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A Direct Method for the Existence of Evolution Operators Associated with Functional Evolutions in General Banach Spaces

By

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1. Introduction and preliminaries

In what follows, the symbol X stands for a real Banach space. We denote by $C[a, b]$ ($Lip[a, b]$) the space of all continuous (Lipschitz continuous) functions $f: [a, b]\to X$. In this paper we study functional evolution problems of the type

(FDE) $$x' + A(t)x = G(t, x_t), \quad t\in[0, T], \qquad x_0 = \phi,$$

where, for a function $f: [-r, T]\to X$, $f_t(s)=f(t+s)$, $t\in[0, T]$, $s\in[-r, 0]$. Here r, T are positive constants. An operator $A: D\subset X\to X$ is called "accretive" if

$$\|x-y\| \leq \|x-y+\lambda(Ax-Ay)\|$$

for every $\lambda>0$ and every $x, y\in D$. It is called "m-accretive" if it is accretive and $R(I+\lambda A)=X$ for all $\lambda>0$. If A is m-accretive, we set

$$|Ax| = \lim_{\lambda\downarrow 0} \|A_\lambda x\|, \quad x\in X,$$

where $A_\lambda=AJ_\lambda$ with $J_\lambda=(I+\lambda A)^{-1}$. We also set

$$\hat{D}(A) = \{x\in X;\ |Ax|<\infty\}.$$

We have that $D(A)\subset\hat{D}(A)\subset\overline{D(A)}$, $\|A_\lambda x\|\leq|Ax|$ for all $x\in X$ and $|Ax|\leq \inf\|Ax\|$ for $x\in D(A)$. Moreover, $x_n\in\hat{D}$ and $x_n\to x$ imply $|Ax|\leq\liminf|Ax_n|$. For these facts the reader is referred to Crandall [1]. We let $\Delta=\{(t, s);\ 0\leq s\leq t\leq T\}$. Given a mapping $U: \Delta\times K\to X$ (K a subset of X), we say that U is an "evolution operator" if

i) $U(t, t)=I$ (= the identity operator) for all $t\in[0, T]$;
ii) $U(t, s)x$ is continuous on Δ for all $x\in K$;
iii) $U(t, s)U(s, r)x=U(t, r)x$ for all $x\in K$ and all $s, t, r\in[0, T]$ with $0\leq r\leq s\leq t\leq T$.

The following conditions will be used in the sequel.

(C.1) $A(t)\colon D(A(t))\subset X\to X$ is m-accretive for every $t\in[0, T]$.

(C.2) There exists an increasing continuous function $L\colon [0, \infty)\to[0, \infty)$ such that

$$\|A_\lambda(t)x-A_\lambda(s)x\| \le |t-s|L(\|x\|)(1+\|A_\lambda(t)x\|)$$

for all $\lambda>0$, $x\in X$, s, $t\in[0, T]$.

(C.3) $G\colon [0, T]\times C[-r, 0]\to X$ and there exists a positive constant b such that

$$\|G(s, f_1)-G(t, f_2)\| \le b(|s-t|+\|f_1-f_2\|_\infty)$$

for every s, $t\in[0, T], f_1, f_2\in C[-r, 0]$.

The symbol $\|f\|_\infty$ denotes the sup-norm of f.

The main purpose of this paper is to obtain a "generalized solution" of the problem (FDE) which gives rise to an evolution operator $U(t, s)x$ as above. We will assume that (C.1)–(C.3) hold and that $\phi\in Lip[-r, 0]$ is a given function such that $\phi(0)\in\hat{D}(A(t))\equiv\hat{D}$. The set $\hat{D}(A(t))$ is constant under our assumptions (C.1)–(C.2) and so is $\overline{D(A(t))}\equiv\bar{D}$ (see Evans [4, Lemma 3.1]). If (C.2) above is replaced by a (much stronger) condition without the factor $1+\|A_\lambda(t)x\|$, this problem has been solved, in an otherwise more general settting, by the author and Parrott [6]. *However, the present condition* (C.2) *has much wider applicability in the field of partial differential equations and it does not seem possible that we can apply the fixed point method of* [5] *to the present setting except locally. In this vein, see also the note on* (C.2)′ *below.*

Evolution equations of the type (FDE) were considered by Webb [7]. Webb studied constant operators A and functions $G\equiv G(\phi)$, $\phi\in C[-r, 0]$, and considered an initial value problem with underlying space $C[-r, 0]$ instead of X. Webb's results were extended later to time dependent problems by Dyson and Villella Bressan [2], [3].

Our approach here is direct in the sense that the generalized solution $x(t)$, $t\in[-r, T]$, of (FDE) provides us with an evolution operator $U(t, s)x$ on $\Delta\times\bar{D}$ — and not on $\Delta\times K$ with $K\subset C[-r, 0]$. Moreover, two generalized solutions with initial data ϕ, ψ satisfy the fundamental inequality

$$\|x(t)-y(t)\| \le \|\phi(0)-\psi(0)\| + \int_0^r \|G(s, x_s)-G(s, y_s)\|ds, \quad t\in[0, T]$$

(and other inequalities of Bénilan type), which is of great importance in the study of stability and asymptotic behaviour of (FDE). For a result concerning the existence of generalized solutions on $\boldsymbol{R}$, the reader is referred to the author's paper [5].

It should be noted that if an operator $A(t)\colon D\subset X\to X$ (D constant) satisfies the condition (C.1) and G satisfies (C.3), then the condition

(C.2)′ $$\|A(t)x-A(s)x\| \le |t-s|L(\|x\|)(1+\|A(t)x\|),$$

for $(s, t, x) \in [0, T]^2 \times D$ and L as in (C.2), can be used instead of (C.2). In fact, (C.2)′ can be used to reduce the problem (FDE) to a problem where (C.1)–(C.3) are satisfied.

In order to show this, fix $x_0 \in D$ and consider the operators $A_1(t): D \to X$ with $A_1(t)x = A(t)x - A(t)x_0$, $t \in [0, T]$, $x \in D$. Then $A_1(t)x_0 = 0$ and $(I+\lambda A_1(t))x_0 = x_0$, giving $J^1_\lambda(t)x_0 \equiv (I+\lambda A_1(t))^{-1}x_0 = x_0$. Moreover,

$$\begin{aligned}\|A_1(t)x-A_1(s)x\| &\le |t-s|L(\|x\|)(1+\|A(t)x\|)\\ &\quad + |t-s|L(\|x_0\|)(1+\|A(t)x_0\|)\\ &\le |t-s|L(\|x\|)(1+\|A_1(t)x\|+\|A(t)x_0\|) + |t-s|L(\|x_0\|)(1+\|A(t)x_0\|)\\ &\le |t-s|\,(L(\|x\|)+L(\|x_0\|))(1+\|A_1(t)x\|+\|A(t)x_0\|),\end{aligned}$$

where we have used (C.2)′ and $\|A(t)x\| \le \|A(t)x-A(t)x_0\| + \|A(t)x_0\|$. We also have

$$\|A(t)x_0\| \le \|A(0)x_0\| + TL(\|x_0\|)(1+\|A(0)x_0\|).$$

Letting K denote the right-hand side above, we obtain

$$\|A_1(t)x-A_1(s)x\| \le |t-s|L_1(\|x\|)(1+\|A_1(t)x\|),$$
$$(s, t, x) \in [0, T]^2 \times D,$$

where $L_1(\|x\|) = (1+K)(L(\|x\|)+L(\|x_0\|))$. Thus, we can now apply Lemma 3.2 of Evans [4] to obtain that the operator $A_1(t)$ satisfies (C.2). If we let $G_1(t, \phi) \equiv G(t, \phi) - A(t)x_0$, we see that the function G_1 satisfies (C.3). This proves our assertion.

2. Main results

We have the following three lemmas.

Lemma 1. *Let the operators* $A(t)$ *satisfy* (C.1) *and* (C.2). *Assume that* $f \in Lip[0, T]$. *Then, for every* $x_0 \in \hat{D}(A(t)) \equiv \hat{D}$, $s \in [0, T)$, *the Crandall-Pazy solution of the problem*

(1) $$x' + A(t)x = f(t), \quad t \in [s, T],$$
$$x(s) = x_0$$

is equal to the Evans solution.

By the Crandall-Pazy solution of (1) we mean the function $u(t)$, $t \in [s, T]$,

$u(s)=x_0$, whose value at $t\in(s, T]$ is given by

$$(2) \qquad u(t) = U(t, s)x_0 = \lim_{n\to\infty} \Big(\prod_{j=1}^{n} \hat{J}_{(t-s)/n}(s+j(t-s)/n)\Big)x_0$$

where $\hat{J}_\lambda(t)=[I+\lambda\hat{A}(t)]^{-1}$, $\hat{A}(t)=A(t)-f(t)$. On the other hand, the Evans solution of (1) is the function $\bar{u}(t)=\lim_{n\to\infty} u_n(t)$, uniformly on $[s, T]$, where the function $u_n(\tau)$, $\tau\in(s, t]$, is given by

$$(3) \qquad u_n(\tau) = z_{nj}, \quad \tau\in(s+(j-1)(t-s)/n,\ s+j(t-s)/n],$$

where $A(t_{nj})z_{nj}+(z_{nj}-z_{n,j-1})/h=f(t_{nj})$, $j=1, 2,\cdots, n$, $h=(t-s)/n$, $z_{n0}=x_0$, $t_{nj}=s+jh$. Moreover, $u_n(s)=x_0$. In particular, $u_n(t)=z_{nn}$.

Proof of Lemma 1. It suffices to let in (2) $z_{n0}=x_0$ and

$$z_{nj} = \Big(\prod_{i=1}^{j} \hat{J}_{(t-s)/n}(s+i(t-s)/n)\Big)x_0 .$$

Then we have

$$z_{nj} = \hat{J}_{(t-s)/n}(s+j(t-s)/n)z_{n,j-1} = [I+[(t-s)/n]\hat{A}(t_{nj})]^{-1}z_{n,j-1}$$

or

$$A(t_{nj})z_{nj} + (z_{nj}-z_{n,j-1})/h = f(t_{nj}).$$

Thus, $z_{nn}=u_n(t)\to U(t, s)x_0=\bar{u}(t)$. This completes the proof because t is arbitrary in $(s, T]$.

The reason we gave this lemma is the fact that from now on we are only making use of Evans' paper and Evans' solution. The reader should have thus in mind that we are actually dealing with the same solution as that of Crandall and Pazy.

Lemma 2. *Let f_n: $[0, T]\to[0, \infty)$ be continuous and such that*

$$f_{n+1}(t) \le b\int_0^t f_n(s)ds, \quad t\in[0, T], \quad n = 1, 2,\cdots,$$

where b is a positive constant. Then

$$f_{n+1}(t) \le [(bt)^n/n!]\,\|f_1\|_\infty^t, \quad t\in[0, T], \quad n = 1, 2,\cdots,$$

where $\|f\|_\infty^t=\sup_{s\in[0,t]}|f(s)|$.

Proof. The statement to prove is obviously true for $n=1$. Assume that it is true for $n=k$. Then we have

$$f_{k+2}(t) \leq b\int_0^t f_{k+1}(s)ds \leq (b^{k+1}/k!)\int_0^t s^k ds\|f_1\|_\infty^t$$
$$= [(bt)^{k+1}/(k+1)!]\,\|f_1\|_\infty^t .$$

Lemma 3. *Let the conditions* (C.1)–(C.3) *be satisfied. Then the Evans solution* $x(t)$ *of the problem*

$$(\mathrm{E_g})\qquad x' + A(t) = G(t, g_t), \quad t \in [0, T],$$
$$x(0) = x_0 \in \hat{D}$$

belongs to $Lip[0, T]$ *for every* $g \in Lip[-r, T]$.

Proof. We follow the notation of Theorems 1 and 2 and their proofs in Evans [4]. We set $h(t) \equiv t$, $f(t) \equiv G(t, g_t)$ there. The Evans solution of $(\mathrm{E_g})$, $u(t)$, satisfies

$$\begin{aligned}
&\|u(s)-u(t)\| \\
&\leq \omega_\delta(t-s) + \int_0^t [K|t-s| + \|G(\alpha, g_\alpha) - G(t-s+\alpha, g_{t-s+\alpha})\|]d\alpha \\
&\leq \omega_\delta(t-s) + \int_0^t [K|t-s| + b(|t-s| + \|g_\alpha - g_{t-s+\alpha}\|_\infty)]d\alpha \\
&\leq \omega_\delta(t-s) + \int_0^t [K|t-s| + b(|t-s| + M|t-s|)]d\alpha \\
&\leq \omega_\delta(t-s) + M_1|t-s|, \quad 0 \leq s \leq t \leq T,
\end{aligned}$$

where K, b, M, M_1 are appropriate constants. Here we have used the fact that $\|g_t - g_s\|_\infty \leq M|t-s|$, where M is a Lipschitz constant for g on $[-r, T]$. The function $\omega_\delta(t-s)$ is given by

$$\omega_\delta(t-s) \equiv K_1(t-s) + \int_0^{t-s} [\|G(u, g_u)\| + K_2|u|]du + 2\|x_\delta - x_0\|,$$

where K_1, K_2 are constants. We now observe that since $x_0 \in \hat{D}$, we may choose $x_\delta = x_0$ in Evans [4, proofs of Theorems 1, 2] in order to obtain that $\omega_\delta(t-s) \leq K_3|t-s|$, $0 \leq s \leq t \leq T$, where K_3 is another positive constant. This proves the Lipschitz continuity of $u(t)$, $t \in [0, T]$.

We are now ready for our first theorem.

Theorem 1. *Let* (C.1)–(C.3) *be satisfied and let* $\phi \in Lip[-r, 0]$, $\phi(0) \in \hat{D}$ *be fixed. Assume that* $y^0 \in Lip[-r, T]$ *is a given function such that* $y^0(t) = \phi(t)$, $t \in [-r, 0]$. *Consider the sequence* $\{y^n(t)\}$, $t \in [-r, T]$, $n = 1, 2, \cdots$, *such that* $y^n(t)$ *is the Evans solution of the problem*

$$(\mathrm{E}_n) \qquad \begin{aligned} y' + A(t)y &= G(t, y_t^{n-1}), \\ y(0) &= \phi(0) \end{aligned}$$

on $[0, T]$ *and equals* $\phi(t)$ *on* $[-r, 0)$ *(the sequence is well defined by induction, in view of Lemma 3 and the Lipschitz continuity of* $y^0(t)$*).*

Then $y^n(t) \to x(t)$ *uniformly as* $n \to \infty$ *on* $[-r, T]$. *This function* $x(t)$ *is independent of the initial function* $y^0(t)$. *Moreover,* $x(t)$ *is the extension on* $[-r, T]$ *of a local Evans solution of the problem*

$$(4) \qquad \begin{aligned} u' + A(t)u &= G(t, x_t) \\ u(0) &= \phi(0). \end{aligned}$$

Proof. 1. *Convergence of the approximants.* Given a function $f \in Lip[-r, T]$, we let $U(t, 0, f)\phi(0)$ be the Evans solution of the problem

$$\begin{aligned} u'(t) + A(t)u &= G(t, f_t), \quad t \in [0, T], \\ u(0) &= \phi(0). \end{aligned}$$

We have $y^n(t) = U(t, 0, y^{n-1})\phi(0)$, $t \in [0, T]$. At this point, recalling Theorem 3 of Evans [4], we observe that

$$\begin{aligned} \|y^{n+1}(t) - y^n(t)\| &\leq \int_0^t \|G(s, y_s^n) - G(s, y_s^{n-1})\| ds \\ &\leq b \int_0^t \|y_s^n - y_s^{n-1}\|_\infty ds, \quad t \in [0, T], \quad n = 1, 2, \cdots. \end{aligned}$$

If we look at the functions $y^{n+1}(t+\tau) - y^n(t+\tau)$ for $t+\tau \in [-r, T]$, $n=0, 1, \cdots, \tau \in [-r, 0]$, $t \in [0, T]$, we see that

$$\|y^{n+1}(t+\tau) - y^n(t+\tau)\| = \|\phi(t+\tau) - \phi(t+\tau)\| = 0 \qquad \text{for} \quad t + \tau \leq 0, \quad n \geq 0,$$

$$\begin{aligned} \|y^{n+1}(t+\tau) - y^n(t+\tau)\| &\leq b \int_0^{t+\tau} \|y_s^n - y_s^{n-1}\|_\infty ds \\ &\leq b \int_0^t \|y_s^n - y_s^{n-1}\|_\infty ds, \quad n \geq 1, \quad t + \tau \geq 0, \end{aligned}$$

which imply,

$$\|y_t^{n+1}(\tau) - y_t^n(\tau)\| \leq b \int_0^t \|y_s^n - y_s^{n-1}\|_\infty ds, \quad n \geq 1, \quad t \in [0, T], \quad \tau \in [-r, 0]$$

and, finally, by Lemma 2,

$$\|y_t^{n+1} - y_t^n\|_\infty \leq [(bt)^n/n!] \, \|f_1\|_\infty^t, \quad t \in [0, T], \quad n \geq 1,$$

where $\|f_1\|_\infty^t = \sup_{s \in [0,t]} \|y_s^1 - y_s^0\|_\infty$.

To show that $\{y^n(t)\}$ converges uniformly on $[-r, T]$, let $m>n\geq 1$. Then we have

$$\begin{aligned}\|y_t^m - y_t^n\|_\infty &\leq \|y_t^m - y_t^{m-1}\|_\infty + \cdots + \|y_t^{n+1} - y_t^n\|_\infty \\ &\leq \|f_1\|_\infty^t \sum_{n+1}^{m} [(bt)^{i-1}/(i-1)!] \\ &\leq \|f_1\|_\infty^T \sum_{n+1}^{m} [(bT)^{i-1}/(i-1)!].\end{aligned}$$

Since $\sum_{i=0}^{\infty} (bT)^i/i! = e^{bT}$, we have that

$$\sup_{t\in[0,T]} \|y_t^m - y_t^n\| \longrightarrow 0 \quad \text{as} \quad m, n \longrightarrow \infty,$$

which implies easily that there exists a continuous function $x(t)$, $t\in[-r, T]$, such that $y^n(t)\to x(t)$ uniformly on $[-r, T]$ as $n\to\infty$.

2. *Independence of the first approximant.* Let us assume that $y^0(t)$, $\bar{y}^0(t)$ are two different initial functions in Step 1 with corresponding approximatns $\{y^n(t)\}$, $\{\bar{y}^n(t)\}$, $t\in[-r, T]$. Then, according to Theorem 3.1 of [4], we have

$$\begin{aligned}\|y^n(t) - \bar{y}^n(t)\| &\leq \int_0^t \|G(s, y_s^{n-1}) - G(s, \bar{y}_s^{n-1})\| ds \\ &\leq b\int_0^t \|y_s^{n-1} - \bar{y}_s^{n-1}\|_\infty ds, \quad t\in[0, T], \quad n\geq 1.\end{aligned}$$

From this inequality we obtain, from Lemma 2,

$$\|y_t^n - \bar{y}_t^n\|_\infty \leq [(bT)^n/n!] \sup_{t\in[0,T]} \|y_t^0 - \bar{y}_t^0\|_\infty,$$

i.e., $\|y_t^n - \bar{y}_t^n\|_\infty \to 0$ uniformly on $[0, T]$ as $n\to\infty$. This shows that the sequences $\{y^n(t)\}$, $\{\bar{y}^n(t)\}$ have the same limit function as $n\to\infty$.

3. *Coincidence with the Evans solution.* In this step we are going to show that there is a number $T_1\in(0, T]$ such that Problem (4), with $x(t)$ as in Step 1, has an Evans solution on the interval $[0, T_1]$ and that $x(t)$ coincides with this solution on $[0, T_1]$. To this end, we show first that if T_1 is taken to be sufficiently small, then the Lipschitz constant of the approximant $y^n(t)$ can be considered fixed for all $n\geq 0$ and $t\in[-r, T_1]$.

Since $y^n(t)\to x(t)$ uniformly on $[-r, T]$ as $n\to\infty$, we must have $\|y^n(t)\|\leq C$ uniformly in t, n, where C is a positive constant. We may assume that the Lipschitz constant ℓ_0 of $y^0(t)$ on $[-r, T]$ satisfies $\ell_0 > K_1 + C_1$, where

$$C_1 = \sup\{\|G(t, \phi)\|;\ t\in[0, T],\ \phi\in C[-r, 0],\ \|\phi\|_\infty\leq C\}$$

and K_1 is specified in the next paragraph.

It is easy to see, from Lemma 3, that

$$\|y^1(t)-y^1(s)\| \le K_1(t-s) + (C_1+K_2T)(t-s) + \int_0^s [K(t-s)+b((t-s)+\ell_0(t-s))]d\alpha, \quad (t, s)\in \Delta, \tag{5}$$

where the constants K_1, K_2 depend only on $|A(0)\phi(0)|$, $\|\phi(0)\|$ and the constant $K=K(\ell_0, T, \|\phi(0)\|, |A(0)\phi(0)|, C, C_1)$. The nature of this constant K can be confirmed after a careful study of the proof of Theorem 2 of Evans [4] in view of the fact that the sequence $\{\|x_k^n\|\}$ there has an upper bound depending only on K_1, K_2, C_1 and T, and the sequence

$$\|f_k^n+(x_{k-1}^n-x_k^n)/(t_k^n-t_{k-1}^n)\|$$

there has an upper bound depending on the previous parameters as well as the variation of the function $f(t)$ (which in our case is bounded above by $\ell_0(T+r)$).

It is obvious now that if the variable t is restricted on a sufficiently small interval $[0, T_1]$, for some $T_1\in(0, T]$, then $y^1(t)$ has a Lipschitz constant ℓ_0 on the interval $[0, T_1]$ as well. From this point on we restrict the functions $y^n(t)$, $n=1, 2,\cdots$, so that they are defined just on the interval $[-r, T_1]$. We also observe that a Lipschitz constant of $y^1(t)$ on $[-r, 0]$ is ℓ_0. Consequently,

$$\|y^1(t)-y^1(s)\| \le \ell_0|t-s|, \quad t, s\in[-r, T_1].$$

Since

$$\|y_t^n-y_s^n\|_\infty = \sup_{u\in[-r,0]}\|y^n(t+u)-y^n(s+u)\|, \quad s, t\in[0, T_1]$$

and $t+u$, $s+u\in[-r, T_1]$ for such s, t, a repetition of the above argument implies by induction that ℓ_0 is a Lipschitz constant for all functions $y^n(t)$, $t\in[-r, T_1]$, $n=0, 1,\cdots$.

Taking the limit as $n\to\infty$ of the left-hand side of

$$\|y^n(t)-y^n(s)\| \le \ell_0|t-s|$$

we obtain

$$\|x(t)-x(s)\| \le \ell_0|t-s|, \quad s, t\in[0, T_1].$$

This Lipschitz continuity of $x(t)$ implies the existence of a unique Evans solution $u(t)$ of the problem (4) on the interval $[0, T_1]$.

From the inequality

$$\|u(t)-y^n(t)\| \le \int_0^t \|G(s, x_s)-G(s, y_s^{n-1})\|ds, \quad t\in[0, T_1],$$

we obtain immediately $u(t)\equiv x(t)$, $t\in[0, T_1]$. This completes the proof of the theorem.

The function $x(t)$, $t \in [-r, T]$, obtained in Theorem 1 is called the "generalized solution" of the problem (FDE).

3. The evolution operator

The next result shows that the "generalized solution" to the problem

$$(6) \qquad \begin{aligned} &u' + A(t)u = G(t, x_t), \quad t \in [s, T],\\ &u(s) = u_0, \end{aligned}$$

where $x(t)$, $t \in [-r, T]$ is the generalized solution of Theorem 1 and $s \in [0, T]$, gives rise to an evolution operator $U(t, s)u_0$.

Theorem 2. *Assume that the conditions of Theorem 1 on A, G, ϕ are satisfied. Let $z_n(t)$, $t \in [s, T]$, be the Evans solution of the differential equation of* (E_n) *on $[s, T]$ such that $z_n(s) = u_0 \in \hat{D}$. Then $z_n(t)$ converges uniformly on $[s, T]$ as $n \to \infty$ to a function $z(t)$ which is independent of the initial function $y^0 \in Lip[-r, T]$. If we let $U(t, s)u_0 \equiv z(t)$, $t \in [s, T]$, then $U(t, s)u$, $0 \leq s \leq t \leq T$, $u \in \hat{D}$, satisfies the first two properties of an evolution operator. $U(t, s)$ can also be extended to $\bar{D}$, where it is an evolution operator.*

Proof. We first remark that $z_n(t)$ exists because the conditions on $A(t)$ hold on the interval $[s, T]$, $u_0 \in \hat{D}$ and the function $G(t, y_t^n)$ is a global Lipschitzian on $[s, T]$. The uniform convergence of $z_n(t)$ on $[s, T]$ follows immediately from

$$(7) \qquad \|z_m(t) - z_n(t)\| \leq b \int_s^t \|y_v^{m-1} - y_v^{n-1}\|_\infty dv, \quad s \leq t \leq T.$$

To show that $z(t)$ is independent of the initial function y^0, let $\bar{y}^0$ be another such function in $Lip[-r, T]$ and let $\bar{z}_n(t)$ be the corresponding solutions on $[s, T]$ with $\bar{z}_n(s) = u_0$. Then

$$\|\bar{z}_n(t) - z_n(t)\| \leq b \int_s^t \|\bar{y}_v^{n-1} - y_v^{n-1}\|_\infty dv, \quad s \leq t \leq T$$

implies immediately our assertion. If we let $U_n(t, s)u_0 \equiv z_n(t)$, $t \in [s, T]$, we thus have $U_n(t, s)u_0 \to U(t, s)u_0$ uniformly in $t \in [s, T]$. From (7) we actually obtain that this convergence is uniform with respect to $(t, s) \in \Delta$. Since $U_n(t, t)u_0 = u_0$ and $U(t, s)U_0$ is uniformly continuous in $(t, s) \in \Delta$, because $U_n(t, s)u_0$ has this property, it remains to show the evolution identity:

$$U(t, s)U(s, \tau)u_0 = U(t, \tau)u_0, \quad 0 \leq \tau \leq s \leq t \leq T.$$

This identity follows from the triangle inequality provided that $U(s, \tau)u_0 \in \hat{D}$.

We now extend $U_n(t, s)$ and $U(t, s)$ to the operators on $\bar{D}$ by uniform continuity for each $0 \leq s \leq t \leq T$. It follows (cf. Evans [4]) that the extension of U_n, denoted again by U_n, satisfies the properties of the evolution operator on $\Delta \times \bar{D}$ for each n.

We show that for any $u_0 \in \bar{D}$, and $0 \leq s \leq t \leq T$,

$$(*) \qquad \|U_m(t, s)u_0 - U(t, s)u_0\| \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.$$

In fact, taking a sequence $u_n \in \hat{D}$ with $u_n \to u_0$, we have

$$\begin{aligned} &\|U_m(t, s)u_0 - U(t, s)u_0\| \\ &\quad \leq \|U_m(t, s)u_0 - U_m(t, s)u_n\| + \|U_m(t, s)u_n - U(t, s)u_n\| \\ &\qquad + \|U(t, s)u_n - U(t, s)u_0\| \\ &\quad \leq 2\|u_n - u_0\| + b\int_s^t \|y_\tau^{m-1} - x_\tau\|_\infty d\tau, \end{aligned}$$

so that

$$\limsup_{m \to \infty} \|U_m(t, s)u_0 - U(t, s)u_0\| \leq 2\|u_n - u_0\| \quad \text{for all} \quad n.$$

This shows that $(*)$ holds.

Using the convergence in $(*)$, we see that

$$\|U_m(t, s)U_m(s, \tau)u_0 - U(t, s)U(s, \tau)u_0\| \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty,$$

and that $U(t, s)u_0 \in \bar{D}$ for $u_0 \in \bar{D}$ and U is an evolution operator $\Delta \times \bar{D} \to \bar{D}$.

We state separately a theorem concerning the fact that $U(t, s)u \in \bar{D}$ for every $u \in \bar{D}$ as shown above.

Theorem 3. *The evolution operator $U(t, s)$ of Theorem 2 maps $\bar{D}$ into $\bar{D}$ for every $(t, s) \in \Delta$. In particular, the generalized solution $x(t)$ lies in $\bar{D}$ for all $t \in [0, T]$.*

The following theorem also follows easily from the proofs of Theorems 1 and 2.

Theorem 4. *Assume that we have two generalized solutions $x(t)$, $y(t)$, $t \in [-r, T]$ with associated initial functions ϕ, $\psi \in Lip[-r, 0]$, $\phi(0) \in \hat{D}$, $\psi(0) \in \hat{D}$, and evolution operators $U(t, s)$, $V(t, s)$, respectively. Then*

$$(10) \qquad \|x(t) - y(t)\| \leq \|\phi(0) - \psi(0)\| + \int_0^t \|G(s, x_s) - G(s, y_s)\| ds, \quad t \in [0, T],$$

$$(11) \qquad \|U(t, s)u - V(t, s)v\| \leq \|u - v\| + \int_s^t \|G(\tau, x_\tau) - G(t, y_\tau)\| d\tau, \quad (t, s) \in \Delta, \quad u, v \in \bar{D}.$$

4. Example

Let Ω be an open, bounded subset of $\boldsymbol{R}^n$ and $X=C(\bar{\Omega})$. Assume that A: $D\subset C(\bar{\Omega})\to C(\bar{\Omega})$ is any m-accretive operator. Let m: $[0, T]\times\bar{\Omega}\to\boldsymbol{R}$ be such that

i) $x\to m(t, x)$ is continuous for each t;

ii) there exists a constant $\varepsilon>0$ such that

$$\varepsilon \le m(t, x), \quad (t, x)\in[0, T] \times \bar{\Omega};$$

iii) there exists a constant $\lambda>0$ such that

$$|m(t, x)-m(s, x)| \le \lambda|t-s|, \quad s, t\in[0, T], \quad x\in\bar{\Omega}.$$

Then the operator $A(t)$: $D\to C(\bar{\Omega})$ defined by $m(x, t)A$ satisfies (C.1) and (C.2). For this fact see Evans [4, Proposition 11.1].

We now state the following conditions:

iv) K: $[0, T]^2\times\bar{\Omega}\to\boldsymbol{R}$ is continuous and such that

$$\|K(t_1, s, x)-K(t_2, s, x)\| \le b_1|t_1-t_2|, \quad (t, s, x)\in[0, T]^2 \times \bar{\Omega}.$$

v) g: $\boldsymbol{R}\to[-M, M]$, $M>0$ constant, is Lipschitz continuous with Lipschitz constant $C>0$.

If iv), v) are true, then the function G: $[0, T]\times C[-r, 0]\to X$ defined by

$$(G(t, \phi))(x) = \int_{-r}^{0} K(t, s, x)g(\phi(x, s))ds$$

satisfies (C.3). In fact, for $\phi, \psi\in C[-r, 0]$, $t_1, t_2\in[0, T]$,

$$\begin{aligned}
&|(G(t_1, \phi))(x)-(G(t_2, \psi))(x)| \\
&\le \int_{-r}^{0} |K(t_1, s, x)g(\phi(x, s))-K(t_2, s, x)g(\phi(x, s))|ds \\
&\quad + \int_{-r}^{0} |K(t_2, s, x)g(\phi(x, s))-K(t_2, s, x)g(\psi(x, s))|ds \\
&\le Mb_1r|t_1-t_2| + Cb_2r\|\phi-\psi\|_\infty \le b(|t_1-t_2|+\|\phi-\psi\|_\infty),
\end{aligned}$$

where

$$\begin{aligned}
b_2 &= \max\{|K(t, s, x)|;\ (t, s, x)\in[0, T]^2\times\bar{\Omega}\}, \\
\|\phi-\psi\|_\infty &= \sup_{s\in[-r,0]}\sup_{x\in\bar{\Omega}}|\phi(x, s)-\psi(x, s)|, \\
b &= r\max\{Mb_1, Cb_2\}.
\end{aligned}$$

It follows that the abstract problem (FDE), with $A(t)$, G as above, possesses a generalized solution on the interval $[-r, T]$.

5. The ordinary case. Discussion

The results of this paper can of course be applied to ordinary problems with $G \equiv G(t, x)$. Actually, in this case, whenever $D(A(t)) = D = \text{const.}$, the problem

$$\text{(ODE)} \qquad \begin{aligned} x' + A(t)x &= G(t, x), \quad t \in [0, T], \\ x(0) &= x_0 \in \hat{D} \end{aligned}$$

can be reduced to a problem of the type

$$\begin{aligned} u' + A_1(t)u &= bu + f(t), \quad t \in [0, T], \\ u(0) &= x_0, \end{aligned}$$

where b is the Lipschitz constant of G from (C.3)′ (which is (C.3) modified in the obvious way) and f: $[0, T] \to X$ is a Lipschitz continuous function. In fact, we first let $\hat{A}_1(t)x \equiv A(t)x - G(t, x) + bx$, $x \in D$. It is obvious that $\hat{A}_1(t)x$ is accretive in x for all $t \in [0, T]$. To show it is m-accretive, it suffices to show that the mapping $x \to \hat{A}_1(t)x + x$ is onto, or that the equation $[A(t) + (b+1)I]x = G(t, x) + v$ is solvable in x for each $t \in [0, T]$, $v \in X$. In fact, the operator

$$x \longrightarrow [A(t) + (b+1)I]^{-1}[G(t, x) + v]$$

is a contraction on X with contraction constant $b/(b+1) < 1$. The unique fixed point of this operator is the solution of the above equation.

Now we compute

$$\begin{aligned} \|\hat{A}_1(t)x - \hat{A}_1(s)x\| &\le |t-s|L(\|x\|)(1 + \|A(t)x\|) + b|t-s| \\ &\le |t-s|\,(b + L(\|x\|)(1 + \|A(t)x - G(t, x) + bx\| + \|G(t, x) - bx\|) \\ &\le |t-s|L_1(\|x\|)(1 + \|\hat{A}_1(t)x\| + \|G(t, 0)\| + 2b\|x\|) \\ &\le |t-s|L_2(\|x\|)(1 + \|\hat{A}_1(t)x\|), \end{aligned}$$

where L_i: $[0, \infty) \to [0, \infty)$, $i = 1, 2$, are continuous increasing functions.

On the basis of these facts, the problem (ODE) is reduced to the problem

$$\begin{aligned} x' + \hat{A}_1(t)x &= bx, \quad t \in [0, T], \\ x(0) &= x_0, \end{aligned}$$

where $\hat{A}_1(t)$ satisfies (C.1) and (C.2) with $\hat{A}_1$ instead of A. If we let $A_1(t)x \equiv \hat{A}_1(t)x - \hat{A}_1(t)\bar{x}_0$, $t \in [0, T]$, for some $\bar{x}_0 \in D$, $f(t) \equiv -\hat{A}_1(t)\bar{x}_0$, we have our assertion.

Unfortunately we could not show that for $t \in [0, T]$ the generalized solution

$x(t)$ of Theorem 1 lies in $\hat{D}$, except when t is in the initial interval $[0, T_1]$ where $x(t)$ is Lipschitz continuous. In view of the properties of the function $x \to |A(t)x|$, $x \in X$, the value $x(t)$ lies in $\hat{D}$ if $|A(t)y^n(t)| \leq K(t)$, where $K(t)$ is a constant independent of n.

Problems with perturbations $G(t, x)$ which are locally defined with respect to x are genuine extensions of the present as well as the Crandall-Pazy-Evans theory.

If $A(t)$ is strongly accretive ($\|x-y+\lambda(A(t)x-A(t)y)\| \geq (1+\lambda\alpha)\|x-y\|$, $\alpha > 0$), then inequalities involving the exponential $e^{-\alpha t}$ like (10) and (11) hold with the usual expressions on their right-hand sides. To see this for (10), one solves first the equation

$$u' + A_m(t)u = G(t, y_t^{n-1}), \quad t \in [0, T],$$
$$u(0) = \phi(0),$$

to get such an inequality for its solution $u_{mn}(t)$, where $A_m(t) \equiv A_{1/m}(t)$, and then takes the limit as $m \to \infty$ to obtain the same inequality for $y^n(t)$ (cf. Crandall and Pazy [1, Remark following Lemma 4.2]). Taking the limit again as $n \to \infty$, we obtain our assertion.

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