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On the Global Center of Generalized Liénard Equation and its Application to Stability Problems II

By

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In this note we consider the Liénard system

$$\begin{aligned} x' &= y - F(x) \\ y' &= -g(x) \end{aligned} \tag{1}$$

where F and g are continuous functions on $\boldsymbol{R}^1$ satisfying

$$F(0)=0 \quad \text{and} \quad xg(x)>0 \qquad \text{for} \quad x \neq 0.$$

We assume the regularity for $F(x)$ and $g(x)$ which ensures the existence of unique solution to the initial value problem.

Recently, the authors [2] have developed a new method to investigate the asymptotic behavior of the solutions of (1), and gave theorems on the oscillation, the stability and the boundedness. This note is a continuation of [2]. We will present the sufficient conditions under which the zero solution of (1) is asymptotically stable and the solutions of (1) are ultimately bounded. For these definitions, refer to [3] and use the same notations as in [2] to avoid overlapping.

Before beginning to state our results, let us introduce the Poincaré-Bendixson theorem. Consider the two-dimensional autonomous system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{2}$$

where f and g are continuous on $\boldsymbol{R}^2$, and we assume that for any $(x_0, y_0) \in \boldsymbol{R}^2$ there exists a unique solution $(x(t), y(t))$ of (2) such that $(x(0), y(0))=(x_0, y_0)$. If the solution $(x(t), y(t))$ is defined for all $t \geqq 0$, the set $T^+_{(2)}(x_0, y_0)=\{(x(t), y(t)) \mid t \geqq 0\}$ is called the semi-orbit of the solution of (2). Let $L(T^+_{(2)})$ be the positive limit set of the solution $(x(t), y(t))$. The following theorem is well known (see, for example, [1]).

Poincaré-Bendixson Theorem. *Let $(x(t), y(t))$ be a solution of (2), and suppose that $T^+_{(2)}$ is bounded. Then*

(i) *$L(T^{+}_{(2)})$ is a nonempty, closed and connected set.*
(ii) *If $L(T^{+}_{(2)})$ contains no critical point, then $L(T^{+}_{(2)})$ is a periodic orbit.*
(iii) *If $L(T^{+}_{(2)})$ contains a critical point, then $L(T^{+}_{(2)})$ consists of critical points.*

Now we give a result on the asymptotic stability of the zero solution of (1).

Theorem 1. *Suppose that $-F(-x)$ and $-g(-x)$ satisfy* (A_3) *for $x \geqq 0$, and for some $k>0$*

$$(A_{14}) \qquad F(G^{-1}(-w)) < F(G^{-1}(w)) \qquad \text{for} \quad 0 < w \leqq k.$$

Then the zero solution of (1) *is uniformly asymptotically stable.*

The uniform stability of the zero solution of (1) follows from [2, Theorem 6.1]. By the uniqueness of solutions of (1), we can apply the Poincaré-Bendixson Theorem to (1). Hence there exists a neighborhood N of the origin such that for any solution $(x(t), y(t))$ of (1) with $(x(0), y(0))=(x_0, y_0) \in N$, either $L(T^{+}_{(1)})$ is a periodic orbit or $L(T^{+}_{(1)})$ consists of the origin. Therefore the proof of Theorem 1 is completed by showing that the system (1) has no non-trivial periodic orbit. In order to prove this we give a simple lemma (cf. [2], Lemma 6.1).

Lemma 1. *Let $H(x, y)$ and $\tilde{H}(x, y)$ be continuous on $[a, b] \times \boldsymbol{R}^1$. Suppose that solutions of scalar equations*

$$(3) \qquad \frac{dy}{dx} = H(x, y)$$

and

$$(4) \qquad \frac{dy}{dx} = \tilde{H}(x, y)$$

are unique to the right and

$$H(x, y) < \tilde{H}(x, y) \quad \text{for all} \quad (x, y) \in [a, b] \times \boldsymbol{R}^1.$$

Let $y(x)$ and $\tilde{y}(x)$ be solutions of (3) *and* (4) *on $[a, b]$ satisfying $y(a) \leqq \tilde{y}(a)$, respectively. Then $y(x) < \tilde{y}(x)$ for all $x \in (a, b]$.*

Proof of Theorem 1. Define the function $\tilde{F}(x)$ by $\tilde{F}(x)=F(x)$ for $x \leqq 0$ and $\tilde{F}(G^{-1}(w))=F(G^{-1}(-w))$ for $w \geqq 0$, and consider the system

$$(5) \qquad \begin{aligned} x' &= y - \tilde{F}(x) \\ y' &= -g(x). \end{aligned}$$

Then the origin is a local center of (5), and as in the proof of [2, Theorem 6.1] we can show that there exists a nieghborhood N of the origin such that for any $(x_0, y_0) \in N$ the solution $(x(t), y(t))$ of (1), with $(x(0), y(0))=(x_0, y_0)$, remains in the closed region $R_{(5)}(x_0, y_0)$ enclosed by the orbit $T_{(5)}(x_0, y_0)$.

Suppose that there exists a non-trivial periodic orbit $T_{(1)}$ which is contained in N. Choose a point (x_0, y_0) on $T_{(1)}$ such that $x_0>0$ and $y_0>F(x_0)$ and let $(x_1, F(x_1))$ be the intersecting point of $T_{(1)}(x_0, y_0)$ and the characteristic curve. Let $(x(t), y(t))$ and $(\tilde{x}(t), \tilde{y}(t))$ be the solutions of (1) and (5) satisfying $(x(0), y(0))=(\tilde{x}(0), \tilde{y}(0))=(x_0, y_0)$, respectively. Then as in the proof of [2, Lemma 6.2], these define the solutions $y(x)$ and $\tilde{y}(x)$ on $[x_0, x_1]$ of the scalar equations

$$\frac{dy}{dx} = \frac{g(x)}{F(x)-y} \quad \text{and} \quad \frac{dy}{dx} = \frac{g(x)}{\tilde{F}(x)-y}$$

satisfying $y(x_0)=\tilde{y}(x_0)$, respectively. Hence by (A_{14}) and Lemma 1, we have

$$y(x) < \tilde{y}(x) \quad \text{for all} \quad x \in (x_0, x_1].$$

Then $(x_0, y_0) \notin R_{(5)}(x_1, F(x_1))$ and $(x(t), y(t))$ remains in the region $R_{(5)}(x_1, F(x_1))$, which contradicts that the solution $(x(t), y(t))$ is periodic. The proof of Theorem 1 is now complete.

Corollary 1. *Suppose that* $F(-x)\leqq 0$ *for* $0\leqq x\leqq k$, $F(x)$ *and* $g(x)$ *satisfy* (A_3) *for* $0\leqq x\leqq k$, *and* (A_{14}) *hold. Then the zero solution of* (1) *is uniformly asymptotically stable.*

Proof. The proof is reduced to that of Theorem 1 by the transformation $(x, y)\to(-x, -y)$.

We next give our results on the utimate boundedness of the solutions of (1).

Theorem 2. *Suppose that* $-F(-x)$ *and* $-g(-x)$ *satisfy* (A_4) *and* $((A_5)$ *or* $(A_6))$ *for* $x\geqq 0$, (A_{13}) *holds and there exists* $K>0$ *such that*

$$(A_{15}) \qquad F(G^{-1}(-w)) < F(G^{-1}(w)) \quad \text{for all} \quad w \geqq K.$$

Then the solutions of (1) *are uniformly ultimately bounded.*

Proof. By [2, Theorem 6.2] any solution of (1) is uniformly bounded, and hence it follows from the Poincaré-Bendixson Theorem that the positive limit set $L(T^+_{(1)})$ of any solution of (1) is either a periodic orbit or the origin. Therefore it suffices to show that for any $(x_0, y_0) \in \boldsymbol{R}^2$ with sufficiently large $x_0^2+y_0^2$, the solution $(x(t), y(t))$ of (1) such that $(x(0), y(0))=(x_0, y_0)$ is not periodic.

Define the function $\tilde{F}(x)$ by $\tilde{F}(x)=F(x)$ for $x\leqq 0$ and $\tilde{F}(G^{-1}(w))=\tilde{F}(G^{-1}(-w))$ for $w\geqq 0$, and consider the system

$$(6)\qquad \begin{aligned} x' &= y - \tilde{F}(x) \\ y' &= -\, g(x). \end{aligned}$$

We have shown in the proof of [2, Theorem 6.2] that there exists $M>0$ such that for any $(x_0, y_0) \in \boldsymbol{R}^2$ with $x_0^2+y_0^2>M$, the orbit $T_{(6)}(x_0, y_0)$ is an oval surrounding the origin and that any solution $(x(t), y(t))$ of (1) remains in the region $R_{(6)}(x_0, y_0)$ as $t\to\infty$ if $(x(0), y(0)) \in R_{(6)}(x_0, y_0)$. Choose $c>G(K)$ such that $c^2+F(c)^2>M$.

Suppose that there exists $(x_0, y_0) \notin R_{(6)}(c, F(c))$ such that the solution $(x(t), y(t))$ of (1) with $(x(0), y(0))=(x_0, y_0)$ is periodic. Then it is clear that $(x(t), y(t)) \notin R_{(6)}(c, F(c))$ for all $t\geqq 0$ and there exists $t_1 \geqq 0$ such that $x(t_1)=c$. Let $(x_1, y_1)=(x(t_1), y(t_1))$. Comparing the solutions $y(x)$ and $\tilde{y}(x)$

$$\frac{dy}{dx} = \frac{g(x)}{F(x)-y} = \text{ and } \frac{dy}{dx} = \frac{g(x)}{\tilde{F}(x)-y}$$

such that $y(x_1)=\tilde{y}(x_1)=y_1$, we can show by (A_{15}) and Lemma 1 that the solution $(x(t), y(t))$ crosses the characteristic curve at a time $t_2>t_1$ and a point $(x_2, F(x_2))$ in the interior of $R_{(6)}(x_1, y_1)$. Then the solution $(x(t), y(t))$ remains in the region $R_{(6)}(x_2, F(x_2))$ for all $t\geqq t_2$, which contradicts that $(x(t), y(t))$ is periodic, since $R_{(6)}(x_2, F(x_2))$ is strictly contained in $R_{(6)}(x_1, y_1)$. Thus the proof is complete.

By the transformation $(x, y)\to(-x, -y)$, we can show the following corollary by the same arguments in the proof of Theorem 2.

Corollary 2. *Suppose that* $F(-x)\leqq 0$ *for large* $x>0$, $F(x)$ *and* $g(x)$ *satisfy* (A_4) *and* $((\mathrm{A}_5)$ *or* $(\mathrm{A}_6))$ *for* $x\geqq 0$, *and* (A_{13}) *and* (A_{15}) *hold. Then the solutions of* (1) *are uniformly ultimately bounded.*

Finally, we give a result on the global asymptotic stability of the zero solution of (1), the proof of which follows by those of Theorems 1 and 2.

Theorem 3. *Suppose that* $-F(-x)$ *and* $-g(-x)$ *satisfy* (A_3), (A_4) *and* $((\mathrm{A}_5)$ or $(\mathrm{A}_6))$ *for* $x\geqq 0$, *and*

$$(\mathrm{A}_{16})\qquad F(G^{-1}(-w)) < F(G^{-1}(w)) \quad \textit{for all} \quad w>0.$$

Then the zero solution of (1) *is globally asymptotically stable.*

References

[1] Coddington, E. A. and Levinson, N., *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.