha|=m} \| x^\alpha(t)u_j(t) \|_2 \leq \sum_{|\alpha|=m} \| x^\alpha u_{0j} \|_2 e^{ct}.
\]

Next let \(2 < p < \infty\). Recalling (3.1), we have
\[
\sum_{|\alpha|=m} \| x^\alpha(t)u_j(t) \|_p \\
\leq \sum_{|\alpha|=m} \| S(t)x^\alpha u_{0j} \|_p + \int_0^t \sum_{|\alpha|=m} \| S(t-s)x^\alpha(s)F(u_j(s)) \|_p \, ds \\
\leq \sum_{|\alpha|=m} \| S(t)x^\alpha u_{0j} \|_p + c \int_0^t (t-s)^{-\delta(p)} \sum_{|\alpha|=m} \| x^\alpha(s)F(u_j(s)) \|_p \, ds.
\]

This together with (3.18) yields
\[
\sum_{|\alpha|=m} \| x^\alpha(t)u_j(t) \|_p \\
\leq \sum_{|\alpha|=m} \| S(t)x^\alpha u_{0j} \|_p + c \int_0^t (t-s)^{-\delta(p)} \sum_{|\alpha|=m} \| x^\alpha(s)u_j(s) \|_p \, ds.
\]

Using (3.2) we obtain
\[
\left[ \int_0^t \left\{ \sum_{|\alpha|=m} \| x^\alpha(\tau)u_j(\tau) \|_p \right\}^q \, d\tau \right]^{1/q} \\
\leq c \sum_{|\alpha|=m} \| x^\alpha u_{0j} \|_2 \\
+ c \left[ \int_0^t \right. \left( t-s \right)^{-\delta(p)} \sum_{|\alpha|=m} \| x^\alpha(s)u_j(s) \|_p \, ds \left. \right\}^{q} \left[ \right]^{1/q}.
\]

An argument similar to that which led from (3.15) to (3.16) then shows that
\[
\int_0^t \left\{ \sum_{|\alpha|=m} \| x^\alpha (\tau) u_j(\tau) \|_p \right\}^q d\tau \\
\leq c \left\{ \sum_{|\alpha|=m} \| x^\alpha u_{0j} \|_2 \right\}^q + c \int_0^t d\tau \int_0^\tau \left\{ \sum_{|\alpha|=m} \| x^\alpha (s) u_j(s) \|_p \right\}^q ds.
\]

Thus by Gronwall's inequality we find that

\[(3.20) \quad \int_0^t \left\{ \sum_{|\alpha|=m} \| x^\alpha (\tau) u_j(\tau) \|_p \right\}^q d\tau \leq c \left\{ \sum_{|\alpha|=m} \| x^\alpha u_{0j} \|_2 \right\}^q e^{ct}.
\]

We shall show that this result implies

\[(3.21) \quad \sum_{|\alpha|=m} \| x^\alpha (t) u_j(t) \|_2 \leq (1 + ct^{(q-2)/qe^{ct}}) \sum_{|\alpha|=m} \| x^\alpha u_{0j} \|_2.
\]

Now, again from the equation \( u = Gu, (3.17) \) and (3.6), we have

\[\sum_{|\alpha|=m} \| x^\alpha (t) u_j(t) \|_2 \leq \sum_{|\alpha|=m} \| x^\alpha u_{0j} \|_2 + \sum_{|\alpha|=m} \left\| \int_0^t S(t-s) x^\alpha(s) F(u_j(s)) ds \right\|_2.
\]

Using (3.4), (3.18) and (3.20) we have for the second term on the right hand side,

\[\sum_{|\alpha|=m} \left\| \int_0^t S(t-s) x^\alpha(s) F(u_j(s)) ds \right\|_2 \leq c \sum_{|\alpha|=m} \left\{ \int_0^t \| x^\alpha(s) F(u_j(s)) \|_p^{q'} ds \right\}^{1/q'} \leq c \sum_{|\alpha|=m} \left\{ \int_0^t \| x^\alpha(s) u_j(s) \|_p^{q'} ds \right\}^{1/q'} \leq ct^{(q-2)/qe^{ct}} \sum_{|\alpha|=m} \| x^\alpha u_{0j} \|_2.
\]

Thus we obtain (3.21).

We now let \( j \to \infty \). Then \( x^\alpha u_{0j} \to x^\alpha u_0 \) strongly in \( L^2 \) whenever \( |\alpha| \leq m \), and we know that \( u_j(t) \to u(t) \) strongly in \( L^2 \) uniformly in \( t \) on \([0, T']\). Furthermore, the operator \( x^\alpha(t) \) is self-adjoint in \( L^2 \) and in particular weakly closed in \( L^2 \) for any \( \alpha \) and \( t \). Therefore from (3.19) and (3.21) it follows that, whenever \( |\alpha| = m \) and \( 0 < t < T' \), \( u(t) \) is in the domain in \( L^2 \) of \( x^\alpha(t) \) with

\[(3.22) \quad \sum_{|\alpha|=m} \| x^\alpha(t) u(t) \|_2 \leq (1 + ct^{(q-2)/qe^{ct}}) \sum_{|\alpha|=m} \| x^\alpha u_0 \|_2,
\]

and \( x^\alpha(t) u_j(t) \to x^\alpha(t) u(t) \) weakly in \( L^2 \) uniformly in \( t \). In particular, \( x^\alpha(\cdot) u(\cdot) \) is weakly continuous in \( L^2 \) on \([0, T']\) whenever \( |\alpha| = m \), and this fact implies that
Hence, Schrödinger's equation also completes the problem following the same field of continuous unbounded evolution. To prove this result, we need to use a Cauchy-like result for the Cauchy problem [(3.22)] for approximating data \( u_{0j}^{*} \in H_{m}^{m} \). Since every \( u_{0j}^{*} \) is continuous in \( L^{2} \) at \( t=0 \) when \( |\alpha|=m \). To prove the same result at any \( t_{0} \) in \([0, T']\), consider the solution \( u^{*} \) of the Cauchy problem for (2.1) for \( t \geq t_{0} \) with initial value \( u(t_{0}) \). In this case we choose as approximating data those \( u_{0j}^{*} \in H_{m}^{m}, j=1,2,... \), such that \( u_{0j}^{*} \rightarrow u(t_{0}) \) in \( H_{k}^{k} \) and \( x^{*}u_{0j}^{*} \rightarrow x^{*}u(t_{0})u(t_{0}) \) for all \( \alpha \) with \( |\alpha|=m \). Then by replacing \( x^{*}u_{0j}^{*} \) by \( x^{*}(t_{0})u_{0j}^{*} \) everywhere in the other proof, we conclude that \( x^{*}(\cdot)u^{*}(\cdot) \) is continuous in \( L^{2} \) at \( t=t_{0} \) whenever \( |\alpha|=m \). But \( u^{*}(t) \) coincides with \( u(t) \) for all \( t \) in \([t_{0}, T']\) by the uniqueness result. Hence \( x^{*}(\cdot)u(\cdot) \) is right continuous in \( L^{2} \) at \( t=t_{0} \) for every such \( \alpha \). Since the equation (2.1) is reversible in \( t \) in an obvious sense, \( x^{*}(\cdot)u(\cdot) \) must be also left continuous at \( t=t_{0} \). Thus \( x^{*}(\cdot)u(\cdot) \in C([0, T']; L^{2}) \) whenever \( |\alpha|=m \). This completes the proof of the theorem since \( T' \) is an arbitrary number in \((0, T)\).

4. Applications

We shall apply our main results to the following nonlinear Schrödinger equations:

\[
(H) \quad i \frac{\partial u}{\partial t} = -\Delta u + (\mathcal{V}^{*}|u|^{2})u,
\]

where \( \mathcal{V} \in L^{r_1} + L^{r_2} \) with \( n/2 < r_1 \leq r_2 \leq \infty \), \( r_1, r_2 > 1 \), and

\[
(\mathcal{V}^{*}|u|^{2})(t, x) = \int_{\mathbb{R}^{n}} \mathcal{V}(x-y)|u(t, y)|^{2}dy;
\]

\[
(P) \quad i \frac{\partial u}{\partial t} = -\Delta u + g|u|^{\ell-1}u,
\]

where \( g \in \mathbb{R} \) and \( \ell \in \mathbb{N} \). The equation (H) is the classical approximation to the field equation for a non-relativistic quantum mechanical many-body system, interacting through a two-body potential \( \mathcal{V} \). Equations of the form (P) are also widely used in several domains of physics. For example, the case \( g < 0 \), \( \ell = 3 \) and \( n=2 \) arises in modeling the propagation of a laser beam through a medium (see, e.g., [5] for a blow-up result). To study (H) and (P) we recall the Gagliardo-Nirenberg-Sobolev inequality:

\[
\|\partial^{j} \phi\|_{r} \leq c \|\partial^{m} \phi\|_{p}^{a} \|\phi\|_{q}^{1-a},
\]

where \( 1/r - j/n = a(1/p - m/n) + (1-a)/q \), \( j/m \leq a \leq 1 \) (if \( m - n/p \) is a nonnegative integer, only \( a < 1 \) is allowed), and
\[ \| \partial^k \phi \|_s = \{ \sum_{|\alpha|=m} \| \partial^\alpha \phi \|_s \}^{1/s}. \]

This yields
\[ (4.1) \quad \| \partial^j \phi \|_r \leq c \| \partial^m \phi \|_p \| \phi \|_q^{1-j/m}, \]
where \( 1/r = (j/m)/p + (1-j/m)/q \). We will also use the notation
\[ \| x^k \phi \|_s = \{ \sum_{|\alpha|=k} \| x^\alpha \phi \|_s \}^{1/s}. \]

For (H) we have the following result.

**Theorem 3.** Let \( k \) and \( m \) be given nonnegative integers with \( k \geq m \) and \( k \geq 1 \). Let \( r_1 \) and \( r_2 \) be in the range indicated in (H). Choose any \( s_1 \) with \( 1 \leq s_1 \leq \infty \) and
\[ 1/2 - \frac{k}{n} < \frac{1}{s_1} < \frac{1}{2} - \frac{1}{2r_1} + \frac{4}{r_2}, \]
and let \( 1/p = 1 - 1/2r_1 - 1/s_1 \). Let \( s_2 \) be such that \( 1/s_2 = 1/s_1 + 1/r_1 - 1/r_2 \). With those \( k, m, p \) and \( V = L^{s_1} \cap L^{s_2} \), the conclusions of Theorem 1 are valid for (H). If we replace the condition \( k \geq m \) by \( k < m \) in the above assumptions, then the conclusions of Theorem 2 are valid for (H), with \( q_1 = s_1 \) and \( q_2 = s_2 \).

**Proof.** We have \( 2 \leq p, 1/2 - 1/n < 1/p \) and \( H^k \subset L^{s_1} \cap L^{s_2} \). By assumption, \( \mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 \) with \( \mathcal{V}_1 \in L^{s_1} \) and \( \mathcal{V}_2 \in L^{s_2} \). Then (4.1) implies that for \( \ell = 1, 2, j \geq 1 \) and \( \phi \in \mathcal{S} \),
\[ \| \partial^j \{ \mathcal{V}_1 (\phi_1 \phi_2) \} \phi_3 \|_p \leq \| \mathcal{V}_1 \|_{r_{1j_1}} \sum_{j_1 = j}^3 \| \partial^j \phi_1 \|_{((r_1/j)/p + (1-j_1/j)/s_1)}^{1-s_1} \]
\[ \leq c \| \mathcal{V}_1 \|_{r_{1j_1}} \sum_{j_1 = j}^3 \| \partial^j \phi_1 \|_p \| \phi_1 \|_s^{1-j_1/j}. \]
We also have for \( \ell = 1, 2, j \geq 0 \) and \( \phi \in \mathcal{S} \),
\[ \| x^j \{ \mathcal{V}_1 (\phi_1 \phi_2) \} \phi_3 \|_p \leq \| \mathcal{V}_1 \|_{r_1} \| \phi_1 \|_s \| \phi_2 \|_s \| x^j \phi_3 \|_p. \]

Notice that the assumptions also imply \( W^{k_1} \subset L^{s_1} \), \( \ell = 1, 2 \). Thus \( F(\phi) = (\mathcal{V} \phi)^2 \) satisfies the required conditions.

For (P) we have the following.

**Theorem 4.** Let \( k \) and \( m \) be given nonnegative integers with \( k \geq m \) and \( k \geq 1 \). Assume \( \ell \) is an odd number with \( 1/(\ell^2 - 1) > (n-2k)/8 \). Choose any \( q \) with \( \ell+1 \leq q \leq \infty \) and
\[
\frac{1}{2} - \frac{k}{n} < \frac{1}{q} < \frac{4}{n(\ell^2 - 1)},
\]

and let \(1/p = 1/2 - (\ell - 1)/2q\). With those \(k, m, \ell, p\) and \(V = L^q\), the conclusions of Theorem 1 are valid for (P). If we replace the condition \(k \geq m\) by \(k < m\) in the above assumptions, the conclusions of Theorem 2 are valid for (P), with \(q_1 = q_2 = q\).

Proof. We have \(2 \leq p, 1/2 - 2/n(\ell + 1) < 1/p\) and \(H^k \subset L^q\). (4.1) implies that for any \(j \geq 1\) and \(\phi \in \mathcal{S}\),

\[
\| \partial j \prod_{i=1}^k \phi_i \|_{p'} \\
\leq c \sum_{j_1 + \ldots + j_k = j} \prod_{i=1}^k \| \partial^{j_i} \phi_i \|_{p^{j_i}} \| \phi_i \|_{q}^{1 - j_i/j},
\]

We also have for any \(j\) and \(\phi \in \mathcal{S}\),

\[
\| x^j \prod_{i=1}^k \phi_i \|_{p'} \leq \prod_{i=1}^{k-1} \| \phi_i \|_{q} \| x^j \phi_i \|_{p}.
\]

The assumptions imply \(W^{k,p} \subset L^q\). Hence \(F(\phi) = g|\phi|^{j-1}\phi\) satisfies the required conditions.

In Theorems 3 and 4, \((s_1, s_2)\) and \(q\) can be chosen uniformly for large values of \(m\). This means that in (H) and (P) satisfying our hypotheses, any solution with compactly supported \(H^k\)-data can become \(C^\infty\). Under some additional conditions on their nonlinearities, (H) and (P) generate global \(H^k\)-solutions for all \(H^k\)-data (and so the solutions of Theorems 3 and 4 with \(H^k_m\)-data extend to global ones), provided \(k\) is appropriately chosen \([3]\) \([6]\).

In a forthcoming paper, it will be shown that the condition \(1/2 \leq 1/p < 1/2 - 2/n(\ell + 1)\) in Theorems 1 and 2 can be weakened.

References


nuna adreso:
Nakao Hayashi
Masayoshi Tsutsumi
Department of Applied Physics
Waseda University
Tokyo 160
Japan

Kuniaki Nakamitsu
Faculty of Science and Engineering
Tokyo Denki University
Saitama 350–03
Japan

(Ricevita la 15-an de oktobro, 1986)