Funkcialaj Ekvacioj, **31** (1988) 227–231

Set-Valued Solutions of the Pexider Functional Equation

By

Kazimierz NIKODEM
(Silesian University, Poland)

In the present paper we characterize set-valued solutions of the Pexider functional equation

$$(1) \qquad F(x+y) = G(x) + H(y)$$

with three unknown functions F, G and H. It generalizes a classical theorem giving the solutions of the Pexider equation by means of additive functions (cf. [1]).

Let Y be a topological vector space satisfying the T_0 separation axiom. For real numbers s, t and sets A, $B\subset Y$ we put $sA+tB:=\{y\in Y;\ y=sa+tb,\ a\in A,\ b\in B\}$. We assume that the space 2^Y of all subsets of Y is endowed with the Hausdorff topology (cf. [4]). In this topology the sets $N_W(A):=\{B\subset Y;\ B\subset A+W,\ A\subset B+W\}$, where W runs a base of neighbourhoods of zero in Y, form a base of neighbourhoods of a set $A\subset Y$. The symbol $A_n\to A$ means that the sequence $(A_n)_{n\in N}$ is converging to A in the Hausdorff topology. A set-valued function (s.v. function) $F\colon X\to 2^Y$, where X is a semigroup, is said to be additive if it satisfies the Cauchy functional equation $F(x_1+x_2)=F(x_1)+F(x_2)$, $x_1, x_2\in X$. By $CC(Y)$ we denote the family of all non-empty, convex and compact subsets of Y. The symbols $\boldsymbol{R}$, $\boldsymbol{Q}$, $\boldsymbol{N}$ denote the sets of all real, rational and positive integer numbers, respectively.

Theorem 1. *Assume that* $(X, +)$ *is an abelian semigroup with zero and* Y *is a* T_0 *topological vector space. If s.v. functions* $F\colon X\to CC(Y)$, $G\colon X\to CC(Y)$ *and* $H\colon X\to CC(Y)$ *satisfy the equation* (1), *then there exist an additive s.v. function* $F_0\colon X\to CC(Y)$ *and sets* $A, B\in CC(Y)$ *such that*

$$(2) \qquad F(x) = F_0(x) + A + B, \quad G(x) = F_0(x) + A \quad \text{and} \quad H(x) = F_0(x) + B$$

for all $x\in X$.

In the proof of this theorem we will use some facts which we list here as lemmas.

Lemma 1 (Rådström [3]). *Let* A, B *be subsets of* Y *and assume that* B *is* *closed and convex. If there exists a bounded and non-empty set $C \subset Y$ such that $A+C \subset B+C$, then $A \subset B$.*

This cancellation law is formulated in [3] for a real normed space, but the proof given there holds in topological vector space, too.

Lemma 2. *If $(A_n)_{n\in N}$ and $(B_n)_{n\in N}$ are decreasing sequences of compact subsets of Y, then $\bigcap_{n\in N}(A_n+B_n)=\bigcap_{n\in N}A_n+\bigcap_{n\in N}B_n$.*

Lemma 3. *If $(A_n)_{n\in N}$ is a decreasing sequence of compact subsets of Y, then $A_n \to \bigcap_{n\in N} A_n$.*

Lemma 4. *If A is a bounded subset of Y and $(s_n)_{n\in N}$ is a real sequence converging to an $s\in \boldsymbol{R}$, then $s_nA\to sA$.*

Lemma 5. *If $A_n\to A$ and $B_n\to B$, then $A_n+B_n\to A+B$.*

Lemma 6. *If $A_n\to A$ and $A_n\to B$, then* $\operatorname{cl} A=\operatorname{cl} B$.

Lemmas 2–6 are rather known and can be easily verified. Short proofs of some of them can be found in [2].

Proof of Theorem 1. Assume first that $0\in G(0)$ and $0\in H(0)$. Then, for every $x\in X$ we have

$$F(2x) = F(x) + H(x) \subset G(x) + H(0) + H(x) + G(0) = F(x) + F(x) = 2F(x),$$

which implies that the sequence $(2^{-n}F(2^nx))_{n\in N}$ is decreasing. Put $F_0(x):=\bigcap_{n\in N}2^{-n}F(2^nx)$, $x\in X$. It is clear that $F_0(x)\in CC(Y)$ for all $x\in X$. Using three times the equation (1) we get

$$\begin{aligned} G(2x) + H(0) &= F(2x) = G(x) + H(x) \subset G(x) + H(x) + G(0) \\ &= G(x) + G(x) + H(0) = 2G(x) + H(0). \end{aligned}$$

In view of Lemma 1 this implies that $G(2x)\subset 2G(x)$, and consequently the sequence $(2^{-n}G(2^nx))_{n\in N}$ is decreasing.
Applying Lemma 2 and the equality $F(2^nx)=G(2^nx)+H(0)$, $n\in \boldsymbol{N}$, we obtain

$$F_0(x) = \bigcap_{n\in N} 2^{-n}F(2^nx) = \bigcap_{n\in N} 2^{-n}G(2^nx) + \bigcap_{n\in N} 2^{-n}H(0).$$

But $\bigcap_{n\in N}2^{-n}H(0)=\{0\}$, because the set $H(0)$ is bounded. Therefore $F_0(x)=\bigcap_{n\in N}2^{-n}G(2^nx)$ for all $x\in X$. In an analogous way we show that the sequence $(2^{-n}H(2^nx))_{n\in N}$ is decreasing and $F_0(x)=\bigcap_{n\in N}2^{-n}H(2^nx)$ for all $x\in X$. Hence, using once more Lemma 2, we get

$$F_0(x_1+x_2) = \bigcap_{n\in N} 2^{-n}F(2^n x_1+2^n x_2) = \bigcap_{n\in N} 2^{-n}[G(2^n x_1)+H(2^n x_2)]$$
$$= \bigcap_{n\in N} 2^{-n}G(2^n x_1) + \bigcap_{n\in N} 2^{-n}H(2^n x_2) = F_0(x_1) + F_0(x_2),\ x_1,\ x_2\in X,$$

which means that the s.v. function F_0 is additive.

Now observe that

$$(3)\qquad F(nx) + (n-1)H(0) = F(x) + (n-1)H(x)$$

for all $x\in X$ and $n\in N$. Indeed, for $n=1$ the equality is trivial. Assume that it holds for a natural number k. Then, in virtue of (1), we obtain

$$F((k+1)x) + kH(0) = G(kx) + H(x) + (k-1)H(0) + H(0)$$
$$= F(kx) + H(x) + (k-1)H(0) = F(x) + (k-1)H(x) + H(x) = F(x) + kH(x),$$

which proves that (3) holds for $n=k+1$. Thus, by induction, it holds for all $n\in N$. In particular we have

$$F(2^n x) + (2^n-1)H(0) = F(x) + (2^n-1)H(x),$$

whence

$$2^{-n}F(2^n x) + (1-2^{-n})H(0) = 2^{-n}F(x) + (1-2^{-n})H(x).$$

In virtue of Lemma 3, $2^{-n}F(2^n x)\to\bigcap_{n\in N} 2^{-n}F(2^n x)=F_0(x)$. On the other hand, by Lemma 4, $(1-2^{-n})H(0)\to H(0)$, $2^{-n}F(x)\to\{0\}$ and $(1-2^{-n})H(x)\to H(x)$. Therefore, using Lemma 5 and Lemma 6, we obtain $\mathrm{cl}\,[F_0(x)+H(0)]=\mathrm{cl}\,H(x)$, whence $H(x)=F_0(x)+H(0)$ for all $x\in X$. In an analogous way one can prove that $G(x)=F_0(x)+G(0)$, $x\in X$. Let $A:=G(0)$ and $B:=H(0)$. Then $G(x)=F_0(x)+A$ and $H(x)=F_0(x)+B$ for all $x\in X$. Moreover, $F(x)=G(x)+H(0)=F_0(x)+A+B$, $x\in X$. This finishes our proof in the case where $0\in G(0)$ and $0\in H(0)$. In the opposite case, fix arbitrarily points $a\in G(0)$ and $b\in H(0)$ and consider the s.v. functions F_1, G_1, H_1: $X\to CC(Y)$ defined by $F_1(x):=F(x)-a-b$, $G_1(x):=G(x)-a$ and $H_1(x):=H(x)-b$, $x\in X$. These s.v. functions satisfy the equation (1) and moreover $0\in G_1(0)$ and $0\in H_1(0)$. Therefore, by what we have proved previously, they are of the form (2). Returning to the s.v. functions F, G and H we see that they are of the form (2), too. This completes the proof.

Remark. If X is a group, then the s.v. function F_0 occuring in the assertion of Theorem 1 is in fact single valued. Indeed, the set $F_0(0)$ is bounded and $F_0(0)+F_0(0)=F_0(0)$; so $F_0(0)=\{0\}$. Hence, for arbitrarily fixed $x\in X$ we get $F_0(x)+F_0(-x)=F_0(0)=\{0\}$, which implies that $F_0(x)$ is a one-point set.

In [4] Rådström has proved that a s.v. function F_0: $[0, \infty)\to CC(Y)$, where Y

is a locally convex Hausdorff space, is additive if and only if there exist an additive function f: $[0, \infty) \to Y$ and a set $K \in CC(Y)$ such that $F_0(x) = f(x) + xK$, $x \in [0, \infty)$. Applying this result to Theorem 1 we obtain the following

Theorem 2. *Let Y be a locally convex Hausdorff space. S.v. functions F: $[0, \infty) \to CC(Y)$, G: $[0, \infty) \to CC(Y)$ and H: $[0, \infty) \to CC(Y)$ satisfy the equation (1) if and only if there exist an additive function f: $[0, \infty) \to Y$ and sets $K, A, B \in CC(Y)$ such that*

$$F(x) = f(x) + xK + A + B, \quad G(x) = f(x) + xK + A \quad \text{and} \quad H(x) = f(x) + xK + B \tag{4}$$

for all $x \in [0, \infty)$.

Finally, we give some examples showing that the assertions of our theorems do not hold without the assumption that the values of F, G and H are convex and compact. Consider the triples (F_i, G_i, H_i) of s.v. functions from $[0, \infty)$ into $2^{\boldsymbol{R}}$ defined by the formulas:

1. $F_1(x) := [0, 2], \quad G_1(x) = H_1(x) := \begin{cases} C \ , & x \in [0, \infty) \cap \boldsymbol{Q} \\ [0, 1], & x \in [0, \infty) \setminus \boldsymbol{Q} \end{cases},$

where C denotes the Cantor set;

2. $F_2(x) = G_2(x) := \boldsymbol{R}, \quad H_2(x) := \begin{cases} \{0\}, & x \in [0, \infty) \cap \boldsymbol{Q} \\ \boldsymbol{R} \ , & x \in [0, \infty) \setminus \boldsymbol{Q} \end{cases};$

3. $F_3(x) = G_3(x) := (0, x+1), \quad H_3(x) := \begin{cases} [0, x], & x \in [0, \infty) \cap \boldsymbol{Q} \\ (0, x) \ , & x \in [0, \infty) \setminus \boldsymbol{Q} \end{cases}.$

It is easily seen that these triples satisfy the equation (1) but they are not of the form (4) (or of the form (2)).

References

[1] Kuczma, M., *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality.* PWN-Uniwersytet Slaski, Warszawa-Kraków-Katowice, 1985.

[2] Nikodem, K., On Jensen's functional equation for set-valued functions, Radovi Mat., **3** (1987), 23–33.

[3] Rådström, H., An embedding theorem for space of convex sets, Proc. Amer. Math. Soc., **3** (1952), 165–169.

[4] Rådström, H., One-parameter semigroups of subsets of a real linear space, Ark. Mat., **4** (1960), 87–97.

nuna adreso:
Institute of Mathematics
Silesian University
40–007 Katowice
Poland

(Ricevita la 17-an de julio, 1986)
(Reviziita la 13-an de majo, 1987)