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On the Inverse Sturm-Liouville Problem with Spatial Symmetry

By

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§ 1. Introduction

The inverse Sturm-Liouville problem on a finite interval was first considered by V. Ambarzumian [1], and then by G. Borg [2] in some detail. Since then this problem has attracted many author's attentions. Especially I. M. Gel'fand and B. M. Levitan [6] discovered a constructive method which reduces the problem to solving an integral equation called after their names, though their attention was chiefly directed to the problem on a half-infinite interval. See also the paper by B. M. Levitan and G. M. Gasymov [14].

In the present paper, our concern will be restricted to the inverse Sturm-Liouville problem with spatial symmetry, though our method will be clearly applicable to other problems, which will be reported upon in another paper. We shall start with stating the definition and some elementary properties of spatial symmetric Sturm-Liouville problem.

Let SBV^1 denote the set of functions $p(x)$ defined in the interval $0 \leqq x \leqq 1$ and satisfying the symmetric condition $p(1-x)=p(x)$ whose first derivatives are of bounded variation, similarly SH^1 denote the set of simmetric functions having the square integrable derivatives. For a potential $p(x)$ in SBV^1 (or SH^1) and for a real number $h \in \boldsymbol{R}$, the spatially symmetric boundary value problem (B.V.P. for short) $\langle p, h\rangle$ of the third kind is defined by

$$\text{(1.1)} \qquad \langle p, h\rangle : \left(\begin{array}{l} -y'' + p(x)y = \lambda y; \quad 0 < x < 1 \\ y'(0) - hy(0) = y'(1) + hy(1) = 0, \end{array} \right.$$

Similarly, for a potential $p(x)$, the symmetric boundary value problem $\langle p, \infty\rangle$ with Dirichlet boundary condition is defined by

$$\text{(1.2)} \qquad \langle p, \infty\rangle : \left(\begin{array}{l} -y'' + p(x)y = \lambda y; \quad 0 < x < 1 \\ y(0) = y(1) = 0. \end{array} \right.$$

It is well-known (see e.g. Levitan and Gasymov [6]) that the asymptotic distribution of the eigenvalues $\{\lambda_n(p, h)\}_{n=0}^{\infty} = \{\lambda_n\}_{n=0}^{\infty}$ of the B.V.P. $\langle p, h\rangle$ is given by

$$(1.3)\quad \begin{cases} \lambda_n(p,\, h)^{1/2} = \pi n + a_0(p,\, h)/(\pi n) + \gamma_n \\ \text{with either } \gamma_n = O(1/n^3) \text{ as } n \longrightarrow \infty \text{ if } p(x) \in SBV^1 \\ \text{or } \sum_{n=0}^{\infty} (n^2\gamma_n)^2 < +\infty \text{ if } p(x) \in SH^1, \end{cases}$$

where

$$a_0(p,\, h) = 2h + (1/2)\int_0^1 p(t)dt.$$

The distribution of the eigenvalues $\{\lambda_n(h,\, \infty)\}_{n=1}^{\infty} = \{\mu_n\}_{n=1}^{\infty}$ of the B.V.P. $\langle p,\, \infty\rangle$ is slightly different from (1.3) and given by

$$(1.4)\quad \begin{cases} \lambda_n(p,\, \infty)^{1/2} = \pi n + b_0(p)/(\pi n) - b_0(p)b_1(p)/(\pi n)^2 + \bar{\gamma}_n \\ \text{with either } \bar{\gamma}_n = O(1/n^3) \text{ as } n \longrightarrow \infty \text{ if } p(x) \in SBV^1 \\ \text{or } \sum_{n=1}^{\infty} (n^2\bar{\gamma}_n)^2 < +\infty \text{ if } p(x) \in SH^1, \end{cases}$$

where

$$b_0(p) = (1/2)\int_0^1 p(t)dt$$

$$b_1(p) = (1/4)\left\{p(0) + p(1) - (1/2)\left(\int_0^1 p(t)dt\right)^2\right\}.$$

Next let us define the solutions $c_\nu(x,\, \lambda)$ and $s_\nu(x,\, \lambda)$ $(\nu = 0,\, 1)$ of the differential equation $-y'' + p(x)y = \lambda y$ by imposing the following initial condition under which they satisfy the given boundary condition at one of the endpoints $x = \nu$ $(\nu = 0,\, 1)$,

$$(1.5)\quad \begin{aligned} &c_\nu(x,\, \lambda) = 1, \quad c_\nu'(x,\, \lambda) = (-1)^\nu h \\ &s_\nu(x,\, \lambda) = 0, \quad s_\nu'(x,\, \lambda) = (-1)^\nu. \end{aligned} \qquad \nu = 0,\, 1.$$

Then the symmetric property of the potential implies the relation:

$$(1.6)\quad c_1(x,\, \lambda) = c_0(1-x,\, \lambda), \quad s_1(x,\, \lambda) = s_0(1-x,\, \lambda).$$

Furthermore the eigenfunctions satisfy the following relations:

Lemma 1.1.

$$(1.7)\quad c_1(x,\, \lambda_n) = c_0(1-x,\, \lambda_n) = (-1)^n c_0(x,\, \lambda_n)$$

$$(1.8)\quad s_1(x,\, \mu_n) = s_0(1-x,\, \mu_n) = (-1)^{n+1} s_0(x,\, \mu_n).$$

Proof. This formula is elementary. See T. Suzuki [21] for the proof.

Now let us consider two B.V.P's $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$, the former of which will be regarded as known and will be refered to as "reference problem", while the latter being as unknown. Our purpose in this paper is to introduce a new algorythm for expressing the difference $\tilde{p}(x)-p(x)$, $\tilde{h}-h$ of the two potentials and two boundary conditions. In the case where $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ share their eigenvalues in common with at most finite number of exceptions, such case will be callded "degenerate case", the attempts to finding an effective expression of $\tilde{p}(x)-p(x)$ and $\tilde{h}-h$ were done by H. Hochstadt [8] [9] [10], O. Hald [7], T. Suzuki [21] [22] [23] and the author (umpublished, see Remark 2.2). The Hochstadt-Hald method has a disadvantage that their expression contains the solutions of the differential equation associated with the unknown problem. On the other hand, in Suzuki's method, though the formula was establised only for the case where the number of the eigenvalues which may differ from each other was two, the formula for the difference of the two potentials is more explicit and contains no unknown quantities. It is because he made full use of deformation formulas which date back to the A. Povzner's work [19] and which is also used by Gel'fand-Levitan [6]. So we shall go on this line.

The main idea of this paper is to introduce a new integral equation of the Gel'fand-Levitan type which is different from the original one. Let $c_\nu = c_\nu(x, \lambda)$ $(\nu=0, 1)$ be the solution for $\langle p, h\rangle$ defined by (1.5), while $\tilde{c}_\nu = \tilde{c}_\nu(x, \lambda)$ for $\langle \tilde{p}, \tilde{h}\rangle$. Then the deformation formulas say that there exist Volterra-type integral operators K, L such that $\tilde{c}_\nu$ $(\nu=0, 1)$ are expressed as

$$\tilde{c}_0 = c_0 + Kc_0, \quad \tilde{c}_1 = c_1 + Lc_1. \tag{1.9}$$

Now consider the kernel $F=F(x, t)$ defined by

$$I + F = (I+K)^{-1}(I+L), \tag{1.10}$$

from which a Gel'fand-Levitan type equation:

$$K(x, t) + F(x, t) + \int_0^x K(x, s)F(s, t)ds = 0 \tag{1.11}$$

is derived. Then the formulas for $\tilde{p}(x)-p(x)$, $\tilde{h}-h$ are given in terms of the Fredholm determinant of $I+F$ (see Theorem 3.4). Details of these ideas will be discussed in Section 3.

In the "degenerate case", our integral equation (1.11) has an advantage that the kernel $F(x, t)$ is reduced to a very simple form (see Theorem 4.5), so that the formulas for $\tilde{p}(x)-p(x)$, $\tilde{h}-h$ can be explicitly written down (see Remark 3.5 and Theorem A in §2). Such simplicity cannot be expected in the ordinary Gel'fant-Levitan equation. Also note that our method can avoid the complicated non-linear calculations needed in Suzuki [23] and the author's previous work, where the kernel $K(x, t)$ was considered directly not by-passing $F(x, t)$, and treat the problem within the framework of linear calculation. The inverse problem in the degenerate case will be treated in §4.

Furthermore our method can be applicable to the general case where infinitely many eigenvalues of $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ may differ from each other. Especially it is suitable to the discussion of the well-posedness problem, hence it will be made use of in this area as an alternative of the methods by Hochstadt [8] [9], Hald [7], A. Mizutani [17] and J. R. McLaughlin [16]. The well-posedness theorem (Theorem B) state that max $|\tilde{p}(\cdot)-p(\cdot)|$ and $|\tilde{h}-h|$ can be compared with $\sum_{n=0}^{\infty}|\lambda_n(\tilde{p}, \tilde{h})-\lambda_n(p, h)|$. Combining the results of the degenerate problem and the well-posedness problem (i.e. Theorem A, B), we can obtain a useful approximation formula (Theorem C) which gives an approximation $\langle p_N, h_N\rangle$ of $\langle p, h\rangle$ only in terms of the first N eigenvalues λ_n $(n=0, 1,\dots, N)$ of $\langle p, h\rangle$ and elementary functions, (trigonometric functions). The inverse problem in the general case will be considered in Section 6, while well-posedness and approximation problem will be treated in Section 7.

We shall conclude the section by emphasizing that, in the original Gel'fand-Levitan equation, one is likely to pay a one-sided attention to the boundary condition put on a only one end-point, while in ours we shall take, to a certain extent, into account of the both boundary conditions on the both end-points, thus this equation seems more suitable, in some respects, than the original one is, when the inverse Sturm-Liouville problem on a finite interval is considered.

§ 2. Statement of the main results

For the statement of our main results we have to define the other solutions $d_n(x, \lambda)$ and $r_n(x, \lambda)$ of the equation $-y''+p(x)y=\lambda y$, in addition to the solutions $c_\nu(x, \lambda)$ and $s_\nu(x, \lambda)$ defined in Section 1:

$$d_n(x, \lambda) := c_0(x, \lambda) + (-1)^{n+1}c_1(x, \lambda), \tag{2.1}$$

$$r_n(x, \lambda) := s_1(x, \lambda) + (-1)^{n}s_1(x, \lambda). \tag{2.2}$$

Note that, by virtue of Lemma 1.1, these solutions vanish identically when the λ is equal to the n-th eigenvalue of the B.V.P. $\langle p, h\rangle$ or $\langle p, \infty\rangle$ respectively.

Throughout the paper the differentiation with respect to the space variable x will be denoted by dash, while the differentiation with respect to the spectral parameter λ will be denoted by dot, i.e. "$'$"$=d/dx$, "$\cdot$"$=d/d\lambda$.

Our first theorem is concerned with the representation of the boundary value problem such that all but finite number of its eigenvalues coincide with those of a prescribely given boundary value problem. We shall hereafter refer to such given problem as the "reference problem". For the problem of the third kind, we have

Theorem A. *Let a potential $p(x) \in SBV^1$ (resp. SH^1) and a number $h \in \boldsymbol{R}$ be given. Set $\lambda_n(p, h) = \lambda_n (n=0, 1, 2, \cdots)$. Let Λ be a finite subset of non-negative integers $\{0, 1, 2, \cdots\}$ and let an increasing sequence $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ satisfy the condition:*

$$\tilde{\lambda}_n \neq \lambda_n \quad if \quad n \in \Lambda, \quad \tilde{\lambda} = \lambda_n \quad if \quad n \notin \Lambda.$$

Then there exist the unique potential $\tilde{p}(x) \in SBV^1$ (resp. SH^1) and the unique number $\tilde{h} \in \boldsymbol{R}$ such that

$$\lambda_n(\tilde{p}, \tilde{h}) = \tilde{\lambda}_n \quad (n=0, 1, 2, \cdots).$$

Such pair $\langle \tilde{p}, \tilde{h} \rangle$ is represented by the following formula:

$$\tilde{p}(x) = p(x) - 2(d/dx)^2 \log \tau(x),$$
$$\tilde{h} = h - [(d/dx) \log \tau(x)]_{x=0},$$

where the function $\tau(x)$ is given by

$$\tau(x) = \det \left\{ \delta_{m,n} + \int_0^x d_m(t, \tilde{\lambda}_m) \sum_{k \in \Lambda_2^c} a_{n,k} c_0(t, \lambda_k) dt \right\}_{m,n \in \Lambda_1^c}$$
$$= \det \left\{ \delta_{m,n} + \int_0^x \sum_{k \in \Lambda_1^c} a_{k,m} d_k(t, \tilde{\lambda}_k) c_0(t, \lambda_n) dt \right\}_{m,n \Lambda_2^c}.$$

Moreover, for the boundary condition $\tilde{h}$, there exists a simpler expression:

$$\tilde{h} = h - \sum_{m \in \Lambda_1^c} [1 + (-1)^{m+1} c_0(1, \tilde{\lambda}_m)] / \sigma_m.$$

Here we used the following notations:

$$a_{m,n} = (-1)^m \tilde{w}(\lambda_n) / \{(\tilde{\lambda}_m - \lambda_n) \tilde{w}^{\cdot}(\tilde{\lambda}_m) \dot{w}(\lambda_n)\}$$
$$= \begin{cases} \left(\dfrac{\tilde{\lambda}_n - \tilde{\lambda}_n}{\tilde{\lambda}_0 - \lambda_n}\right) \prod_{k \in \Lambda \setminus \{0\}} \left(\dfrac{\tilde{\lambda}_k - \lambda_n}{\lambda_k - \lambda_n}\right) \prod_{k=1}^{\infty} \left(\dfrac{(\pi k)^2}{\tilde{\lambda}_k - \tilde{\lambda}_m}\right) \quad (m=0) \\ \dfrac{(-1)^m (\pi m)^2}{\tilde{\lambda}_0 - \lambda_n} \left(\dfrac{\tilde{\lambda}_n - \lambda_n}{\tilde{\lambda}_m - \lambda_n}\right) \prod_{k \in \Lambda \setminus \{0\}} \left(\dfrac{\tilde{\lambda}_k - \lambda_n}{\lambda_k - \lambda_n}\right) \\ \cdot \prod_{k(\neq 0, m)} \left(\dfrac{(\pi k)^2}{\tilde{\lambda}_k - \tilde{\lambda}_m}\right) \quad (m>0), \\ \qquad \sigma_m = (-1)^{m+1} \tilde{w}^{\cdot}(\tilde{\lambda}) \end{cases}$$

where

$$w(\lambda) := c_0'(1, \lambda) + h c_0(1, \lambda) = (\lambda_0 - \lambda) \prod_{k=1}^{\infty} [(\lambda_k - \lambda)/(\pi k)^2],$$

$$\tilde{w}(\lambda) := (\tilde{\lambda}_0 - \lambda) \prod_{k=1}^{\infty} [(\tilde{\lambda}_k - \lambda)/(\pi k)^2].$$

Moreover

$$\Lambda_1^c = \Lambda \setminus \Lambda_1, \quad \Lambda_2^c = \Lambda \setminus \Lambda_2,$$

where

$$\Lambda_1 := \{m \in \Lambda;\ \text{there exists a } f(m) \in \Lambda \text{ such that } m - f(m) \text{ is even and } \tilde{\lambda}_m = \lambda_{f(m)}\},$$

$$\Lambda_2 := \{n \in \Lambda;\ \text{there exists a } g(n) \in \Lambda \text{ such that } n - g(n) \text{ is even and } \lambda_n = \tilde{\lambda}_{g(n)}\}.$$

As for the Dirichlet problem we have

Theorem A*. *Let a potential $p(x) \in SBV^1$ (resp. SH^1) be given. Set $\lambda_n(p, \infty) = \mu_n$ ($n = 1, 2, 3, \cdots$). Let Λ be a finite subset of the set of the positive integers $\{1, 2, 3, \cdots\}$ and let an increasing sequence $\{\tilde{\mu}\}_{n=1}^{\infty}$ satisfy the condition:*

$$\tilde{\mu}_n \neq \mu_n \quad \text{if} \quad n \in \Lambda, \quad \tilde{\mu}_n = \mu_n \quad \text{if} \quad n \notin \Lambda.$$

Then there exists the unique potential $\tilde{p}(x) \in SBV^1$ (resp. SH^1) such that

$$\lambda_n(\tilde{p}, \infty) = \tilde{\mu}_n \quad (n = 1, 2, 3, \cdots).$$

Such potential is represented by the formula:

$$\tilde{p}(x) = p(x) - 2(d/dx)^2 \log \omega(x),$$

where the function $\omega(x)$ is given by

$$\omega(x) = \det\left\{\delta_{m,n} + \int_0^x r_m(t, \tilde{\mu}_m) \textstyle\sum_{k \in \Lambda_2^c} b_{n,k} s_0(t, \mu_k) dt\right\}_{m,n \in \Lambda_1^c}$$

$$= \det\left\{\delta_{m,n} + \int_0^x \textstyle\sum_{k \in \Lambda_1^c} b_{k,m} r_k(t, \tilde{\mu}_k) s_0(t, \mu_n) dt\right\}_{m,n \in \Lambda_2^c}.$$

Here we used the following notations:

$$b_{m,n} = (-1)^{m+1} \tilde{v}(\tilde{\mu}_m)/((\tilde{\mu}_m - \mu_n)\tilde{v}^{\cdot}(\tilde{\mu}_m)\dot{s}_0(1, \mu_n))$$

$$= (-1)^{m+1}(\pi m)^2 \left(\frac{\tilde{\mu}_n - \mu_n}{\tilde{\mu}_m - \mu_n}\right) \prod_{k \in \setminus\{0\}} \left(\frac{\tilde{\mu}_k - \mu_n}{\mu_k - \mu_n}\right) \prod_{k(\neq m)} \left(\frac{(\pi k)^2}{\tilde{\mu}_k - \tilde{\mu}_m}\right)$$

where

$$\tilde{v}(\lambda) := \textstyle\prod_{n=1}^{\infty} [(\tilde{\mu}_k - \lambda)/(\pi k)^2].$$

The subsets Λ_1^c and Λ_2^c of Λ are defined in the same way as in Theorem A.

Remark 2.1. By the definition, note that the mappings

$$f: \Lambda_1 \longrightarrow \Lambda_2, \quad g: \Lambda_2 \longrightarrow \Lambda_1$$

are bijective and monotonically increasing, and one is the inverse mapping of the other. Of course, we may replace Λ_1^c and Λ_2^c in the representations of $\tau(x)$ and $\omega(x)$ by Λ, though the sizes of the matrices and the summations in the formulas get larger. Moreover apparently these theorems are valid for the wider potential class SL^1, the set of real valued integrable functions $p(x)$ in $0 \leqq x \leqq 1$ satisfying $p(1-x) = p(x)$.

Remark 2.2. In Suzuki [23], Theorem A and Theorem A* have been proved only in the case where $\sharp\Lambda$ (the number of the elements in Λ) is one or two. Also the Author proved the same results by a different method (unpublished). But it seems difficult to derive the general formulas such as presented here by those methods.

In this occasion, we shall anounce the key theorem by way of which, in the unpublished work, the author derived the formulas in the special case $\sharp\Lambda = 1$ or 2, since the theorem seems interesting in itself.

Theorem 2.3. *Let $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ be two symmetric B.V.P.'s of the third kind.*

1) (G. Borg [2]). *Suppose*

$$\lambda_n(p, h) = \lambda_n(\tilde{p}, \tilde{h}) \quad (n = 0, 1, 2, \cdots),$$

then

$$\langle p, h\rangle = \langle \tilde{p}, \tilde{h}\rangle.$$

2) *For a fixed non-negative integer N, suppose that*

$$\lambda_n(p, h) = \lambda_n(\tilde{p}, \tilde{h}) \qquad \text{for} \quad n \neq N \quad \text{with} \quad n = 0, 1, 2, \cdots,$$
$$\lambda_N(p, h) \neq \lambda_N(\tilde{p}, \tilde{h}).$$

Let an integer M be defined by

$$M := \begin{cases} N-1 \\ N \\ N+1 \end{cases} \quad \text{according to } (\tilde{\lambda}_N - \lambda_N)(\tilde{h} - h) \begin{cases} > 0 \\ = 0 \\ < 0, \end{cases}$$

where we write as $\lambda_N := \lambda_N(p, h)$, $\tilde{\lambda}_N := \lambda_N(\tilde{p}, \tilde{h})$ for brevity. Then the following facts hold:

$$\lambda_n\left(p,\, h+\frac{\tilde{\lambda}_N-\lambda_N}{\tilde{h}-h}\right)=\lambda_n\left(\tilde{p},\, \tilde{h}+\frac{\tilde{\lambda}_N-\lambda_N}{\tilde{h}-h}\right)$$

for $n\neq M$ *with* $n=0, 1, 2,\cdots$, *if* $h\neq\tilde{h}$ (*resp.* $n=1, 2, 3,\cdots$, *if* $h=\tilde{h}$), *and*

$$\lambda_N(p,\, h)=\lambda_M\left(\tilde{p},\, \tilde{h}+\frac{\tilde{\lambda}_N-\lambda_N}{\tilde{h}-h}\right),$$

$$\lambda_N(\tilde{p},\, \tilde{h})=\lambda_M\left(p,\, h+\frac{\tilde{\lambda}_N-\lambda_N}{\tilde{h}-h}\right).$$

In view of the statement 1), *it cannot happen that* $M\leqq -1$ (*resp.* $M\leqq 0$) *when* $h\neq\tilde{h}$ (*resp.* $h=\tilde{h}$).

In paricular we obtain

Corollary 2.4. (if and only if part→G. Borg [2]). *Suppose*

$$\lambda_n(p,\, h)=\lambda_n(\tilde{p},\, \tilde{h})\quad (n=1, 2, 3,\cdots),$$

then

$$\{\lambda_0(\tilde{p},\, \tilde{h})-\lambda_0(p,\, h)\}(\tilde{h}-h)\leqq 0.$$

The equality holds if and only if $\langle p,\, h\rangle=\langle\tilde{p},\, \tilde{h}\rangle$.

Corollary 2.5. (Suzuki [21]). *Let a boundary condition h be fixed and suppose that*

$$\lambda_n(p,\, h)=\lambda_n(\tilde{p},\, h)\qquad \textit{for}\quad n\neq N\quad \textit{with}\quad n=0, 1, 2,\cdots,$$

then the N-th eigenvalue of the $\langle p,\, h\rangle$ (*resp.* $\langle\tilde{p},\, h\rangle$) *is equal to the N-th eigenvalue of the Dirichlet problem* $\langle\tilde{p},\, \infty\rangle$ (*resp.* $\langle p,\, \infty\rangle$).

As immediate corollaries to Theorem A and Theorem A*, we obtain

Corollary to Theorem A. *In particular, if we choose* ⟨*the zero potential, the Neumann boundary condition*⟩ *as the reference problem, then the n-th eigenvalue of the reference problem is* $(\pi n)^2$ $(n=0, 1, 2,\cdots)$. *Thus let a sequence* $\{\lambda_n\}_{n=0}^{\infty}$ *be such that*

$$\lambda_0<\lambda_1<\cdots<\lambda_N<\{\pi(N+1)\}^2,\quad \lambda_n=(\pi n)^2\quad (n\geqq N+1)$$

for a fixed N $(\geqq 1)$. *Then the B.V.P.* $\langle p,\, h\rangle$ *such that*

$$\lambda_n(p,\, h)=\lambda_n\quad (n=0, 1, 2,\cdots)$$

is given by the formula:

$$\begin{cases} p(x) = -2(d/dx)^2 \log \tau(x), \\ h = -(d/dx) \log \tau(x)|_{x=0}, \end{cases}$$

where $\tau(x)=$

$$= \det\Big(\delta_{m,n} + \sum_{k=0}^{N} a_{n,k}\Big(\frac{\sin(\lambda_m^{1/2}+\pi kx)+(-1)^m[\sin\{\lambda_m^{1/2}(1-x)+\pi kx\}-\sin\lambda_m^{1/2}]}{2(\lambda_m^{1/2}+\pi k)}$$

$$+\frac{\sin(\lambda_m^{1/2}x-\pi kx)+(-1)^m[\sin\{\lambda_m^{1/2}(1-x)+\pi kx\}-\sin\lambda_m^{1/2}]}{2(\lambda_m^{1/2}-\pi k)}\Big)\Big)_{m,n=0}^{N},$$

where the $a_{m,n}$ *can be written as the following finite products:*

$$a_{m,n} = \frac{(-1)^m}{\lambda_m^{1/2}\sin\lambda_m^{1/2}} \frac{[\lambda_m-(\pi m)^2][\lambda_n-(\pi n)^2]}{\lambda_m-(\pi n)^2}$$

$$\cdot \prod_{\substack{k=0 \\ (k\neq m)}}^{N}\left(\frac{\lambda_m-(\pi k)^2}{\lambda_m-\lambda_k}\right)\prod_{\substack{k=0 \\ (k\neq n)}}^{N}\left(\frac{(\pi n)^2-\lambda_k}{(\pi n)^2-(\pi k)^2}\right).$$

Moreover the boundary condition admits another representation:

$$h = \sum_{k=0}^{N} \frac{1+(-1)^{m+1}\cos\lambda_m^{1/2}}{\dfrac{(-1)^m\lambda_m^{1/2}\sin\lambda_m^{1/2}}{\lambda_m-(\pi m)^2}\displaystyle\prod_{\substack{k=0 \\ (k\neq m)}}^{N}\left(\frac{\lambda_m-\lambda_k}{\lambda_m-(\pi k)^2}\right)}.$$

Corollary to Theorem A*. *In particular, if we choose* $\langle$*the zero potential,* $\infty\rangle$ *as the reference problem, then the n-th eigenvalue of the reference problem is* $(\pi n)^2$ $(n=1, 2, 3,\cdots)$. *Thus let a sequence* $\{\mu_n\}_{n=1}^{\infty}$ *be such that*

$$\mu_1 < \mu_2 < \cdots < \mu_N < \{\pi(N+1)\}^2, \quad \mu_n = (n\pi)^2 \quad (n \geqq N+1)$$

for a fixed N $(\geqq 1)$. *Then the B.V.P.* $\langle p, \infty\rangle$ *such that*

$$\lambda_n(p, \infty) = \mu_n \quad (n=1, 2, 3,\cdots)$$

is represented by the formula:

$$p(x) = -2(d/dx)^2 \log \omega(x),$$

where $\omega(x)=$

$$\det\Big(\delta_{m,n} + \sum_{k=1}^{N} b_{n,k}\Big(\frac{\sin(\mu_m^{1/2}x+\pi kx)+(-1)^m[\sin\{\mu_m^{1/2}(1-x)+\pi kx\}-\sin\mu_m^{1/2}]}{2(\mu_m^{1/2}-\pi k)}$$

$$-\frac{\sin(\mu_m^{1/2}x-\pi kx)+(-1)^m[\sin\{\mu_m^{1/2}(1-x)-\pi kx\}-\sin\mu_m^{1/2}]}{2(\mu_m^{1/2}+\pi k)}\Big)\Big)_{m,n=1}^{N},$$

where the $b_{n,k}$ *can be written as the following finite products:*

$$b_{m,n} = \frac{(-1)^m \mu_m^{1/2}}{\sin \mu_m^{1/2}} \frac{[\mu_n = (\pi n)^2][\mu_m - (\pi m)^2]}{\mu_m - (\pi n)^2}$$

$$\cdot \prod_{\substack{k=1 \\ (k \neq m)}}^{N} \left(\frac{\mu_m - (\pi k)^2}{\mu_m \mu_k} \right) \prod_{\substack{k=1 \\ (k \neq n)}}^{N} \left(\frac{(\pi n)^2 - \mu_k}{(\pi n)^2 - (\pi k)^2} \right).$$

The second theorem is concerned with the well-posedness of the inverse problem.

Theorem B. (Well-posedness Theorem). *Let a reference problem* $\langle p, h \rangle \in SBV^1 \times \boldsymbol{R}$ *(resp.* $SH^1 \times \boldsymbol{R}$*) be given. Set* $\lambda_n := \lambda_n(p, h)$ $(n = 0, 1, 2, \cdots)$. *Then there exist positive constants* C *and* C' *depending only on* $\langle p, h \rangle$ *such that, for any unknown problem* $\langle \tilde{p}, \tilde{h} \rangle \in SBV^1 \times \boldsymbol{R}$ *(resp.* $SH^1 \times \boldsymbol{R}$*) with eigenvalues* $\{\tilde{\lambda}_n\}_{n=0}^{\infty} := \{\lambda_n(\tilde{p}, \tilde{h})\}_{n=0}^{\infty}$, *the following estimates are valid provided that the sum* $\sum_{m=0}^{\infty} |\tilde{\lambda}_m - \lambda_m|$ *is smaller than* C':

$$\|\tilde{p}(\cdot) - p(\cdot)\|_\infty \leqq C \sum_{m=0}^{\infty} |\tilde{\lambda}_m - \lambda_m|,$$

$$|\tilde{h} - h| \leqq C \sum_{m=0}^{\infty} (1+m)^{-1} |\tilde{\lambda}_m - \lambda_m|,$$

where $\|\cdot\|_\infty$ *denotes the maximum norm on the interval* $0 \leqq x \leqq 1$. *Similarly let the* $\langle p, \infty \rangle \in SBV^1$ *(resp.* SH^1*) be given. Set* $\mu_n := \lambda_n(p, \infty)$ $(n = 1, 2, 3, \cdots)$. *Then there exist positive constants* D *and* D' *depending only on* $p(x)$ *such that, for any unknown problem* $\langle \tilde{p}, \infty \rangle \in SBV^1$ *(resp.* SH^1*) with eigenvalues* $\{\tilde{\mu}_n\}_{n=1}^{\infty} := \{\lambda_n(\tilde{p}, \infty)\}_{n=1}^{\infty}$, *the following estimate holds*:

$$\|\tilde{p}(\cdot) - p(\cdot)\|_\infty \leqq D \sum_{m=1}^{\infty} |\tilde{\mu}_m - \mu_m|,$$

provided that $\sum_{m=1}^{\infty} |\tilde{\mu}_m - \mu_m|$ *is smaller than* D'.

Remark 2.6. The similar result was proved by A. Mizutani [17] for more general setting where the simmetricity condition does not need to be assumed, though in his work the boundary conditions were not perturbed. He made use of the slightly modified version of the original Gel'fand-Levitan's inversion procedure. See also J. R. MacLauhlin [16] in this connection. We shall later prove the theorem in an alternative way by appealing to a different type of Gel'fand-Levitan equation, which seems more suitable in this kind of arguments. On the other hand, O. Hald [7] also proved the analogous theorem, where the statement is more global but it has the decisive disadvantage that the constants C and D depend also on the unknown problems.

Finally the third theorem will be concerned with the approximation of the potential (and the boundary condition) of the given B.V.P. in terms of the quantities defined by using elementary functions. Before mentioning the theorem, we must provide some notations.

Let a potential $p(x) \in SBV^1$ (or SH^1) and a number $h \in \boldsymbol{R}$ be given. For the brevity, we write as

$$\text{(2.3)} \qquad \begin{aligned} \lambda_n &:= \lambda_n(p, h) \quad (n=0, 1, 2,\cdots) \\ \mu_n &:= \lambda_n(p, \infty) \quad (n=1, 2, 3,\cdots). \end{aligned}$$

Then recall (see (1.3) and (1.4)) that the asymptotic distributions of these sequences are given by

$$\text{(2.4)} \qquad \lambda_n = (\pi n)^2 + \alpha_0 + \varepsilon_n, \quad \mu_n = (\pi n)^2 + \beta_0 + \delta_n,$$

with $\varepsilon_n, \delta_n \to 0$ as $n \to \infty$, where

$$\text{(2.5)} \qquad \alpha_0 = 4h + \int_0^1 p(t)dt, \quad \beta_0 = \int_0^1 p(t)dt.$$

Hence let the sequences $\{\lambda_n^*\}_{n=0}^{\infty}$ and $\{\mu_n^*\}_{n=1}^{\infty}$ be defined by

$$\text{(2.6)} \qquad \lambda_n = \lambda_n^* + \alpha_0, \quad \mu_n = \mu_n^* + \beta_0,$$

then, by (2.4), there exists an integer N_0 such that the inequality $\lambda_N^* < \{\pi(N+1)\}^2$ and $\mu_N^* < \{\pi(N+1)\}^2$ hold for all $N \geqq N_0$. With $\{\lambda_0^*, \lambda_1^*,\cdots, \lambda_N^*\}$ (resp. $\{\mu_1^*, \mu_2^*,\cdots, \mu_N^*\}$) in place of $\{\lambda_0, \lambda_1,\dots, \lambda_N\}$ (resp. $\{\mu_1, \mu_2,\dots, \mu_N\}$) in the statement of Corollary to Theorem A (resp. Corollary to Theorem A*), we calculate the function $\tau(x)$ (resp. $\omega(x)$) and the real number h on the basis of the formulas given in these corollaries, and we will denote these by $\tau_N(x)$ and h_N (resp. $\omega_N(x)$) in order to emphasize the dependence on the N. Moreover let $p_N(x)$ and $\bar{p}_N(x)$ be defined by

$$\text{(2.7)} \qquad \begin{aligned} p_N(x) &= \alpha_0 - 2(d/dx)^2 \log \tau_N(x), \\ \bar{p}_N(x) &= \beta_0 - 2(d/dx)^2 \log \omega_N(x) \end{aligned}$$

Then the B.V.P.'s $\langle p_N, h_N \rangle$ (resp. $\langle \bar{p}_N, \infty \rangle$) approximate the original B.V.P. $\langle p, h \rangle$ (resp. $\langle p, \infty \rangle$). More strictly speaking, we have

Theorem C. (Approximation Theorem). *If a potential $p(x)$ is in SBV^1, then as $N \to \infty$*

$$\|p(\cdot) - p_N(\cdot)\|_\infty = O(N^{-1}), \quad \|p(\cdot) - \bar{p}_N(\cdot)\|_\infty = O(N^{-1}).$$

As for the boundary condition, we have

$$|h - h_N| = O(N^{-2}) \quad as \quad N \longrightarrow \infty.$$

If a potential $p(x)$ is in SH^1, then as $N \to \infty$

$$\|p(\cdot)-p_N(\cdot)\|_\infty = o(N^{-1/2}), \quad \|p(\cdot)-\bar{p}_N(\cdot)\|_\infty = o(N^{-1/2}).$$

As for the boundary condition, we have

$$|h-h_N| = o(N^{-3/2}) \quad as \quad N \longrightarrow \infty.$$

Remark 2.7. Note that the approximation potentials $p_N(x)$, $\bar{p}_N(x)$ and the approximation boundary condition h_N are written only interms of the elementary functions. Hence this theorem has a merit in practice. In fact, let a certain physical problem be subjected to an unknown boundary value problem $\langle p, h\rangle$, and suppose that we have obtained the values $\lambda_0, \lambda_1, \cdots, \lambda_N$ as the first, second,$\cdots$, the N-th eigenvalues of this B.V.P. by an experimental measurament. If the number N is sufficiently large, we may determine, from the experimental deta $\{\lambda_0, \lambda_1, \cdots, \lambda_N\}$, the second term α_0 in the asymptotic distribution formula (2.4). With these values $\lambda_0, \lambda_1, \cdots, \lambda_N$ and α_0, we calculate the function $p_N(x)$ and the number h_N, thereby we can get an approximation of the unknown problem in terms of the elementary functions.

Remark 2.8. Though, in the statement of Theorem C, we choose the B.V.P.'s $\langle$zero potential, Neumann boundary condition$\rangle$ and $\langle$zero potential, Dirichlet boundary condition$\rangle$ as the base points, it is obvious that the similar assertion well be valid if we choose the general B.V.P.'s $\langle p, h\rangle$ and $\langle p, \infty\rangle$ as the base points.

Theorem A will be proved in Section 4 and 5, and Theorem B and C will be proved in Section 7. The proof of Theorem 2.3 will be left until Section 8, since it digresses from our main subject.

§3. Deformation formulas and the Gel'fand-Levitan type equation

In the inverse Sturm-Liouville problems and other related fields, it has produced many fruitful results to utilize the deformation kernels of Volterra type which transform the solution of one differential equation containing a spectral parameter into that of another differential equation, and that does not depend on the spectral parameter. On these matters, the reader should refer the article by Suzuki [24] and literatures cited there.

Now we shall provide several deformation formulas which will be made use of to construct an integral equation of Gel'fand-Levitan type. In order to avoid the confusion whether the potentials under consideration belong to SBV^1 or SH^1, we shall introduce the function class W^1 collecting the real-valued functions on $0 \leqq x \leqq 1$ which have the integrable first derivative, then $SBV^1 \subset W^1$ and $SH^1 \subset W^1$.

For a function $p(x)$ in the class W^1 and for a real number $h \in \boldsymbol{R}$, we shall define the solutions $c_\nu(x, \lambda)$ and $s_\nu(x, \lambda)$ $(\nu=0, 1)$ to the differential equation $-y''+p(x)y=\lambda y$ by requiring the initial conditions (1.5). Similarly we define the solutions $\tilde{c}_\nu(x, \lambda)$ and $\tilde{s}_\nu(x, \lambda)$ corresponding to another function $\tilde{p}(x)$ in W^1 and another number $\tilde{h} \in \boldsymbol{R}$. Then we have the following lemma.

Lemma 3.1. *There exist unique integral kernels* $K(x, t)$, $\tilde{K}(x, t)$, $M(x, t)$, $\tilde{M}(x, t) \in W^2(\{0 \leqq t \leqq x \leqq 1\})$ *and* $L(x, t)$, $N(x, t) \in W^2(\{0 \leqq x \leqq t \leqq 1\})$, *where* $W^2(\Omega)$ *denotes the set of real functions in* Ω *which have integrable deivatives up to the second order inclusive, such that the following deformation formulas hold*:

$$(3.1)_0 \qquad \tilde{c}_0(x, \lambda) = c_0(x, \lambda) + \int_0^x K(x, t)c_0(t, \lambda)dt,$$

$$(3.1)_1 \qquad \tilde{c}_1(x, \lambda) = c_1(x, \lambda) + \int_x^1 L(x, t)c_1(t, \lambda)dt,$$

$$(3.2)_0 \qquad \tilde{s}_0(x, \lambda) = s_0(x, \lambda) + \int_0^x M(x, t)s_0(t, \lambda)dt,$$

$$(3.2)_1 \qquad \tilde{s}_1(x, \lambda) = s_1(x, \lambda) + \int_x^1 N(x, t)s_1(t, \lambda)dt,$$

$$(3.3) \qquad c_0(x, \lambda) = \tilde{c}_0(x, \lambda) + \int_0^x \tilde{K}(x, t)\tilde{c}_0(t, \lambda)dt,$$

$$(3.4) \qquad s_0(x, \lambda) = \tilde{s}_0(x, \lambda) + \int_0^x \tilde{M}(x, r)\tilde{s}_0(t, \lambda)dt.$$

These integral kernels are characterized as the (*unique*) *solutions of the following hyperbolic problems*:

$$(3.5) \qquad \begin{cases} K_{xx} - K_{tt} = [\tilde{p}(x) - p(t)]K & \text{in } \ 0 < t < x < 1, \\ K(x, x) = \tilde{h} - h + (1/2)\int_0^x [\tilde{p}(t) - p(t)]dt, \\ K_t(x, t)|_{t=0} = hK(x, 0), \end{cases}$$

$$(3.6) \qquad \begin{cases} L_{xx} - L_{tt} = [\tilde{p}(x) - p(t)]L & \text{in } \ 0 < x < t < 1, \\ L(x, x) = \tilde{h} - h + (1/2)\int_x^1 [\tilde{p}(t) - p(t)]dt, \\ L_t(x, t)|_{t=1} = -hL(x, 1), \end{cases}$$

$$(3.7) \qquad \begin{cases} M_{xx} - M_{tt} = [\tilde{p}(x) - p(t)]M & \text{in } \ 0 < t < x < 1, \\ M(x, x) = (1/2)\int_0^x [\tilde{p}(t) - p(t)]dt, \\ M(x, 0) = 0, \end{cases}$$

$$(3.8)\qquad \begin{cases} N_{xx} - N_{tt} = [\tilde{p}(x) - p(t)]N & in \quad 0 < x < t < 1, \\ N(x, x) = (1/2)\int_x^1 [\tilde{p}(t) - p(t)]dt, \\ N(x, 1) = 0, \end{cases}$$

$$(3.9)\qquad \begin{cases} \tilde{K}_{xx} - \tilde{K}_{tt} = [p(x) - \tilde{p}(t)]\tilde{K} & in \quad 0 < t < x < 1, \\ \tilde{K}(x, x) = h - \tilde{h} + (1/2)\int_0^x [p(t) - \tilde{p}(t)]dt, \\ \tilde{K}_t(x, t)|_{t=0} = \tilde{h}\tilde{K}(x, 0), \end{cases}$$

$$(3.10)\qquad \begin{cases} \tilde{M}_{xx} - \tilde{M}_{tt} = [p(x) - \tilde{p}(t)]\tilde{M} & in \quad 0 < t < x < 1, \\ \tilde{M}(x, x) = (1/2)\int_0^x [p(t) - \tilde{p}(t)]dt, \\ \tilde{M}(x, 0) = 0. \end{cases}$$

Proof. This sort of lemma is well-known. See Suzuki [20] for the proof.

In what follows, unless stated otherwise, the B.V.P's $\langle p, h\rangle$ and $\langle p, \infty\rangle$ will be regarded as known and referred to as "reference problem", while the B.V.P.'s $\langle \tilde{p}, \tilde{h}\rangle$ and $\langle \tilde{p}, \infty\rangle$ will be regarded as unknown. In the previous works by Suzuki [20] [21] [22] [23] [24], the kernel $K(x, t)$ itself was put in the subject of investigation. But such direct approach involved many difficulties, since the kernel $K(x, t)$ depends on the unknown potentials $\tilde{p}(x)$ to a undefiable extent as is implied by the partial differential equation (3.5). Thus, instead of considering the kernel $K(x, t)$ directly, we shall adopt the following device:

The integral kernels $K(x, t)$, $L(x, t)$, $M(x, t)$, $N(x, t)$, $\tilde{K}(x, t)$ and $\tilde{M}(x, t)$ which are defined on the triangular domains $\{0 \leqq t \leqq x \leqq 1\}$ or $\{0 \leqq x \leqq t \leqq 1\}$ will be extended to the functions on $\{0 \leqq x, t \leqq 1\}$ by putting the values zero outside the original domains of definition, and the integral operators on the interval $0 \leqq x \leqq 1$ corresponding to these kernels will be denoted by K, L, M, N, $\tilde{K}$ and $\tilde{M}$ respectively. Then we shall consider the integral operator F and G defined by the formula:

$$(3.11)\qquad I + F = (I + K)^{-1}(I + L) = (I + \tilde{K})(I + L),$$

$$(3.12)\qquad I + G = (I + M)^{-1}(I + N) = (I + \tilde{M})(I + N),$$

where I denotes the identity mapping. The corresponding integral kernels are given by

$$(3.13)\qquad F(x, t) = \tilde{K}(x, t) + \int_0^t \tilde{K}(x, s)L(s, t)ds \quad ; x > t,$$

$$(3.14)\qquad F(x, t) = L(x, t) + \int_0^x \tilde{K}(x, s)L(s, t)ds \quad ; x < t,$$

(3.15) $$G(x, t) = \tilde{M}(x, t) + \int_0^t \tilde{M}(x, s)N(s, t)ds;\ x > t,$$

(3.16) $$G(x, t) = N(x, t) + \int_0^x \tilde{M}(x, s)N(s, t)ds\ ;\ x < t.$$

Note that the kernels $F(x, t)$ and $G(x, t)$ are of the class W^2 of the respective domains $\{0 \leqq t \leqq x \leqq 1\}$ and $\{0 \leqq x \leqq t \leqq 1\}$, but they may have gaps on the diagonal line $x = t$. Some properties of these kernels are stated in the following

Lemma 3.2. *In each domain $\{0<t<x<1\}$ and $\{0<x<t<1\}$, the kernels $F(x, t)$ and $G(x, t)$ satisfy the following inhomogeneous hyperbolic equations respectively*:

(3.17) $$\Box_p F = \tilde{K}(x, 0)\,[L_x(x, t) - \tilde{h}L(x, t)]_{x=0},$$

(3.18) $$\Box_p G = -\tilde{M}_t(x, t)\,|_{t=0}\,N(0, t),$$

where $\Box_p$ denotes the modified d'Alembertian:

(3.19) $$\Box_p = (\partial/\partial x)^2 - (\partial/\partial t)^2 - p(x) + p(t).$$

Furthermore the gaps of the kernels $F(x, t)$ and $G(x, t)$ on the diagonal line are expressed as

(3.20) $$F(x, x+0) - F(x, x-0) = 2(\tilde{h} - h) + (1/2)\int_0^1 [\tilde{p}(t) - p(t)]dt$$

(3.21) $$G(x, x+0) - G(x, x-0) = (1/2)\int_0^1 [\tilde{p}(t) - p(t)]dt.$$

Proof. This lemma is shown by elementary but somewhat complicated calculations. Indeed the equations (3.17) and (3.18) follows from the applications of (3.5)–(3.10) to (3.13)–(3.16) with twice integration by parts. While (3.20) and (3.21) also follows from (3.13)–(3.16) and (3.5)–(3.10).

Remark 3.3. The hyperbolic equations (3.17) and (3.18) are simpler than (3.5)–(3.10) in that the coefficients of these equations consist of the known function $p(\cdot)$, even though the inhomogeneous terms appear newly, but these terms are separated to the functions in x and in t, so that their treatment will be turn out to be easy.

In view of the asymptotic formulas of eigenvalues (1.3) and (1.4), the equations (3.20) and (3.21) imply that the kernels $F(x, t)$ (resp. $G(x, t)$) have no gap on the diagonal line if and only if the second terms in the asymptotic formulas for two B.V.P's $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ (resp. $\langle p, \infty\rangle$ and $\langle \tilde{p}, \infty\rangle$) coincide, that is, $a_0(p, h) = a_0(\tilde{p}, \tilde{h})$ (resp. $b_0(p) = b_0(\tilde{p})$).

From (3.11) and (3.12) the integral equations of Gel'fand-Levitan type follows immediately:

$$K(x, t) + F(x, t) + \int_0^x K(x, s)F(s, t)ds = 0: 0 < t < x, \tag{3.22}$$

$$M(x, t) + G(x, t) + \int_0^x K(x, s)G(s, t)ds = 0: 0 < t < x. \tag{3.23}$$

Let F_x denote the integral operator defined by the kernel $F(s, t)$ on the interval $[0, x]$ with $0<x\leqq 1$, similarly G_x denote the operator on $[0, x]$ associated with $G(s, t)$. Then the differences of the two potentials and the two boundary conditions are given in the following theorem:

Theorem 3.4. *Let* $p(x)$ *and* $\tilde{p}(x)$ *be in* W^1. *Suppose that*

$$F(x, x+0) = F(x, x-0), \quad G(x, x+0) = G(x, x-0).$$

By Remark 3.3., *these assumption is equivalent to the conditions*

$$a_0(p, h) = a_0(\tilde{p}, \tilde{h}), \quad b_0(p) = b_0(\tilde{p}),$$

in the asymptotic distribution formulas (1.3) *and* (1.4). *Then the "Gel'fand-Levitan equation"* (3.22) *and* (3.23) *are uniquely solvable, that is,* $\det(I+F_x)\neq 0$, $\det(I+G_x)\neq 0$, *and the following formulas hold*:

$$\begin{cases} \tilde{p}(x) = p(x) - 2(d/dx)^2 \log\det(I+F_x), \\ \tilde{h} = h - [(d/dx)\log\det(I+F_x)]_{x=0}, \end{cases} \tag{3.24}$$

$$\tilde{p}(x) = p(x) - 2(d/dx)^2 \log\det(I+G_x), \tag{3.25}$$

where "det" *denotes the Fredholm determinant.*

Proof. We shall only show (3.24). Let F^ξ be an operator defined by

$$I + F^\xi = (I+\xi K)^{-1}(I+\xi L),$$

where ξ is a complex parameter. Then, as is easily seen, $u_x(t)=\xi K(x, t)$ satisfies the Fredholm equation:

$$u_x(t) + F^\xi(x, t) + \int_0^x u_x(s)F^\xi(s, t)ds = 0, \quad 0 < t < x, \tag{3.26}$$

for each fixed $x\in(0, 1]$, where $F^\xi(x, t)$ denotes the kernel of the operator F^ξ. An elementary argument in the theory of Volterra's integral equations shows that F^ξ is entire, regarded as an operator-valued function of ξ, and that the integral kernel $F^\xi(x, t)$ satisfies the equality:

$$F^{\xi}(x, x+0) - F^{\xi}(x, x-0) = \xi K(x, x) + \xi L(x, x),$$

which holds for any $\xi \in \boldsymbol{C}$. But, using (3.5), (3.9), (3.13) and (3.14), the assumption $F(x, x-0)=F(x, x+0)$ is rewritten as $K(x, x)+L(x, x)=0$. Therefore the kernel $F^{\xi}(x, t)$ has no gap on the diagonal line, and is a continuous kernel, whence a Fredholm theory is applicable. By the definition, for $|\xi|<\varepsilon$ (sufficiently small), we may assume that $\|F_x^{\xi}\| \leqq \delta$ (<1) for $0 \leqq x \leqq 1$, hence $\det(I+F_x^{\xi}) \neq 0$ for such ξ's. Then, applying a Fredholm formula to (3.26), we obtain

$$\begin{aligned}\log\det(I+F_x^{\xi}) &= \operatorname{tr}\log(1+F_x^{\xi}) = \textstyle\sum_{n=1}^{\infty}(-1)^{n+1}n^{-1}\operatorname{tr}(F_x^{\xi})\\ &= \textstyle\sum_{n=1}^{\infty}(-1)^{n+1}n^{-1}\displaystyle\int_0^x dx_1 \cdots \int_0^x dt_{n-1} F^{\xi}(x, t_1)F^{\xi}(x, t_2)\cdots F^{\xi}(t_{n-1}, x).\end{aligned}$$

Differentiating the both sides of this formula with respect to x, we obtain:

$$\begin{aligned}(3.27)\quad -(d/dx)\log\det(I+F_x^{\xi}) &= F^{\xi}(x, x)\\ &+ \textstyle\sum_{n=1}^{\infty}(-1)^{n}\displaystyle\int_0^x dt_1 \cdots \int_0^x dt_{n-1} F^{\xi}(x, t_1)F^{\xi}(t_1, t_2)\cdots F^{\xi}(t_{n-1}, x).\end{aligned}$$

On the other hand, for $|\xi|<\varepsilon$, the "Gel'fand-Levitan equation" (3.26) has a unique solution $u_x(t)=\xi K(x, t)$ which can be also expressed as a C. Neumann series. In particular, putting $x=t$, the diagonal element of this C. Neumann series is seen to be equal to the right-hand side of (3.27). Hence we obtain

$$-(d/dx)\log\det(I+F_x^{\xi}) = \xi K(x, x) \qquad \text{for} \quad |\xi| < \varepsilon,$$

or equivalently

$$\det(I+F_x^{\xi}) = \exp\Big(-\xi\int_0^x K(t, t)\,dt\Big) \qquad \text{for} \quad |\xi| < \varepsilon.$$

Since the both sides of the above formula are entire in ξ, putting $\xi=1$, we have

$$(3.28)\qquad \det(I+F_x) = \exp\Big(-\int_0^x K(t, t)\,dt\Big) \neq 0.$$

Hence the integral equation (3.22) is uniquely solvable. The derivation of (3.24) is now immediate by using (3.5) and (3.28). Hence the theorem is proved.

Remark 3.5. When the integral kernel is of the degenerate form, i.e.

$$F(x, t) = \textstyle\sum_{n=1}^{N} a_n(x)b_n(t),$$

then the Fredholm determinant $\det(I+F_x)$ is written in the explicit form:

$$\det(I+F_x) = \det\Big(\delta_{i,j} + \int_0^x a_i(t)b_j(t)\,dt\Big)_{i,j=1}^{N}.$$

One of our main purpose is to show that, when the number of the eigenvalues of the two B.V.P.'s $\langle p, H\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ (resp. $\langle p, \infty\rangle$ and $\langle \tilde{p}, \infty\rangle$) which are different from each other is at most finite, then the kernel $F(x, t)$ reduces to a degenerate form, and to find its expicit form. After this, Lemma 3.4 and Remark 3.5 will be used in order to express the unknown problem $\langle \tilde{p}, \tilde{h}\rangle$ (resp. $\langle \tilde{p}, \infty\rangle$) in terms of the quantities associated with the reference problem $\langle p, h\rangle$ (resp. $\langle p, \infty\rangle$).

We shall end this section with providing a simple lemma which will be often used in the subsequent sections.

Lemma 3.6. *Let ℓ be a linear operator acting on a $\boldsymbol{C}$-linear function space defined on the interval J, and let $f_1(x), f_2(x), \dots, f_N(x)$ be the functions in this space each of which is not identically zero on a subinterval $\tilde{J}$. Suppose that for mutually different numbers $\alpha_1, \alpha_2, \cdots, \alpha_N \in \boldsymbol{C}$, the equations $\ell\,(f_n) = \alpha_n f_n$ $(n=1, 2, \cdots, N)$ hold, then the functions $f_1(x), f_2(x), \cdots, f_N(x)$ are liniearly independent on the interval $\tilde{J}$.*

Proof. This lemma is an immediate consequence of the regularity of the van der Monde determinant.

§ 4. Degenerate problem and formula for the difference of two potentials

In this section, under the assumption that two B.V.P.'s $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ are in the "degenerate case", that is, $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ share their eigenvalues in common with only finite number of exceptions, we shall determine the forms fo the kernels $F(x, t)$ and $G(x, t)$. Such problem will be refered to as "degenerate problem". Subsequently, on the basis of Theorem 3.4, we shall derive the formulas for the differences of the two potentials and the two boundary conditions which are stated in Theorem A and A*. But the proof of the converse that the $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ defined by these formuls actually have the prescribed eigenvalues will be postponed until Section 5. The consideration will be made chiefly for the problem of the third kind, and that for the Dirichlet problem will be mentioned briefly in the last part of this section.

Let $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ be two symmetric problems, Λ be a finite subset of the set of non-negative integers, and suppose that

$$(4.1)\qquad \begin{aligned} \lambda_n(p, h) &\neq \lambda_n(\tilde{p}, \tilde{h}) \qquad \text{for} \quad n \in \Lambda, \\ \lambda_n(p, h) &= \lambda_n(\tilde{p}, \tilde{h}) \qquad \text{for} \quad n \notin \Lambda. \end{aligned}$$

In the sequel, for brevity, we shall write as

$$(4.2)\qquad \lambda_n := \lambda_n(p, h), \quad \tilde{\lambda}_n := \lambda_n(\tilde{p}, \tilde{h}) \quad (n=0, 1, 2, \cdots)$$

Note that the assumption (4.1) combined with (1.3) and the second paragraph in Remark 3.3 implies that the kernel $F(x, t)$ has no gap on the diagonal line:

$$(4.3)\quad F(x, x+0) - F(x, x-0) = 2(\tilde{h}-h) + \frac{1}{2}\int_0^1 [\tilde{p}(t)-p(t)]dt = 0.$$

By virtue of the symmetricity relation (1.7) in Lemma 1.1, putting $\lambda=\lambda_n$ in the deformation formula $(3.1)_1$ in Lemma 3.1, we find that

$$(-1)^n\tilde{c}_1(x, \lambda_n) = c_0(x, \lambda_n) + \int_x^1 L(x, t)c_0(t, \lambda_n)dt.$$

Hence, subtracting this equation from $(3.1)_0$ with $\lambda=\lambda_n$ to obtain

$$\tilde{c}_0(x, \lambda_n) + (-1)^{n+1}\tilde{c}_1(x, \lambda_n) = \int_0^1 \{K(x, t)-L(x, t)\}c_0(t, \lambda_n)dt,$$

where we regard $K(x, t)$ (resp. $L(x, t)$) as a function of $0\leqq t\leqq 1$ by putting $K(x, t)=0$ for $x<t<1$ (resp. $L(x, t)=0$ for $0<t<x$). Since $\{c_0(\cdot, \lambda_n)\}_{n=0}^{\infty}$ forms a completely orthogonal system in $L^2(0, 1)$, it follows that

$$(4.4)\quad K(x, t) = \sum_{n\in\Lambda} \frac{\tilde{d}_n(x, \lambda_n)c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda_n)\|^2} \quad (0<t<x<1),$$

$$(4.5)\quad L(x, t) = -\sum_{n\in\Lambda} \frac{\tilde{d}_n(x, \lambda_n)c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda_n)\|^2} \quad (0<x<t<1),$$

where $\tilde{d}_n(x, \lambda)$ is defined by

$$(4.6)\quad \tilde{d}_n(x, \lambda) := \tilde{c}_0(x, \lambda) + (-1)^{u+1}\tilde{c}_1(x, \lambda).$$

Note that Lemma 1.1 and assumption (4.1) imply that $\tilde{d}_n(x, \lambda)$ vanishes identically for $\lambda=\lambda_n$ with $n\notin\Lambda$. The exachange of the role of the two B.V.P.'s $\langle p, h\rangle$ and $\langle\tilde{p}, \tilde{h}\rangle$ in the formula (4.4) yields

$$(4.7)\quad \tilde{K}(x, t) = \sum_{n\in\Lambda} \frac{d_n(x, \tilde{\lambda}_n)\tilde{c}_0(t, \tilde{\lambda})}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_n)\|^2} \quad (0<t<x<1)$$

where $d_n(x, \lambda)$ is defined by

$$(4.8)\quad d_n(x, \lambda) := c_0(x, \lambda) + (-1)^{n+1}c_1(x, \lambda).$$

By virtue of (4.7), we can obtain a useful formula expressing the boundary condition:

Lemma 4.1.

$$(4.9)\quad \begin{aligned}\tilde{h} &= h - \textstyle\sum_{m\in\Lambda} d_m(0, \tilde{\lambda}_m)/\|\tilde{c}_0(\cdot, \tilde{\lambda}_n)\|^2\\ &= h - \textstyle\sum_{m\in\Lambda} [1+(-1)^{m+1}c_0(1, \tilde{\lambda}_m)]/\sigma_m,\end{aligned}$$

similarly

$$\tilde{h} - h = \textstyle\sum_{n\in\Lambda} \tilde{d}_n(0, \lambda_n)/\|c_0(\cdot, \lambda_n)\|^2, \tag{4.9'}$$

where σ_m *is given by*

$$\sigma_m := -\tilde{c}_0(1, \tilde{\lambda}_m)\tilde{w}^{\cdot}(\lambda_m) = (-1)^{m+1}\tilde{w}^{\cdot}(\lambda_m), \tag{4.10}$$

with

$$w(\lambda) := c_0'(1, \lambda) + hc_0(1, \lambda) = (\lambda_0 - \lambda)\textstyle\prod_{k=1}^{\infty}[(\lambda_k - \lambda)/(\pi k)^2], \tag{4.11}$$

$$\tilde{w}(\lambda) := \tilde{c}_0'(1, \lambda) + \tilde{h}\tilde{c}_0(1, \lambda) = (\tilde{\lambda}_0 - \lambda)\textstyle\prod_{k=1}^{\infty}[(\tilde{\lambda}_k - \lambda)/(\pi k)^2].$$

Proof. The first equality in (4.9) is obtained by putting $x=t=0$, with (3.9) in mind. The second equality is a direct consequence of the definition of $d_m(x, \lambda)$ and the well-known formula for normalizing constants (see Levitan-Gasymov [14]):

$$\|c_0(\cdot, \lambda_n)\|^2 = -c_0(1, \lambda_n)w^{\cdot}(\lambda_n) = (-1)^{n+1}w^{\cdot}(\lambda_n).$$

Note that this lemma and Lemma 4.2 below proves the last assertion of Theorem A.

Now we can arrive at the first step of determining the explicit form of the kernel $F(x, t)$. Indeed, the substitution of (4.5) and (4.7) into (3.13) gives

$$F(x, t) = \textstyle\sum_{m\in\Lambda} \frac{d_m(x, \tilde{\lambda}_m)u_m(t)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2} \qquad (0 \leqq t \leqq x \leqq 1) \tag{4.12}$$

where the function $u_m(t)$ is defined by

$$u_m(t) = \tilde{c}_0(t, \tilde{\lambda}_m) - \textstyle\sum_{n\in\Lambda} \frac{c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda_n)\|^2}\int_0^t \tilde{c}_0(s, \tilde{\lambda}_m)\tilde{d}_n(s, \lambda_n)ds, \tag{4.13}$$

which implies, in particular

$$u_m(0) = 1, \quad u_m'(0) = \tilde{h} - \textstyle\sum_{n\in\Lambda} \tilde{d}_n(0, \lambda_n)/\|c_0(\cdot, \lambda_n)\|^2 = h, \tag{4.13'}$$

here the last equality follows from (4.9). Similarly, by (4.13), we find that

$$u_m(1) = (-1)^m - \textstyle\sum_{n\in\Lambda} \frac{(-1)^n}{\|c_0(\cdot, \lambda_n)\|^2}\int_0^1 \tilde{c}_0(t, \tilde{\lambda}_m)\tilde{d}_n(t, \lambda_n)dt$$

$$u_m'(1) = (-1)^{m+1}\tilde{h} - \textstyle\sum_{n\in\Lambda} \frac{(-1)^{n+1}h}{\|c_0(\cdot, \lambda_n)\|^1}\int_0^1 \tilde{c}_0(t, \tilde{\lambda}_m)\tilde{d}_n(t, \lambda_n)dt$$

$$-\textstyle\sum_{n\in\Lambda}(-1)^n\cdot(-1)^m\tilde{d}_n(1, \lambda_n)/\|c_0(\cdot, \lambda_n)\|^2$$

$$= (-1)^{m+1}\tilde{h} + \{-hu_m(1)+(-1)^m h\}$$
$$+ (-1)^m \sum_{n\in\Lambda} \tilde{d}_n(0, \lambda_n)/\|c_0(\cdot, \lambda_n)\|^2 = -hu_m(1),$$

where the last equality follows from (4.9′). Thus we obtain

$$u'_m(1) = -hu_m(1) \quad (m\in\Lambda). \tag{4.13″}$$

On the other hand, substituting (4.5) and (4.7) into (3.14), we obtain

$$F(x, t) = -\sum_{n\in\Lambda}\frac{v_n(x)c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda_n)\|^2} \quad (0<x<t<1), \tag{4.14}$$

where the function $v_n(x)$ is defined by,

$$v_n(x) = \tilde{d}_n(x, \lambda_n) + \sum_{m\in\Lambda}\frac{d_m(x, \tilde{\lambda}_m)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2}\int_0^x \tilde{c}_0(s, \tilde{\lambda}_m)\tilde{d}_n(s, \lambda_n)ds. \tag{4.15}$$

In particular, this equality, combined with (4.9), gives

$$v_n(0) = \tilde{d}_n(0, \lambda_n), \quad v'_n(0) = \tilde{d}'_n(0, \lambda_n) + (h-\tilde{h})\tilde{d}_n(0, \lambda_n). \tag{4.15′}$$

For some time, we shall consider the problem when the functions $d_n(x, \tilde{\lambda}_n)$ or $\varphi_m(x)$ vanish identically. To this end, let us introduce the following notations.

$$\begin{aligned} \Lambda_1 &:= \{m\in\Lambda;\ \text{there exists a } f(m)\in\Lambda \text{ such that} \\ &\qquad f(m)\equiv m \pmod 2 \text{ and } \tilde{\lambda}_m=\lambda_{f(m)}\}, \\ \Lambda_2 &:= \{n\in\Lambda;\ \text{there exists a } g(m)\in\Lambda \text{ such that} \\ &\qquad g(n)\equiv n \pmod 2 \text{ and } \lambda_n=\tilde{\lambda}_{g(n)}\}. \end{aligned} \tag{4.16}$$

Note that the mappings $f\colon \Lambda_1\to\Lambda_2$ and $g\colon \Lambda_2\to\Lambda_1$ are bijective, monotonically increasing and the inverse mappings of each other. Then the following lemma holds:

Lemma 4.2. 1) *$d_m(x, \tilde{\lambda}_m)$ is identiaclly zero if and only if $m\in\Lambda_1$.*
2) *$v_n(x)$ is identically zero if and only if $\tilde{d}_n(x, \lambda_n)$ is identically zero, and subsequently, if and only if $n\in\Lambda_2$.*

Proof. By the definition (4.8), the equation $d_m(x, \tilde{\lambda}_m)=0$ $(0\leqq x\leqq 1)$ is equivalent to $c_0(x, \tilde{\lambda}_m)=(-1)^m c_1(x, \tilde{\lambda}_m)$ $(0\leqq x\leqq 1)$, which is, in view of Lemma 1.1, nothing but that $\tilde{\lambda}_m$ is the M-th eigenvalue of $\langle p, h\rangle$ for some M with $M\equiv m \pmod 2$. Hence the assertion 1) follows. In a similar manner, the part of the assertion 2) that the identical vanishingness of $\tilde{d}_n(x, \lambda_n)$ is equivalent to $n\in\Lambda_2$ is also proved. We shall show the remaining part. First, if $v_n(x)=0$

$(0\leqq x\leqq 1)$, then in particular $v_n(0)=v_n'(0)=0$, hence (4.15′) implies that $\tilde{d}_n(0, \lambda_n)=\tilde{d}_n'(0, \lambda_n)=0$. Since $\tilde{d}_n(x, \lambda_n)$ is a solution of the second order differential equation $-y''+\tilde{p}(x)y=\lambda_n y$, we must have $\tilde{d}_n(x, \lambda_n)=0$ $(0\leqq x\leqq 1)$. Conversely, if $\tilde{d}_n(x, \lambda_n)$ is identically zero, it follows immediately from (4.15) that $v_n(x)$ vanishes identically. Hence the lemma is proved.

In the sequel we shall write as $\Lambda_1^c := \Lambda\setminus\Lambda_1$ and $\Lambda_2^c := \Lambda\setminus\Lambda_2$. By virtue of the above lemma, the summations over Λ in the formulas from (4.4) to (4.15) can be replaced by those over Λ_1^c or Λ_2^c.

In view of the formulas (4.12) and (4.14), we shall define the functions $F_0(x, t)$ and $F_1(x, t)$ in the region $0\leqq x, t\leqq 1$, as follows

$$F_0(x, t) := \sum_{m\in\Lambda_1^c} \frac{d_m(x, \tilde{\lambda}_m)u_m(t)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2} \quad (0\leqq x, t\leqq 1), \tag{4.17}$$

$$F_1(x, t) := \sum_{n\in\Lambda_2^c} \frac{v_n(x)c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda)\|^2} \quad (0\leqq x, t\leqq 1). \tag{4.18}$$

Then, by definition, $F_0(x, t)$ and $F_1(x, t)$ are identical with $F(x, t)$ in the regions $\{x\geqq t\}$ and $\{x\geqq t\}$ respectively, i.e.,

$$F_0(x, t) = F(x, t) \quad (x\geqq t), \quad F_1(x, t) = F(x, t) \quad (x\leqq t) \tag{4.19}$$

Now we shall claim:

Lemma 4.3. $F(x, t)=F_0(x, t)=F_1(x, t)$ $(0\leqq x, t\leqq 1)$.

To prove this lemma, we need the following lemma.

Lemma 4.4. *In the domain* $\{0<t<x<1\}$, *the functions* $F_\nu(x, t)$ $(\nu=0, 1)$ *satisfy the hyperbolic equation*:

$$\Box_p F_\nu = \tilde{K}(x, 0)[(\partial/\partial x)L(x, t)-\tilde{h}L(x, t)]_{x=0} \quad (\nu=0, 1), \tag{4.20}$$

along with the boundary conditions:

$$[(\partial/\partial t)F_\nu(x, t)-hF_\nu(x, t)]_{t=0} = 0 \quad (0\leqq x\leqq 1, \nu=0, 1) \tag{4.21}$$

$$F_0(x, x) = F_1(x, x) \quad (0\leqq x\leqq 1), \tag{4.22}$$

where recall that $\Box_p$ *is a modified d'Alembertian*:

$$\Box_p := (\partial/\partial x)^2 - (\partial/\partial t)^2 - p(x) + p(t).$$

Similarly, in the domain $\{0<x<t<1\}$, $F_\nu(x, t)$ $(\nu=0, 1)$ *satisfy the equation*

$$\Box_p F_\nu = \tilde{K}(x, 0)[(\partial/\partial x)L(x, t)-\tilde{h}L(x, t)]_{x=0} \quad (\nu=0, 1), \tag{4.23}$$

along with the boundary conditions:

(4.24) $$[(\partial/\partial t)F(x, t)+hF(x, t)]_{t=1} = 0 \quad (0 \leqq x \leqq 1, \ v=0, 1),$$

(4.22) $$F_0(x, x) = F_1(x, x) \quad (0 \leqq x \leqq 1).$$

Leaving the proof of Lemma 4.4 until later, we shall observe that Lemma 4.3. follows from Lemma 4.4.

Proof of Lemma 4.3. Put $F_2(x, t) := F_1(x, t) - F_0(x, t)$. Then (4.20), (4.21) and (4.22) in Lemma 4.4 imply that $F_2(x, t)$ is a solution of the homogenious hyperbolic problem:

(4.25) $$\begin{cases} \square_p F_2 = 0 & (0<t<x<1), \\ [(\partial/\partial t)F_2(x, t) - hF_2(x, t)]_{t=0} = 0 & (0 \leqq x \leqq 1), \\ F_2(x, x) = 0 & (0 \leqq x \leqq 1). \end{cases}$$

Let $R(x, t; \xi, \eta)$ be the Riemann function associated with $\square_p$, i.e.,

$$\begin{cases} \square_p R = 0 \text{ as a function of the variables } (x, t), \\ R(x, t; \xi, \eta) = 1 \text{ on the characteristic lines } x - \xi = \pm (t - \eta), \end{cases}$$

and let the function $A(\xi; x, t)$ be defined by

$$A(\xi; x, t) := [hR(\xi, \eta; x, t) - (\partial/\partial\eta)R(\xi, \eta; x, t)]_{\eta=0},$$

Moreover let D_j ($j=1, 2, 3, 4$) be the regions indicated by Fig. 1. Then, by a well-known method of Riemann (see e.g. H. Picard [18]), it can be observed that $F_2(x, t)$ ($(x, t) \in D_1$) is expressed as

(4.26) $$F_2(x, t) = \frac{1}{2}\{g(x-t)+g(x+t)\} + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)A(\xi; x, t)d\xi,$$

where $g(x) = F_2(x, t)$ $(0 \leqq x \leqq 1)$, and $g(x)$ solves the homogenious Volterra integral equation:

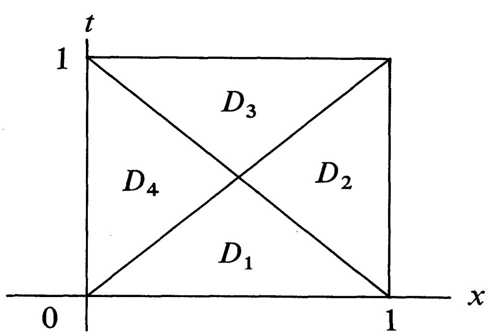

Fig. 1.

(4.27) $$g(x) + \int_0^x g(\xi)A(\xi;\ x/2,\ x/2)d\xi = 0 \quad (0 \leqq x \leqq 1).$$

Hence $g(x)=0$, and therefore, by (4.26), $F_2(x, t)$ is dientically zero in D_1. That $F_2(x, t)$ vanishes identically in D_2 is shown by noting that $F_2(x, t)$ solves the Goursat problem with zero deta on $\{x-1/2=\pm(y-1/2)\} \cap \partial D_2$. Similarly it is shown that $F_2(x, t)$ vanishes identically in $\{0 \leqq x \leqq t \leqq 1\}$, and hence the lemma is proved.

Proof of Lemma 4.4. The validity of (4.21) and (4.24) for $v=1$ is obvious from the definition (4.18), whereas the assertion for $v=0$ follows from (4.13′), (4.13″) and (4.17). The equation (4.22) follows from (4.3) and (4.19). Now we shall show (4.20) and (4.23). By virtue of (4.5) and (4.7), the inhomogeneous term in the hyperbolic equation (3.17) is calculated as follows:

(4.28) $$\tilde{K}(x, 0)\,[(\partial/\partial x)L(x, t) - \tilde{h}L(x, t)]_{x=05}$$
$$= \Big(\sum_{m\in\Lambda_1^c} \frac{d_m(x, \tilde{\lambda}_m)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2}\Big)\Big(\sum_{n\in\Lambda_2^c} \frac{(-1)^n \tilde{w}(\lambda_n)c_0(t, \lambda)}{\|c_0(\cdot, \lambda_n)\|^2}\Big).$$

Substituting (4.12) and (4.28) into the hyperbolic equation (3.17) in Lemma 3.2, we obtain,

$$\sum_{m\in\Lambda_1^c} \frac{\tilde{d}_m(x, \tilde{\lambda}_m)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2}\Big(-u_m''(t) + [p(t)-\tilde{\lambda}_m]u_m(t)$$
$$- \sum_{n\in\Lambda_2^c} \frac{(-1)^n \tilde{w}(\lambda_n)c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda_n\|^2}\Big) = 0 \qquad \text{for} \quad 0 < t < x < 1.$$

Since each $d_m(x, \tilde{\lambda}_m)/\|c_0(\cdot, \tilde{\lambda}_m)\|^2$ $(m \in \Lambda_1^c)$ is not identically zero on the interval $[t, 1]$, Lemma 3.6 with $\ell = -(d/dx)^2 + p(x)$ and $\{\alpha_m\} = \{\tilde{\lambda}_m\}_{m\in\Lambda_1^c}$ implies that they are linearly independent on the interval $[t, 1]$. Hence it follows from the above equation that for $m \in \Lambda_1^c$

(4.29) $$-u_m''(t) + [p(t)-\tilde{\lambda}_m]u_m(t) = \sum_{n\in\Lambda_2^c} \frac{(-1)^n \tilde{w}(\lambda_n)c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda_n)\|^2}$$

Similarly, the substitution of (4.14) and (4.28) into (3.17) yields

$$\sum_{n\in\Lambda_2^c} \frac{c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda_n)\|^2}\Big(-v_n''(x) + [p(x)-\lambda_n]v_n(x)$$
$$- \sum_{m\in\Lambda_1^c} \frac{(-1)^{n+1} \tilde{w}(\lambda_n)d_m(x, \lambda_m)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2}\Big) = 0 \qquad \text{for} \quad 0 < x < t < 1.$$

Hence, by lemma 3.6 with the same ℓ as above, $\{\alpha_n\} = \{\lambda_n\}_{n\in\Lambda_2^c}$, and $\{f_n\} = \{c_0(\cdot, \lambda_n)/\|c_0(\cdot, \lambda_n)\|^2\}_{n\in\Lambda_2^c}$, we obtain for $n \in \Lambda_2^c$

(4.30) $$-v_n''(x) + [p(x)-\lambda_n]v_n(x) = \sum_{m\in\Lambda_1^c} \frac{(-1)^{n+1} \tilde{w}(\lambda_n)d_m(x\ \tilde{\lambda}_m)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2}$$

Now the equations (4.20) and (4.23) are easily shown by inserting (4.28), (4.29) and (4.30) into the definitions (4.17), (4.18) of $F_\nu(x, t)$ $(\nu=1, 2)$. This proves the lemma.

Once Lemma 4.3 is established, the derivation of the explicit form of $F(x, t)$ is now easy. Indeed we obtain

Theorem 4.5. *Let $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ be in a "degenerate case" and suppose that*

$$\lambda_n(p, h)\left(\begin{matrix} = \\ \neq \end{matrix}\right. \lambda_n(\tilde{p}, \tilde{h}) \text{ according to } n \left(\begin{matrix} \notin \\ \in \end{matrix}\right. \Lambda.$$

Then the kernel $F(x, t)$ is written as

$$F(x, t) = \sum_{(m,n)\in \Lambda_1^c \times \Lambda_2^c} a_{m,n} d_m(x, \tilde{\lambda}_m) c_0(t, \lambda_n), \tag{4.31}$$

where the numbers $a_{m,n}$ are given by

$$\begin{aligned} a_{m,n} &= (-1)^{n+1}\tilde{w}(\lambda_n)/\{(\lambda_n - \tilde{\lambda}_m)\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2 \|c_0(\cdot, \lambda_n)\|^2\} \\ &= (-1)^m \tilde{w}(\lambda_n)/\{(\tilde{\lambda}_m - \lambda_n)\tilde{w}^{\cdot}(\tilde{\lambda}_m)\dot{w}(\lambda_n)\}, \end{aligned} \tag{4.32}$$

where $w(\lambda)$ and $\tilde{w}(\lambda)$ are defined by

$$w(\lambda) := c_0'(1, \lambda) + hc_0(1, \lambda) = (\lambda_0 - \lambda)\prod_{k=1}^{\infty} [(\lambda_k - \lambda)/(\pi k^2)],$$
$$\tilde{w}(\lambda) := \tilde{c}_0'(1, \lambda) + \tilde{h}c_0(1, \lambda) - (\tilde{\lambda}_0 - \lambda)\prod_{k=1}^{\infty} [(\tilde{\lambda}_k - \lambda)/(\pi k)^2].$$

Proof. Lemma 4.3 implies that

$$F(x, t) = \sum_{m\in\Lambda_1^c} \frac{d_m(x, \tilde{\lambda}_m)u_m(t)}{\|\tilde{c}_0(\cdot, \tilde{\lambda})\|^2} = -\sum_{k\in\Lambda_2^c} \frac{v_k(x)c_0(t, \lambda_k)}{\|c_0(\cdot, \lambda_k)\|^2}. \tag{4.33}$$

With the orthogonality of $\{c_0(\cdot, \lambda_n)\}_{n=0}^{\infty}$ in $L^2(0, 1)$ in mind, we multiply the both sides on the second equality in (4.33) by $c_0(t, \lambda_n)$ $(n\in\Lambda_2^c)$ and integrate in t over the interval $0<t<1$, to botain

$$v_n(x) = -\sum_{m\in\Lambda_1^c} \frac{d_m(x, \tilde{\lambda}_m)}{\|\tilde{c}_0(\cdot, \tilde{\lambda})\|^2} \int_0^1 u_m(t)c_0(t, \lambda_n)dt,$$

hence each $v_n(x)$ $(m\in\Lambda_2^c)$ are expressed as a linear combination of the $d_m(x, \tilde{\lambda}_m)$'s $(m\in\Lambda_1^c)$. Thus, let us put

$$v_n(x) = -\sum_{m\in\Lambda_1^c} a_{m,n} d_m(x, \tilde{\lambda}_m) \quad (n\in\Lambda_2^c), \tag{4.34}$$

then $F(x, t)$ takes the form of (4.31). To evaluate the values of $a_{m,n}$, we have only to substitute (4.33) and (4.34) into the hyperbolic equation (3.17) given in Lemma

3.2, with the expression of the inhomogeneous term (4.28) took into account. Then we obtain

$$\begin{aligned}\square_p F &= \textstyle\sum_{(m,n)\in\Lambda_1^c\times\Lambda_2^c} a_{m,n}(\lambda_n-\tilde{\lambda}_m)d_m(x,\tilde{\lambda}_m)c_0(t,\lambda_n)\\ &= \textstyle\sum_{(m,n)\in\Lambda_1^c\times\Lambda_2^c} \dfrac{(-1)^{n+1}\tilde{w}(\lambda_n)}{\|\tilde{c}_0(\cdot,\tilde{\lambda}_m)\|^2\|c_0(\cdot,\lambda)\|^2} d_m(x,\tilde{\lambda}_m)c_0(t,\lambda_n).\end{aligned}$$

Applying Lemma 3.6 to this equation twicely, we obtain the first equality in (4.32). Finally the second inequality in (4.32) is deduced by a well-known formula for the normalizing constants:

$$\|c_0(\cdot,\lambda_n)\|^2 = -c_0(1,\lambda_n)\dot{w}(\lambda_n) = (-1)^{n+1}\dot{w}(\lambda_n)$$

and corresponding formula for $\|\tilde{c}_0(\cdot,\tilde{\lambda}_m)\|^2$ (see Levitan-Gasymov [14]). Hence the theorem is proved.

Theorem 4.5 implies that, if two problems $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ are in the degenerate case, then the kernel $F(x, t)$ is reduced to a degenerate kernel in a sense mentioned in Remark 3.5. Hence, by applying Theorem 4.5 to Theorem 3.4 and Remark 3.5, we obtain an explicit form for $\langle \tilde{p}, \tilde{h}\rangle$ in terms of $\langle p, h\rangle$ and $\{\tilde{\lambda}_n\}_{n\in\Lambda}$.

Theorem 4.6. *Let Λ be a finite subset of non-negative integers. If the symmetric problem $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ satisfy the condition:*

$$\text{(4.1)} \qquad \lambda_n(p,h)\begin{pmatrix} = \\ \neq \end{pmatrix} \lambda_n(\tilde{p},\tilde{h}) \text{ according to } n \begin{pmatrix} \notin \\ \in \end{pmatrix} \Lambda,$$

then one problem $\langle \tilde{p}, \tilde{h}\rangle$ is uniquely determined by the other problem $\langle p, h\rangle$ and the values $\lambda_n(\tilde{p}, \tilde{h})$ $(n\in\Lambda)$. Moreover $\langle \tilde{p}, \tilde{h}\rangle$ must admit the representation expressed in the statement of Theorem A.

Proof. This theorem is now evident. Merely we note that the two-way expressions of τ-function in Theorem A are correspondent to the ways of the separation of the variables in the kernel $F(x, t)$, i.e.

$$F(x,t) = \textstyle\sum_{m\in\Lambda_1^c} d_m(x,\tilde{\lambda}_m)\{\sum_{n\in\Lambda_2^c} a_{m,n}c_0(t,\lambda_n)\},$$

$$F(x,t) = \textstyle\sum_{n\in\Lambda_2^c}\{\sum_{m\in\Lambda_1^c} a_{m,n}d_m(x,\tilde{\lambda}_m)\}c_0(t,\lambda_n).$$

Remark 4.7. In the original Gel'fand-Levitan theory, we have to solve the integral equation:

$$\text{(4.35)} \qquad K(x,t) + f(x,t) + \int_0^x K(x,s)f(s,t)ds = 0, \quad 0 < x < t,$$

where the kernel $f(x, t)$ is defined by

$$(4.36)\qquad f(x, t):=\sum_{n=0}^{\infty}\left(\frac{c_0(x, \tilde{\lambda}_n)c_0(t, \tilde{\lambda}_n)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_n)\|^2}-\frac{c_0(x, \lambda_n)c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda_n)\|^2}\right).$$

The kernel $f(x, t)$ is different from our kernel $F(x, t)$, since $f(x, t)$ satisfy the homogeneous hyperbolic equation $\Box_p f(x, t)=0$, while $F(x, t)$ is a solution of the inhomogeneous equation (3.17). Note that this inhomogenious term does not vanish identically. For, otherwise, it can be shown in a similar manner as in the proof of Lemma 4.3 and Lemma 4.4 that $F(x, t)$ vaninishes identically, and hence $\langle p, h\rangle=\langle\tilde{p}, \tilde{h}\rangle$. Hence $F(x, t)$ is actually different from $f(x, t)$, provided $\langle p, h\rangle \neq\langle\tilde{p}, \tilde{h}\rangle$. Thus, even if the degenerate condition (4.1) is assumed, the kernel $f(x, t)$ is not expected to reduce to a degenerate form, as is our kernel.

Now we shall refer to the case of Dirichlet problem. Also in this case, the arguements will be substantially paralell to those in the problem of the third kind. In order to clearify the correspondence of formulas, lemmas, e.t.c. for the Dirichlet problem to the counterparts for the third kind one, we shall call the formers by the same names of the latters followed by superscript "*".

Let $p(x)$ and $\tilde{p}(x)$ be two symmetric potentials, Λ be a finite subset of positive integers, and suppose that

$$(4.1^*)\qquad \lambda_n(p, \infty)\begin{pmatrix}\neq\\ =\end{pmatrix}\lambda_n(\tilde{p}, \infty)\text{ according to } n\begin{pmatrix}\in\\ \notin\end{pmatrix}\Lambda.$$

For brevity, we shall write as

$$(4.2^*)\qquad \mu_n:=\lambda_n(p, \infty),\quad \tilde{\mu}_n:=\lambda_n(\tilde{p}, \infty)\quad (n=1, 2, 3,\cdots).$$

Then the assumption (4.1*) and Remark 3.3 imply

$$(4.3^*)\qquad G(x, x+0)=G(x, x-0).$$

Next, (1.8) and (3.2) lead to

$$(4.4^*)\qquad M(x, t)=\sum_{n\in\Lambda}\frac{\tilde{r}_n(x, \mu_n)s_0(t, \mu_n)}{\|s_0(\cdot, \mu_n)\|^2}\qquad (x>t),$$

$$(4.5^*)\qquad N(x, t)=-\sum_{n\in\Lambda}\frac{\tilde{r}_n(x, \mu_n)s_0(t, \mu_n)}{\|s_0(\cdot, \mu_n)\|^2}\qquad (x<t),$$

where $\tilde{r}_n(x, \lambda)$ is defined by

$$(4.6^*)\qquad \tilde{r}_n(x, \lambda):=\tilde{s}_0(x, \lambda)+(-1)^n\tilde{s}_1(x, \lambda).$$

Similarly we obtain

$$\tilde{M}(x, t) = \sum_{n\in\Lambda} \frac{r_n(x, \tilde{\mu}_n)\tilde{s}_0(t, \tilde{\mu}_n)}{\|\tilde{s}_0(\cdot, \tilde{\mu}_n)\|^2} \quad (x>t), \tag{4.7*}$$

where

$$r_n(x, \lambda) := s_0(x, \lambda) + (-1)^n s_1(x, \lambda). \tag{4.8*}$$

The substitution of (4.5*) and (4.7*) into (3.15) gives

$$G(x, t) = \sum_{n\in\Lambda} \frac{r_n(x, \tilde{\mu}_n)\varphi_n(t)}{\|\tilde{s}_0(\cdot, \tilde{\mu})\|^2} \quad (x>t), \tag{4.12*}$$

where $\varphi_n(t)$ is defined by

$$\varphi_n(t) := \tilde{s}_0(t, \tilde{\mu}_n) - \sum_{n\in\Lambda} \frac{s_0(t, \mu_m)}{\|s_0(\cdot, \mu_m)\|^2} \int_0^t \tilde{s}_0(s, \tilde{\mu}_n)\tilde{r}(s, \mu_m)ds, \tag{4.13}^*$$

which implies in particular

$$\varphi(0) = 0, \quad \varphi'(0) = 1. \tag{4.13'*}$$

Similarly, applying (4.5*) and (4.7*) into (3.16), we obtain

$$G(x, t) = -\sum_{n\in\Lambda} \frac{\psi_n(x)s_0(t, \mu_n)}{\|s_0(\cdot, \mu)\|^2} \quad (x<t), \tag{4.14*}$$

where $\psi_n(x)$ is defined by

$$\psi_n(x) := \tilde{r}_n(x, \mu_n) + \sum_{n\in\Lambda} \frac{r_m(x, \tilde{\mu}_m)}{\|\tilde{s}_0(\cdot, \tilde{\mu}_m)\|^2} \int_0^x \tilde{s}_0(t, \tilde{\mu}_m)r_n(t, \mu_n)dt. \tag{4.15*}$$

In particular, we find that

$$\psi_n(0) = \tilde{r}_n(0, \mu_n), \quad \psi_n'(0) = \tilde{r}_n'(0, \mu_n). \tag{4.15'*}$$

Corresponding to Lemma 4.2, we have:

Lemma 4.2*. *Let Λ_1 and Λ_2 be defined by*

$$\begin{aligned} \Lambda_1 &:= \{m\in\Lambda;\ \textit{there exists a number } f(m)\in\Lambda \textit{ such that} \\ &\qquad m\equiv f(m) \ (\mathrm{mod}\ 2) \textit{ and } \tilde{\mu}_m=\mu_{f(m)}\}, \\ \Lambda_2 &:= \{n\in\Lambda;\ \textit{there exists a number } g(n)\in\Lambda \textit{ such that} \\ &\qquad n\equiv g(n) \ (\mathrm{mod}\ 2) \textit{ and } \mu_n=\tilde{\mu}_{g(n)}\}. \end{aligned} \tag{4.16*}$$

1) *$r_m(x, \tilde{\mu}_m)$ is identically zero if and only if $m\in\Lambda_1$.*
2) *$\psi_n(x)$ is identically zero if and only if $\tilde{r}_n(x, \mu_n)$ is identically zero, and subsequently, if and only if $n\in\Lambda_2$.*

Put $\Lambda_1^c := \Lambda \diagdown \Lambda_1$, and $\Lambda_2^c := \Lambda \diagdown \Lambda_2$. Then summations over Λ in the formulas from (4.4*) to (4.15*) can be replaced by those over Λ_1^c or Λ_2^c.

Now, let the functions $G_0(x, t)$ and $G_1(x, t)$ be defined by

$$G_0(x, t) := \sum_{n \in \Lambda_1^c} \frac{r_n(x, \tilde{\mu}_n)\varphi_m(x)}{\|\tilde{s}_0(\cdot, \tilde{\mu}_n)\|^2} \quad (0 \leqq x, t \leqq 1), \tag{4.17*}$$

$$G_1(x, t) := -\sum_{n \in \Lambda_2^c} \frac{\psi_m(x) s_0(t, \mu_n)}{\|s_0(\cdot, \mu_n)\|^2} \quad (0 \leqq x, t \leqq 1). \tag{4.18*}$$

Then, we have:

$$G_0(x, t) = G(x, t) \quad (x \geqq t), \quad G_1(x, t) = G(x, t) \quad (x \leqq t). \tag{4.19*}$$

The counterparts of Lemma 4.3 and Lemma 4.4 are stated as follows:

Lemma 4.3*. $G(x, t) = G_0(x, t) = G_1(x, t)$ $(0 \leqq x, t \leqq 1)$.

Lemma 4.4.*. *The function $G_\nu(x, t)$ $(\nu = 0, 1)$ satisfy the following hyperbolic equation in the domain $\{0 < t < x < 1\}$,*

$$\Box_p G_\nu = -[\partial/\partial t)\tilde{M}(x, t)]_{t=0} N(0, t) \quad (t < x), \tag{4.20*}$$

along with the boundary conditions:

$$G_\nu(x, t)|_{t=0} = 0 \qquad (0 \leqq x \leqq 1, \nu = 0, 1), \tag{4.21*}$$

$$G_0(x, x) = G_1(x, x) \quad (0 \leqq x \leqq 1). \tag{4.22*}$$

Similarly, in the domain $\{0 < x < t < 1\}$, $G_\nu(x, t)$ satisfy

$$\Box_p G_\nu = -[(\partial/\partial t)\tilde{M}(x, t)]_{t=0} N(0, t) \quad (x < t, \nu = 0, 1), \tag{4.23*}$$

along with the conditions:

$$G_\nu(x, t)|_{t=1} = 0 \qquad (0 \leqq x \leqq 1, \nu = 0, 1), \tag{4.24*}$$

$$G_0(x, x) = G_1(x, x) \quad (0 \leqq x \leqq 1). \tag{4.22*}$$

Proof of Lemma 4.2*. Put $G_2(x, t) := G_1(x, t) - G_0(x, t)$. Then Lemma 4.4* implies that $G_2(x, t)$ solves the homogeneous hypergolic problem:

$$\begin{cases} \Box_p G_2 = 0 & (0 < t < x < 1), \\ G_2(x, t)|_{t=0} = 0 & (0 \leqq x \leqq 1), \\ G_2(x, x) = 0 & (0 \leqq x \leqq 1). \end{cases} \tag{4.25*}$$

Hence $G_2(x, t)$ is identically zero in $0 \leqq t \leqq x \leqq 1$, and similarly in $0 \leqq x \leqq t \leqq 1$.

Proof of Lemma 4.4*. The validity of (4.21*) and (4.24*) follow from (4.13′*), (4.17*) and (4.18*), moreover (4.22*) follows from (4.3*) and (4.19*). By virtue of (4.5*) and (4.7*), the inhomogeneous term in (4.20*), (4.23*) is calculated as follows:

$$(4.28^*) \qquad -\tilde{M}_t(x, t)|_{t=0} N(0, t)$$
$$= \left(\sum_{m \in \Lambda_1^c} \frac{r_m(x, \mu_m)}{\|\tilde{s}_0(\cdot, \tilde{\mu}_m)\|^2}\right)\left(\sum_{n \in \Lambda_2^c} \frac{r_n(0, \mu_n) s_0(t, \mu_n)}{\|s_0(\cdot, \mu_n)\|^2}\right).$$

Applying (4.12*), (4.14*), (4.28*) to (3.18), and taking 3.6 into account, we can prove (4.20*) and (4.23*).

By using Lemma 4.3*, we can determine the form of $G(x, t)$:

Theorem 4.5*. *Let* $\langle p, \infty\rangle$ *and* $\langle \tilde{p}, \infty\rangle$ *satisfy* (4.1*). *Then* $G(x, t)$ *is expressed as follows*:

$$(4.31^*) \qquad G(x, t) = \sum_{(m,n) \in \Lambda_1^c \times \Lambda_2^c} b_{m,n} r_m(x, \tilde{\mu}_m) s_0(t, \mu_n),$$

where

$$(4.32^*) \qquad b_{m,n} = \tilde{r}_n(0, \mu_n)/\{(\mu_n - \tilde{\mu}_m)\|\tilde{s}_0(\cdot, \tilde{\mu}_m)\|^2 \|s_0(\cdot, \mu_n)\|^2\}$$
$$= (-1)^{m+1} \tilde{v}(\mu_m)/\{(\tilde{\mu}_m - \mu_n)\tilde{v}^{\cdot}(\tilde{\mu}_m)\dot{v}(\mu_n)\}.$$

Here the functions $v(\lambda)$ *and* $\tilde{v}(\lambda)$ *are defined by*

$$v(\lambda) := s_0(1, \lambda) = \prod_{k=1}^{\infty} [(\mu_k - \lambda)/(\pi k)^2],$$
$$\tilde{v}(\lambda) := \tilde{s}_0(1, \lambda) = \prod_{k=1}^{\infty} [(\tilde{\mu}_k - \lambda)/(\pi k)^2].$$

§ 5. Completion of the inverse problem in the degenerate case

In this section we shall establish the following theorem:

Theorem 5.1. *Let* $\langle p, h\rangle$ (*resp.* $\langle p, \infty\rangle$) *be a given reference problem whose eigenvalues are* $\{\lambda_n(p, h) = \lambda_n\}_{n=0}^{\infty}$ (*resp.* $\{\lambda_n(p, \infty) = \mu_n\}_{n=1}^{\infty}$). *Let* Λ *be a finite subset of non-negative* (*resp. positive*) *integers. Moreover let* $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ (*resp.* $\{\tilde{\mu}_n\}_{n=1}^{\infty}$) *be a real increasing sequence satisfying*

$$\tilde{\lambda}_n = \lambda_n \quad (\textit{resp. } \tilde{\mu}_n = \mu_n) \qquad \textit{for} \quad n \notin \Lambda.$$

Then the formulas in Theorem A (*resp. Theorem A**) *which gives the definitions of* $\tilde{p}(x)$ *and* $\tilde{h}$ (*resp.* $\tilde{p}(x)$) *are well-defined, and the eigenvalues of the B.V.P.* $\langle \tilde{p}, \tilde{h}\rangle$ (*resp.* $\langle \tilde{p}, \infty\rangle$) *defined by these formulas actually coincide with* $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ (*resp.* $\{\tilde{\mu}_n\}_{n=1}^{\infty}$).

Note that, combined with Theorem 4.5 (resp. Theorem 4.5*), this theorem completes the proof of Theorem A (resp. Theorem A*).

To prove this theorem, of course, we must show that the function $\tau(x)$ (resp. $\omega(x)$) defined in Theorem A (resp. Theorem A*) does not vanish on the interval $0 \leqq x \leqq 1$. We shall consider only for the case of the third kind, the case of the Dirichlet problem being similarly considered.

We first deal with the case $\sharp\Lambda = 1$. Let $\Lambda = \{m\}$, and let $w(\lambda)$ and $\tilde{w}(\lambda)$ be defined by

$$w(\lambda) := (\lambda_0 - \lambda) \prod_{n=1}^{\infty} \left(\frac{\lambda_n - \lambda}{(\pi n)^2} \right), \quad \tilde{w}(\lambda) := (\tilde{\lambda}_0 - \lambda) \prod_{n=1}^{\infty} \left(\frac{\tilde{\lambda}_n - \lambda}{(\pi n)^2} \right).$$

Then we have

$$\tilde{w}(\lambda)/(\lambda - \tilde{\lambda}_m) = w(\lambda)/(\lambda - \lambda_n). \tag{5.1}$$

With (5.1) and Theorem 4.5 in mind, we define a function $F(x, t)$ by

$$F(x, t) = [(-1)^{m+1} \tilde{w}^{\cdot}(\tilde{\lambda}_m)]^{-1} d_m(x, \tilde{\lambda}_m) c_0(t, \lambda_m). \tag{5.2}$$

Then the τ-function introduced in the statement of Theorem A is expressed as follows.

$$\begin{aligned} \tau(x) &= \det(I + F_x) \\ &= 1 + [(-1)^{m+1} \tilde{w}^{\cdot}(\tilde{\lambda}_m)]^{-1} \int_0^x d_m(t, \tilde{\lambda}_m) c_0(t, \lambda_m) dt. \end{aligned} \tag{5.3}$$

Note that the $\tau(x)$ is symmetric:

$$\tau(1 - x) = \tau(x), \quad 0 \leqq x \leqq 1,$$

which is easily seen from the anti-symmetricity of $d_m(x, \tilde{\lambda}_m) c_0(x, \lambda_m)$. In particular we point out that

$$\tau(0) = \tau(1) = 1. \tag{5.3'}$$

Lemma 5.2. *The function* $\tau(x)$ *defined by* (5.3) *is positive on the inerval* $0 \leqq x \leqq 1$.

Proof. In order to show this lemma, we shall rewrite (5.3) as follows. By using

$$\begin{aligned} (\tilde{\lambda}_m - \lambda_m) \int_0^x d_m(t, \tilde{\lambda}_m) c_0(t, \lambda_m) dt &= W_t(d_m(\cdot, \tilde{\lambda}_m), c_0(\cdot, \lambda_m)) |_{t=0}^{x} \\ &= W_x(d_m(\cdot, \tilde{\lambda}_m), c_0(\cdot, \lambda_m)) - (-1)^{m+1} w(\tilde{\lambda}_m), \end{aligned}$$

where $W_x(f, g) := f(x)g'(x) - f'(x)g(x)$ (Wronskian of f, g), we get $\tau(x) = 1 - [(\tilde{\lambda}_m - \lambda_m)\tilde{w}^{\cdot}(\tilde{\lambda}_m)]^{-1}[w(\tilde{\lambda}_m) + (-1)^m W_x(d_m(\cdot, \tilde{\lambda}_m), c_0(\cdot, \lambda_m))]$. Applying (5.1), we obtain:

$$\tau(x) = [(\tilde{\lambda}_m - \lambda_m)(-1)^{m+1}\tilde{w}^{\cdot}(\tilde{\lambda}_m)]^{-1} W_x(d_m(\cdot, \tilde{\lambda}_m), c_0(\cdot, \lambda_m)). \tag{5.3''}$$

Since $(-1)^{m+1}\tilde{w}^{\cdot}(\tilde{\lambda}_m)$ is positive, it suffice to show that $(\tilde{\lambda}_m - \lambda_m)^{-1} W_x(d_m(\cdot, \tilde{\lambda}_m), c_0(\cdot, \lambda_m))$ is positive for $0 \leqq x \leqq 1$. We shall only consider the case $\lambda_m < \tilde{\lambda}_m < \lambda_{m+1}$, another case being similar. Let us put for $n = m, m+1$,

$$\begin{aligned} c_0(x, \lambda_n) &= r_n(x)\sin\theta_n(x), \quad c_0'(x, \lambda_n) = r_n(x)\cos\theta_n(x), \\ d_m(x, \tilde{\lambda}_m) &= \rho(x)\sin\varphi(x), \quad d_m'(x, \tilde{\lambda}_m) = \rho(x)\cos\varphi(x), \end{aligned} \tag{5.4}$$

where we assume that $r_n(x)$ and $\rho(x)$ are positive. Then we find

$$W_x(d_m(\cdot, \tilde{\lambda}_m), c_0(\cdot, \lambda_m)) = r_m(x)\rho(x)\sin(\varphi(x) - \theta_m(x)). \tag{5.5}$$

We designate the intial values of $\theta_n(x)$ $(n=0, 1)$ so as to satisfy $\theta_m(0) = \theta_{m+1}(0)$. Furthermore, since (5.3′), (5.3″) and (5.5) imply that $\sin(\varphi(0) - \theta_m(0))$ is positive, it may be assumed that $\theta_m(0) < \varphi(0) < \theta_m(0) + \pi = \theta_{m+1}(0) + \pi$. Hence it follows from the Prüfer's comparision theorem and the inequality $\lambda_m < \tilde{\lambda}_m < \lambda_{m+1}$ that

$$\theta_m(x) < \varphi(x) < \theta_{m+1}(x) + \pi \quad (0 \leqq x \leqq 1). \tag{5.6}$$

But, since λ_m and λ_{m+1} are the successive eigenvalues of $\langle p, h \rangle$, the relation $\theta_{m+1}(1) = \theta_m(1) + \pi$ holds. Hence, by (5.6), we have

$$\theta_m(1) < \varphi(1) < \theta_m(1) + 2\pi.$$

On the other hand, (5.3′), (5.3″) and (5.5) lead to $\sin(\varphi(1) - \theta_m(1)) > 0$, whence it follows that

$$\theta_m(1) < \varphi(1) < \theta_m(1) + \pi. \tag{5.7}$$

Now we shall claim

$$0 < \varphi(x) - \theta_m(x) < \pi \quad (0 \leqq x \leqq 1). \tag{5.8}$$

The first inequality is already shown in (5.6). Thus we consider the second. If (5.8) does not hold, then there exists a point x_0 $(0 < x_0 \leqq 1)$ such that $\theta_m(x_0) + \pi = \varphi(x_0)$. Hence the Prüfer's comparision theorem combined with the condition $\lambda_m < \tilde{\lambda}_m$ implies that $\theta_m(x) + \pi < \varphi(x)$ holds for $x_0 \leqq x \leqq 1$. In particular $\theta_m(1) + \pi \leqq \varphi(1)$, which contradict (5.7), and hence the lemma is proved.

Remark 5.3. Suzuki [23] proves a similar result, but the proof adopted here is different from that in [23].

By virtue of Lemma 5.2, the Gel'fand-Levitan type integral equation associated with the kernel $F(x, t)$ given by (5.2), i.e.

$$K(x, t) + F(x, t) + \int_0^x K(x, s)F(s, t)ds = 0 \quad (0<t<x),$$

is uniquely solvable and, after a simple calculation, its solution is given by

$$K(x, t) = [(-1)^m \tilde{w}^{\cdot}(\tilde{\lambda}_m)]^{-1} d_m(x, \tilde{\lambda}_m) c_0(t, \lambda_m)/\tau(x). \tag{5.9}$$

Hence, if we define $\tilde{p}(x)$ and $\tilde{h}$, by $\tilde{p}(x) := p(x) - 2(d/dx)^2 \log \tau(x)$, and $\tilde{h} := h - [(d/dx) \log \tau(x)]_{x=0}$, then the solution $\tilde{c}_0(x, \lambda)$ of $-y'' + \tilde{p}(x)y = \lambda y$ which satisfy $y(0) = 1$ and $y'(0) = \tilde{h}$ is to be given by

$$\tilde{c}_0(x, \lambda) = c_0(x, \lambda) + \frac{d_m(x, \tilde{\lambda}_m)}{(-1)^m \tilde{w}^{\cdot}(\tilde{\lambda}_m)\tau(x)} \int_0^x c_0(t, \lambda_m) c_0(t, \lambda) dt. \tag{5.10}$$

In fact, as for the initial condition, we find $\tilde{c}_0(0, \lambda) = 1$, and $\tilde{c}_0'(0, \lambda) = h + d_m(0, \tilde{\lambda}_m)/(-1)^m \tilde{w}^{\cdot}(\tilde{\lambda}_m) = h - \tau'(0)/\tau(0) = \tilde{h}$, as expected. That the $\tilde{c}_0(x, \lambda)$ defined by (5.10) is an actual solution of $-y'' + \tilde{p}(x)y = \lambda y$ is shown by direct computation.

Next we shall observe that $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ are the eigenvalues of $\langle \tilde{p}, \tilde{h} \rangle$. By putting $\lambda = \tilde{\lambda}_n = \lambda_n$ $(n \neq m)$ in (5.10), Lemma 1.1 and the orthogonality of $c_0(\cdot, \lambda_m)$ to $c_0(\cdot, \tilde{\lambda}_n)$ imply that

$$\begin{aligned} \tilde{c}_0(1, \tilde{\lambda}_n) &= c_0(1, \lambda_n) = (-1)^n, \\ \tilde{c}_0'(1, \tilde{\lambda}_n) &= c_0'(1, \lambda_n) + [(-1)^m \tilde{w}^{\cdot}(\tilde{\lambda}_m)\tau(1)]^{-1} d_m(1, \tilde{\lambda}_m) c_0(1, \lambda_m) c_0(1, \lambda_n) \\ &= (-1)^{n+1}\{h + d_m(0, \tilde{\lambda}_m)/(-1)^m \tilde{w}^{\cdot}(\tilde{\lambda}_m)\} = (-1)^{n+1}\tilde{h}, \end{aligned}$$

hence $\{\lambda_n\}_{m \neq n}$ are the eigenvalues of $\langle \tilde{p}, \tilde{h} \rangle$. Further, we see that, by (5.1), $\int_0^1 c_0(t, \lambda_m) c_0(t, \tilde{\lambda}_m) dt = (-1)^m (\lambda_m - \tilde{\lambda}_m)^{-1} w(\tilde{\lambda}_m) = (-1)^{m+1} \tilde{w}^{\cdot}(\tilde{\lambda}_m)$, which we substitute into (5.10) to obtain

$$\begin{aligned} \tilde{c}_0(1, \tilde{\lambda}_m) &= c_0(1, \tilde{\lambda}_m) - d_m(1, \tilde{\lambda}_m)/\tau(1) = (-1)^{m+1}, \\ \tilde{c}_0'(1, \tilde{\lambda}_m) &= c_0'(1, \tilde{\lambda}_m) - d_m'(1, \tilde{\lambda}_m)/\tau(1) + d_m(1, \tilde{\lambda}_m)\tau'(1)/\tau^2(1) \\ &\quad + d_m(1, \tilde{\lambda}_m) c_0(1, \tilde{\lambda}_m)/[\tilde{w}^{\cdot}(\tilde{\lambda}_m)\tau(1)] \\ &= (-1)^{m+1}h - (-1)^{m+1} d_m(0, \tilde{\lambda}_m)/[(-1)^m \tilde{w}^{\cdot}(\tilde{\lambda}_m)] = (-1)^{m+1}\tilde{h}. \end{aligned}$$

Hence $\tilde{\lambda}_m$ is also an eigenvalue of $\langle \tilde{p}, \tilde{h} \rangle$. Since $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ admits the asymptotic formula $\lambda_n^{1/2} = \pi n + O(1/n)$ as $n \to \infty$, the λ_n is precisely of the n-th eigenvalue of $\langle \tilde{p}, \tilde{h} \rangle$. Thus the desired results have been proved for the case $\sharp \Lambda = 1$.

In the general case where $\sharp \Lambda$ is an arbitrary positive integer, on the basis of the argument above, we can construct a finite chain of the B.V.P.'s

$\{\langle p_k, h_k\rangle\}_{k=0}^{M}$, such that $\langle p_0, h_0\rangle$ is the reference problem $\langle p, h\rangle$, every successive two B.V.P.'s share only one eigenvalue, and $\langle p_M, h_M\rangle$ is the B.V.P. $\langle \tilde{p}, \tilde{h}\rangle$ to be sought whose eigenvalues are $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$. Thus the desired B.V.P. $\langle \tilde{p}, \tilde{h}\rangle$ actually exists, and hence, by Theorem 4.6, it must has the representation given by Theorem A, and this completes the proof of Theorem A.

§6. The inverse problem in the general case

In this section we shall devote ourselves to extend the former arguments to the general case where we only assume that the second terms in the asymptotic formulas for the eigenvalues of $\langle p, h\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$ (or $\langle p, \infty\rangle$ and $\langle \tilde{p}, \tilde{h}\rangle$) are the same, i.e.

$$a_0(p, h) = a_0(\tilde{p}, \tilde{h}), \quad b_0(p) = b_0(\tilde{p}). \tag{6.1}$$

See (1.3) and (1.5) for the notations. In the following, we shall only consider the problems of the third kind. Under the assumption (6.1), it follows from (1.3) that the eigenvalues $\lambda_n := \lambda_n(p, h)$ and $\tilde{\lambda}_n := \lambda_n(\tilde{p}, \tilde{h})$ admit the asymptotic formulas:

$$\lambda_n = (\pi n)^2 + \alpha_0 + \varepsilon_n, \quad \tilde{\lambda}_n = (\pi n)^2 + \alpha_0 + \tilde{\varepsilon}_n \tag{6.2}$$

where

$$\alpha_0 := 2a_0(p, h) = 2a_0(p, h),$$
$$\varepsilon_n, \tilde{\varepsilon}_n = O(1/n^2) \quad \text{if} \quad p(x), \quad \tilde{p}(x) \in SBV^1,$$
$$\textstyle\sum_{n=0}^{\infty} (\pi\varepsilon_n)^2 < +\infty, \quad \sum_{n=0}^{\infty} (\pi\tilde{\varepsilon}_n)^2 < +\infty \quad \text{if} \quad p(x), \tilde{p}(x) \in SH^1,$$

We shall require various estimates of the solutions, hence we start with mentioning them. We introduce the following notations:

$$M_1(\lambda) := \max|\cos(\lambda^{1/2}x)|, \quad M_2(\lambda) := \max|\sin(\lambda^{1/2}x)/\lambda^{1/2}|,$$
$$M_3(\lambda) := \max|(d/d\lambda)\sin(\lambda^{1/2}x)/\lambda^{1/2}|$$
$$= \max|-\sin(\lambda^{1/2}x)/(2\lambda^{3/2}) + x\cos(\lambda^{1/2}x)/\lambda^{1/2}|,$$

where the maximums are taken over $0 \leqq x \leqq 1$. And we write as $\|p\|_1 := \int_0^1 |p(t)|dt$. Then the following estimates holds:

Lemma 6.1.

(i) $\|c_0(\cdot, \lambda)\|_\infty \leqq (M_1(\lambda) + |h|M_2(\lambda))\exp(M_2(\lambda)\|p\|_1)$,

$\|c_0'(\cdot, \lambda)\|_\infty \leqq |\lambda|M_2(\lambda) + |h|M_1(\lambda) + M_1(\lambda)\|p_1\|_1\|c_0(\cdot, \lambda)\|_\infty$,

$\|\dot{c}_0(\cdot, \lambda)\|_\infty \leqq \{(1/2)M_2(\lambda) + M_3(\lambda)(|h| + \|p\|_1\|c_0(\cdot, \lambda)\|_\infty)\}$

$$\cdot \exp (M_2(\lambda)\|p\|_1),$$

$$\|\dot{c}_0'(\cdot, \lambda)\|\infty \leqq M_1(\lambda)(1/2+\|p\|_1\|\dot{c}_0(\cdot, \lambda)\|_\infty) + M_2(\lambda)(1/2+|h|+(1/2)\|p\|_1\|c_0(\cdot, \lambda)\|_\infty),$$

$$\dot{c}_0(\cdot, \lambda) = -(1/2)\sin \lambda^{1/2} + \xi(\lambda),$$

where

$$|\xi(\lambda)| \leqq M_3(\lambda)(|h|+\|p\|_1\|c_0(\cdot, \lambda)\|_\infty) + M_2(\lambda)\|p\|_1\|\dot{c}_0(\cdot, \lambda)\|_\infty.$$

(ii) *In particular it follows that as* $\lambda \to \infty$,

$$\|c_0(\cdot, \lambda)\|_\infty = O(1), \quad \|c_0(\cdot, \lambda)\|_\infty = O(\lambda^{1/2}),$$

$$\|\dot{c}_0(\cdot, \lambda)\|_\infty = O(\lambda^{-1/2}), \quad \|\dot{c}_0'(\cdot, \lambda)\|_\infty = O(1),$$

$$\dot{c}_0(1, \lambda) = -(1/2)\sin \lambda^{1/2}/\lambda^{1/2} + O(1/\lambda).$$

(iii) $\|d_n(\cdot, \tilde{\lambda}_n)\|_\infty, \|\tilde{d}_n(\cdot, \lambda_n)\|_\infty = O(n^{-1}|\tilde{\lambda}_n - \lambda_n|)$ *as* $n \longrightarrow \infty$,

$\|d_n'(\cdot, \tilde{\lambda}_n)\|_\infty, \quad \|\tilde{d}_n'(\cdot, \lambda_n)\|_\infty = O(|\tilde{\lambda}_n - \lambda_n|)$ *as* $n \longrightarrow \infty$.

(iv) $\|c_0(\cdot, \lambda_n)\|_2, \|\tilde{c}_0(\cdot, \tilde{\lambda}_n)\|_2 = 1/2 + O(1/n)$ *as* $n \longrightarrow \infty$.

Proof. The proof of (i), (ii), (iv) is standard and achieved by the complicated but elementary calculation. The assetion (iii) follows from the equations $d_0(x, \lambda_n) = 0$, $\tilde{d}_0(x, \tilde{\lambda}_n) = 0$, the mean value theorem and the assertion (ii).

Proposition 6.2. *Let us assume* (6.1). *Then*

$$(6.3) \qquad K(x, t) = \sum_{n=0}^{\infty} \frac{\tilde{d}_n(x, \lambda_n)c_0(t, \lambda_n)}{\|c_0(\cdot, \lambda_n)\|^2} \quad (0 \leqq t \leqq x \leqq 1),$$

$$\tilde{K}(x, t) = \sum_{n=0}^{\infty} \frac{d_n(x, \tilde{\lambda}_n)c_0(t, \tilde{\lambda}_n)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_n)\|^2} \quad (0 \leqq t \leqq x \leqq 1).$$

The series on the right-hand sides converge absolutely and uniformly. In particular, putting $x = t = 0$, *we have*

$$(6.4) \qquad \tilde{h} = h - \sum_{n=0}^{\infty} d_n(0, \tilde{\lambda}_n)/\|\tilde{c}_0(\cdot, \tilde{\lambda}_n)\|^2 = h - \sum_{n=0}^{\infty} [1+(-1)^{n+1}c_0(1, \tilde{\lambda}_n)]/\|\tilde{c}_0(\cdot, \tilde{\lambda}_n)\|^2.$$

Moreover the difference of the two potentials are expressed as

$$(6.5) \qquad \tilde{p}(x) - p(x) = 2\sum_{n=0}^{\infty} (d/dx)[\tilde{d}_n(x, \lambda_n)c_0(x, \lambda_n)]/\|c_0(\cdot, \lambda_n)\|^2 = -2\sum_{n=0}^{\infty} (d/dx)[d_n(x, \tilde{\lambda}_n)c_0(x, \tilde{\lambda}_n)]/\|\tilde{c}_0(\cdot, \tilde{\lambda}_n)\|^2.$$

The series in (6.5) *also converges absolutely and uniformly.*

Remark 6.3. Hochstadt [8] proved the essentially same formulas as (6.5), under the condition that $h=\tilde{h}$, $\lambda_n=\tilde{\lambda}_n$ hold for all n's but finite exceptions. But his method is different from ours and based on a contour integral method. Hald [8] extend the Hochstadt's formula to the general case as stated here, using Hochstadt's mehtod.

Proof. As in the derivation of (4.4) and (4.5), the substitution of $\lambda=\lambda_n$ into $(3.1)_0$-$(-1)^n(3,1)_1$, combined with the expantion theorem, implies that

$$K(x, t) - L(x, t) = \sum_{n=0}^{\infty} \tilde{d}_n(x, \lambda_n)c_0(t, \lambda_n)/\|c_0(\cdot, \lambda_n)\|^2, \tag{6.6}$$

where the convergence of the right-hand side is of L^2-sense in t. Here recall that $K(x, t)$ and $L(x, t)$ are regarded as functions in $0\leqq x, t\leqq 1$ by putting $K(x, t)=0$ $(x<t)$, $L(x, t)=0$ $(x>t)$. We see that, by Lemma 6.1, that the n-th term of the summation in (6.6) is $O(n^{-1}|\tilde{\lambda}_n-\lambda_n|)=O(n^{-1}\delta_n)$, where $\delta_n=O(1/n^2)$ or $\sum_{n=0}^{\infty}(n\delta_n)^2<\infty$ according as $p(x)$, $\tilde{p}(x)\in SBV^1$ or SH^1, hence the series converges absolutely and uniformly. This proves (6.3). Now (6.4) follows by putting $x=t=0$, with $\tilde{h}-h=K(0, 0)$ in mind. By (6.3) and the formula $\tilde{p}(x)-p(x)=(d/dx)K(0, 0)$, (6.5) is formally valid, hence, in order to see the actual validity, it suffice to show the absolute and uniform convergence of the series in (6.5). This is easily shown, since, by Lemma 6.1, the n-th term in the series is of $O(|\tilde{\lambda}_n-\lambda_n|)=O(\delta_n)$.

Corollary to Proposition 6.2. *For a $\langle p, h\rangle\in SBV^1\times \boldsymbol{R}$ (resp. SH^1), if we put*

$$\lambda_n^* := \lambda_n(p-\alpha_0(p, h), h) = \lambda_n - \alpha_0(p, h), \tag{6.7}$$

and define the function $v(\lambda)$ by

$$v(\lambda) := w(\lambda+\alpha_0) = (\lambda_n^*-\lambda)\prod_{n=1}^{\infty}[(\lambda_n^*-\lambda)/(\pi n)^2], \tag{6.8}$$

then the boundary condition h is written as

$$h = -\sum_{n=0}^{\infty}[1+(-1)^{n+1}\cos(\lambda_n^*)^{1/2}]/[(-1)^{n+1}\dot{v}(\lambda_n^*)], \tag{6.9}$$

where the n-th term in the series is of order $O(1/n^6)$ (resp. $O(\delta'_n/n^5)$ with $\{\delta'_n\}\in \ell^2$) as $n\to\infty$.

Proof. This formula is obtained by replacing $p(x)$, $\tilde{p}(x)$, h and $\tilde{h}$ by 0, $p(x)-\alpha_0$, 0 and h respectively in (6.4). The assertion concerning the order follows from the estimate:

$$1 + (-1)^{n+1}\cos(\lambda_n^*)^{1/2} = 1 + (-1)^{n+1}\cos(\pi n+\delta'_n/n^2)$$

with $\delta'_n=O(1/n)$ (resp. $\{\delta'_n\}\in\ell^2$). Hence the lemma is proved.

Next we shall refer to the explicit form of the kernels $G(x, t)$ and $F(x, t)$.

Theorem 6.4. *If we assume*

$$a_0(p, h) = a_0(\tilde{p}, \tilde{h}). \tag{6.1}$$

Then the kernel $F(x, t)$ is expressed as

$$F(x, t) = \sum_{m,n=0}^{\infty} \frac{(-1)^n \tilde{w}(\lambda_n) d_m(x, \tilde{\lambda}_m) c_0(t, \lambda_n)}{(\tilde{\lambda}_m - \lambda_n) \|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2 \|c_0(\cdot, \lambda_n)\|^2}. \tag{6.10}$$

Similarly, if we assume

$$b_0(p) = b_0(\tilde{p}),$$

then

$$G(x, t) = \sum_{m,n=1}^{\infty} \frac{(-1)^{n+1} \tilde{v}(\mu_n) r_m(x, \tilde{\mu}_m) s_0(t, \mu_n)}{(\tilde{\mu}_m - \mu_n) \|\tilde{s}_0(\cdot, \tilde{\mu}_m)\|^2 \|s_0(\cdot, \mu_n)\|^2}. \tag{6.11}$$

The series on the right-hand sides and their termwise first order derivatives with respect to x or t converge absolutely and uniformly. Furthermore, there exists a constant B depending only on the B.V.P. $\langle p, h\rangle$ such that the following estimates:

$$|F(x, t)| \leqq B \sum_{m=0}^{\infty} (1+m)^{-1} |\tilde{\lambda}_m - \lambda_m|, \tag{6.12}$$

$$|F_x(x, t)| + |F_t(x, t)| \leqq B \sum_{m=0}^{\infty} |\tilde{\lambda}_m - \lambda_m|, \tag{6.13}$$

hold for $0 \leqq x, t \leqq 1$, provided that $\sum_{m=0}^{\infty} |\tilde{\lambda}_m - \lambda_m|$ is smaller than $(1/3) \min_{n \geqq 1} |\lambda_n - \lambda_{n-1}|$. Similar statement is also valid for $G(x, t)$.

In order to prove this theorem, we must study the convergence of the series in (6.10). To this end, the following estimates will be required.

Lemma 6.5. *Let the numbers b_1, b_2, $\tilde{b}_1$ and $\tilde{b}_2$ be denoted by*

$$b_1 := \sup_{n \geqq 0} |\lambda_n - (\pi n)^2|, \quad \tilde{b}_1 := \sup_{n \geqq 0} |\tilde{\lambda}_n - (\pi n)^2|,$$

$$b_2 := \sup_{m \neq n} \left| \frac{m^2 - n^2}{\lambda_m - \lambda_n} \right|, \quad \tilde{b}_2 := \sup_{m \neq n} \left| \frac{m^2 - n^2}{\tilde{\lambda}_m - \tilde{\lambda}_n} \right|.$$

Let I_n be the closed interval with end points λ_n and $\tilde{\lambda}_n$, then, by the asymptotic properties (6.2), there exists a non-negative integer N such that, for any $n \geqq N$ and for any $m \neq n$, the interval I_n contain neither λ_m nor $\tilde{\lambda}_m$. For $n \geqq N$, we define b_3 and $\tilde{b}_3$ by

$$b_3 := \sup_{\substack{m\neq n,\, n\geqq N \\ \nu_n \in I_n}} \left|\frac{m^2-n^2}{\lambda_m-\nu_n}\right|, \quad \tilde{b}_3 := \sup_{\substack{m\neq n,\, n\geqq N \\ \nu_n \in I_n}} \left|\frac{m^2-n^2}{\tilde{\lambda}_n-\nu_n}\right|.$$

Note that b_i, $\tilde{b}_i$ $(i=1, 2, 3)$ *are finite. Then the following estimates hold*:

(6.14)
$$\|c_0(\cdot, \lambda_n)\|^{-2} = |\dot{w}(\lambda_n)|^{-1} \leqq 2\exp(4b_1 b_2),$$
$$\|\tilde{c}_0(\cdot, \tilde{\lambda}_n)\|^{-2} = |\tilde{w}^{\cdot}(\tilde{\lambda}_n)|^{-1} \leqq 2\exp(4\tilde{b}_1 \tilde{b}_2).$$

Moreover for any $\nu_n \in I_n (n \geqq N)$,

(6.15)
$$|\dot{w}(\nu_n)| \leqq \{1+2b_3(2b_1+\tilde{b}_1)\}\exp\{2(2b_1+\tilde{b}_1)/\pi^2\},$$
$$|\tilde{w}^{\cdot}(\nu_n)| \leqq \{1+2\tilde{b}_3(2\tilde{b}_2+b_1)\}\exp\{2(2\tilde{b}_1+b_1)/\pi^2\}.$$

Let r *and* R *be defined by*

(6.16)
$$r := \min_{n\geqq 1} |\lambda_n-\lambda_{n-1}|, \quad R := \textstyle\sum_{n=0}^{\infty} |\tilde{\lambda}_n-\lambda_n|.$$

If $R \leqq r/3$, *then*

(6.17)
$$\tilde{b}_1 \leqq R + b_1, \quad N = 0, \quad \tilde{b}_2 \leqq \tilde{b}_3 \leqq 2b_3.$$

Proof. By the definition, we observe that

$$\begin{aligned}
|\dot{w}(\lambda_n)|^{-1} &= (\pi n)^2 \textstyle\prod_{m(\neq n)} |(\pi m)^2/[\lambda_m-\lambda_n]| \\
&= \left|(\pi n)^2 \prod_{m(\neq n)} \left(\frac{(\pi m)^2}{(\pi m)^2-(\pi n)^2}\right)\right| \prod_{m(\neq n)} \left|1+\frac{[\lambda_n-(\pi n)^2]-[\lambda_m-(\pi m)^2]}{\lambda_m-\lambda_n}\right| \\
&= |(d/d\lambda)(\lambda^{1/2}\sin\lambda^{1/2})|_{\lambda=(\pi n)^2}|^{-1} \textstyle\prod_{m(\neq n)} [1+2b_1/|\lambda_n-\lambda_m|] \\
&\leqq 2\exp(\textstyle\sum_{m(\neq n)} 2b_1/|\lambda_n-\lambda_m|) \leqq 2\exp(2b_1 b_2 \textstyle\sum_{m\neq n)} |m^2-n^2|^{-1}).
\end{aligned}$$

Hence, by virtue of the inequality:

(6.18)
$$\textstyle\sum_{m(\neq n)} |n^2-m^2|^{-1} \leqq \dfrac{2}{2n-1} + \left(\displaystyle\int_0^{n-1} + \int_{n+1}^{\infty}\right)\dfrac{dx}{|x^2-n^2|} \leqq 2$$

for $n \geqq 1$ (this is also valid for $n=0$), we obtain $|\dot{w}(\lambda_n)|^{-1} \leqq 2\exp(4b_1 b_2)$. Similarly $|\tilde{w}^{\cdot}(\tilde{\lambda}_n)|^{-1} \leqq 2\exp(4\tilde{b}_1\tilde{b}_2)$.

Next we shall show (6.15). We also observe that

$$\begin{aligned}
|\dot{w}(\nu_n)| &= |\textstyle\sum_{m=0}^{\infty} (\pi m)^{-2} \prod_{k(\neq m)} \{[\lambda_k-\nu_n]/(\pi k)^2\} \\
&\leqq \left|\frac{1}{(\pi n)^2} \prod_{k(\neq n)} \left(\frac{(\pi k)^2-(\pi n)^2}{(\pi k)^2}\right)\right| \prod_{k(\neq n)} \left|1+\frac{[\lambda_k-(\pi k)^2]-[\nu_n-(\pi n)^2]}{(\pi k)_{\tau}-(\pi n)^2}\right| \\
&\quad + \sum_{m(\neq n)} \left|\frac{(\pi m)^2-(\pi n)^2}{\lambda_m-\nu_n}\right| \left|\frac{[\lambda_n-(\pi n)^2]-[\nu_n-(\pi n)^2]}{(\pi m)^2-(\pi n)^2}\right|.
\end{aligned}$$

$$\cdot\left|\frac{1}{(\pi n)^2}\prod_{k,\neq n)}\left(\frac{(\pi k)^2-(\pi n)^2}{(\pi k)^2}\right)\right|\prod_{k(\neq n)}\left|1+\frac{[\lambda_k-(\pi k)^2]-[\nu_n-(\pi n)^2]}{(\pi n)^2-(\pi n)^2}\right|$$

$$\leqq |(d/d\lambda)(\lambda^{1/2}\sin\lambda^{1/2})|_{\lambda=(\pi n)^2}|\prod_{k(\neq n)}\left(1+\frac{2b_1+\tilde{b}_1}{\pi^2|k^2-n^2|}\right)$$

$$+b_3\sum_{m(\neq n)}\frac{2b_1+\tilde{b}_1}{|m^2-n^2|}\left|\frac{d}{d\lambda}(\lambda^{1/2}\sin\lambda^{1/2})\right|_{\lambda=(\pi n)^2}\left|\prod_{k(\neq n)}\left(1+\frac{2b_1+\tilde{b}_1}{\pi^2|k^2-n^2|}\right)\right.$$

$$\leqq \{1+b_3(2b_1+\tilde{b}_1)\sum_{m(\neq n)}|m^2-n^2|^{-1}\}\exp(\pi^{-2}(2b_1+\tilde{b}_1)\sum_{m(\neq n)}|m^2-n^2|^{-1}).$$

Hence, by using (6.18), we obtain $|\dot{w}(\nu_n)|\leqq(1+2b_3(2b_1+\tilde{b}_1)\cdot\exp\{2(2b_1+\tilde{b}_1)\})$ $(n>N)$.

Finally we shall see (6.17). The first inequality is evident. If $R\leqq r/3$, then for $n\neq m$ and $\nu_m\in I_m$ we have $|\lambda_n-\nu_m|\geqq|\lambda_n-\lambda_m|-|\lambda_m-\nu_m|\geqq r-|\lambda_m-\tilde{\lambda}_m|\geqq r-R\geqq 2r/3$. Similarly $|\tilde{\lambda}_n-\nu_m|\geqq 2r/3$, hence $N=0$. Moreover $|\tilde{\lambda}_n-\lambda_m|\leqq R\leqq r/3$, thus it follows that

$$|n^2-m^2|/|\tilde{\lambda}_n-\nu_m|\leqq(|n^2-m^2|/|\lambda_n-\nu_m|)(1-|\tilde{\lambda}_n-\lambda_n|/|\lambda_n-\nu_m|)^{-1}$$
$$\leqq(|n^2-m^2|/|\lambda_n-\nu_m|)\{1-(r/3)/(2r/3)\}\leqq 2(|n^2-m^2|/|\lambda_n-\nu_m|),$$

hence $\tilde{b}_3\leqq 2b_3$. Finally, $\tilde{b}_2\leqq\tilde{b}_3$ is obvious from the definition. Hence the lemma is proved.

Let $f(x, t)$ be the series defined by the right-hand side of (6.10), and $\varphi_m(t)$ be defined by the following formula:

$$\varphi_m^{(\nu)}(t):=\sum_{n=0}^{\infty}\frac{(-1)^n\tilde{w}(\lambda_n)c_0^{(\nu)}(t,\lambda_n)}{(\tilde{\lambda}_m-\lambda_n)\|c_0(\cdot,\lambda_n)\|^2},\quad \nu=0,1. \tag{6.19}$$

Then $f(x, t)$ is written as

$$f(x,t)=\textstyle\sum_{m=0}^{\infty}d_m(x,\tilde{\lambda}_m)\varphi_m(t)/\|\tilde{c}_0(\cdot,\tilde{\lambda}_m)\|^2. \tag{6.20}$$

Now we shall require the following lemma:

Lemma 6.6. *The series* (6.19) *for* $\nu=0, 1$ *converge absolutely and uniformly. Moreover suppose* $R\leqq r/3$ (*see* (6.16)), *then there exist positive constants* B_1 *and* B_2 *which depend only on the B.V.P.* $\langle p, h\rangle$, *such that the following estimates hold*:

$$\|\varphi_m^{(\nu)}(\cdot)\|_\infty\leqq B_1(1+m)^\nu, \tag{6.21}$$

$$\|d_m^{(\nu)}(\cdot,\tilde{\lambda}_m)\|_\infty\leqq B_2(1+m)^{\nu-1}|\tilde{\lambda}_m-\lambda_m|, \tag{6.22}$$

for $\nu=0, 1$ *and* $m=0, 1, 2,\cdots$.

Proof. The mean value theorem and (6.19) imply that there exist $\nu_n\in I_n$

such that the following inequality holds:

$$|\varphi_m^{(\nu)}(t)| \leqq \textstyle\sum_{n=0}^{\infty} |\tilde{w}^{\cdot}(\nu_n)(\tilde{\lambda}_n-\lambda_n)/(\tilde{\lambda}_m-\lambda_n)| \, \|c_0^{(\nu)}(\cdot,\lambda_n)\|_\infty / \|c_0(\cdot,\lambda_n)\|^2$$

$$= |\tilde{w}^{\cdot}(\nu_m)| \, \|c_0^{(\nu)}(\cdot,\lambda_m)\|_\infty / \|c_0(\cdot,\lambda_n)\|^2$$

$$+ \sum_{n(\neq m)} |\tilde{w}^{\cdot}(\nu_n)| \left| \frac{m^3-n^2}{\tilde{\lambda}_m-\lambda_n} \right| \frac{|\tilde{\lambda}_n-\lambda_n|}{|m^2-n^2|} \frac{\|c_0^{(\nu)}(\cdot,\lambda_n)\|_\infty}{\|c_0(\cdot,\lambda_n)\|^2}.$$

Thus, if b_4 be defined by

$$b_4 := \sup_{n\geqq 0} \|c_0(\cdot,\lambda_n)\|_\infty + \sup_{n\geqq 0} (1+n)^{-1} \|c_0'(\cdot,\lambda_n)\|_\infty,$$

which is, by Lemma 6.1, finite and depend only on $\langle p, h\rangle$, then Lemma 6.5 and the above inequality imply the estimate: for $\nu=0, 1$,

$$\|\varphi_m^{(\nu)}(\cdot)\|_\infty \leqq \tilde{B}_1\{(1+m)^\nu + \textstyle\sum_{n(\neq m)} |\tilde{\lambda}_n-\lambda_n|(1+n)^\nu/|m^2-n^2|\},$$

where we can take $\tilde{B}_1=2b_4(1+2b_3)\{1+4b_3(4b_3+b_1)\}\exp\{4b_1b_2+2(r+3b_1)/\pi^2\}$, and hence $\tilde{B}_1$ depends only on $\langle p, h\rangle$. Since we can observe that $(1+n)^\nu/|m^2-n^2|\leqq 1$ for $\nu=0, 1$, $m, n=0, 1, 2,\cdots$ with $m\neq n$, we obtain $\|\varphi_m^{(\nu)}(\cdot)\|_\infty \leqq \tilde{B}_1\{(1+m)^\nu+r\} \leqq B_1\cdot(1+m)^\nu$ with some B_1 depending only on $\langle p, h\rangle$.

Next we shall show (6.22). If we put

$$B_2 := \sup_{\nu=0,1,n\geqq 0,|\lambda-\lambda_n|\leqq r,} (1+n)^{1-\nu}\{\|c_0(\cdot,\lambda)\|_\infty + \|c_1(\cdot,\lambda)\|_\infty\},$$

then Lemma 6.1 implies that B_2 is finite and depends only on $\langle p, h\rangle$. Since $d_m^{(\nu)}(x, \tilde{\lambda}_m)=0$, by the mean value theorem, there exists $\nu_m\in I_m\subset[\lambda_m-r, \lambda_m+r]$ such that

$$\|d_m^{(\nu)}(x,\tilde{\lambda}_m)\| = |\dot{d}_m^{(\nu)}(x,\nu_m)| \, |\tilde{\lambda}_m-\lambda_m| \leqq B_2(1+m)^{\nu-1}|\tilde{\lambda}_m-\lambda_m|.$$

Hence the lemma is proved.

Proof of Theorem 6.4. It is sufficient to show (6.10), i.e. the equation $F(x, t)=f(x, t)$, for, if we can show it, then the other statements in the theorem will be easily proved by virtue of (6.20), Lemma 6.6 and the inequality:

$$\|\tilde{c}_0(\cdot,\tilde{\lambda}_m)\|^{-2} \leqq 2\exp\{(8r/3+b_1)b_3\} =: B_3, \tag{6.23}$$

which follows from (6.14) and (6.17), where B_3 depends only on $\langle p, h\rangle$.

Now we shall prove the equation $f(x, t)=F(x, t)$. An argument similar to that which derives the formulas (4.12), (4.13′) and (4.24), though the question on the convergence of infinite sequences is now added, and Lemma 3.2 show that $F(x, t)$ solves the following hyperbolic initial problem in D_1:

$$(6.24)\qquad \begin{cases} \square_p F = H(x, t), & (x, t) \in D_1, \quad (D_1\text{: see Fig. 1 in §4}) \\ F(x, 0) = h_0(x), & F_t(x, t)|_{x=0} = h_1(x). \end{cases}$$

where

$$(6.25)\qquad \begin{cases} H(x, t) := \sum_{m,n=0}^{\infty} \dfrac{d_m(x, \tilde{\lambda}_m)\cdot(-1)^{n+1}\tilde{w}(\lambda_n)c_0(t, \lambda_n)}{\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2 \|c_0(\cdot, \lambda_n)\|^2}, \\ h_0(x) := \sum_{m=0}^{\infty} d_m(x, \tilde{\lambda}_m)/\|c_0(\cdot, \tilde{\lambda}_m)\|^2, \quad h_1(x) := h\cdot h_0(x). \end{cases}$$

Let $H_N(x, t)$ and $h_{v,N}(x)$ $(v=0, 1)$ be defined by replacing the infinite summations in (6.25) by $\sum_{m,n=0}^{N}$ and $\sum_{m=0}^{N}$ respectively, and let us consider the problem (6.24) with data $\{H(x, t), h_0(x), h_1(x)\}$ replaced by $\{H_N(x, t), h_{0,N}(x), h_{1,N}(x)\}$. As is easily seen, its solution is given by

$$(6.26)\qquad f_N(x, t) := \sum_{m,n=0}^{N} \frac{(-1)^n\, d_m(x, \tilde{\lambda}_m)c_0(t, \lambda_n)}{(\tilde{\lambda}_m - \lambda_n)\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2 \|c_0(\cdot, \lambda_n)\|^2}.$$

By the definition of $f(x, t)$, note that

$$(6.27)\qquad \|f(\cdot,\cdot)-f_N(\cdot,\cdot)\|_{C([0,1]\times[0,1])} \longrightarrow 0, \quad \text{as} \quad N \longrightarrow \infty,$$

while the continuous dependence of the solution to (6.24) on the deta $\{H(x, t), h_0(x), h_1(x)\}$ imply that

$$(6.28)\qquad \|F(\cdot,\cdot)-f_N(\cdot,\cdot)\|_{C(\bar{D}_1)} \longrightarrow 0, \quad \text{as} \quad N \longrightarrow \infty,$$

similarly it follows that

$$(6.28')\qquad \|F(\cdot,\cdot)-f_N(\cdot,\cdot)\|_{C\bar{D}_3)} \longrightarrow 0, \quad \text{as} \quad N \longrightarrow \infty,$$

from which $F(x, t)=f(x, t)$ in $\bar{D}_1 \cup \bar{D}_3$ follows. The equation $F(x, t)=f(x, t)$ in $D_2 \cup D_4$ is obtained by (6.27), (6.28), (6.28′) and by the continuous dependence of the solution of the Goursat problem on the deta which is put on the lines $x=t$ and $x=1-t$. Hence the theorem is proved.

§7. Well-posedess and approximation theorems

In this section, we shall consider a well-posedness problem and an approximation problem, and complete the proofs of Theorem B and Theorem C. To this end, we shall apply the result of Theorem 6.4 to the Gel'fand-Levitan type equation (3.22). Also in this case, we shall only consider a problem of the third kind.

Proof of Theorem B. Recall that $K(x, t)$ satisfy the integral equation (see (3.22)):

$$(7.1)\qquad K(x, t) + F(x, t) + \int_0^1 K(x, s)F(s, t)ds = 0 \quad (0<t<x).$$

Theorem 6.4 implies that, if $\sum_{m=0}^{\infty}|\tilde{\lambda}_m-\lambda_m|<(1/3)r$, with $r:=\min_{n\geqq 0}|\lambda_{n+1}-\lambda_n|$, then it follows that

$$(7.2)\qquad |F(x, t)|,\ |(d/dx)F(x, x)|,\ |F_x(x, t)| \leqq B\sum_{m=0}^{\infty}|\tilde{\lambda}_m-\lambda_m|,$$

where B is a constant depending only on $\langle p, h\rangle$. Hence, if we take a positive constant B' so as to satisfy $B'<\min\{(1/3)r, 1/B\}$, and assume $\sum_{m=0}^{\infty}|\tilde{\lambda}_m-\lambda_m|\leqq B'$, then we have

$$(7.3)\qquad \int_0^x\int_0^x |F(x, t)|^2 dsdt \leqq BB' < 1, \quad (0\leqq x\leqq 1),$$

so that the integral equation (7.1) can be solved by iteration and its solution $K(x, t)$ admits the bound:

$$(7.4)\qquad |K(x, t)| \leqq C\sum_{m=0}^{\infty}|\tilde{\lambda}_m-\lambda_m|,$$

where C is a constant depending only on $\langle p, h\rangle$. In the sequel, without any comment, we shall use the single letter C for the representation of various constants depending only on $\langle p, h\rangle$.

With the bounds (7.2) and (7.4) in mind, we shall make use of the formula:

$$\tilde{p}(x) - p(x) = 2(d/dx)F(x, x) - 2K(x, x)^2 + 4\int_0^x K(x, t)F_x(x, t)dt,$$

which can be shown in a similar manner as in A. Mizutani [17, Lemma 5.1], to obtain the desired estimate:

$$|\tilde{p}(x)-p(x)| \leqq C\sum_{m=0}^{\infty}|\tilde{\lambda}_m-\lambda_m| \quad (0\leqq x\leqq 1).$$

As for the estimate of the difference of the two boundary conditions, by using (6.4), (6.22) and (6.23), we see that

$$\begin{aligned}|\tilde{h}-h| &\leqq \sum_{m=0}^{\infty}|d_m(0, \tilde{\lambda}_m)|/\|\tilde{c}_0(\cdot, \tilde{\lambda}_m)\|^2\\ &\leqq C\sum_{m=0}^{\infty}(1+m)^{-1}\|\tilde{\lambda}_m-\lambda_m\|.\end{aligned}$$

Hence the theorem is proved.

Proof of Theorem C. This theorem is an immediate consequence of the combination of Corollaries to Theorem A and Theorem A* with Theorem B. Indeed, by the definitions of α_0, $p_N(\cdot)$, and h_N (see (2.5)–(2.7)) and by Corollary to Theorem A, we see that

$$\begin{cases} \lambda_n(p(\cdot)-\alpha_0,\ h) = \lambda_n^*, \\ \lambda_n(p_N,\ h_N) = \lambda_n^* \text{ or } (\pi n)^2 \text{ according as } n \leqq N \text{ or } n > N. \end{cases}$$

Recall that the asymptotic distribution of $\{\lambda_n^*\}$ is given by $\lambda_n^*=(\pi n)^2+\varepsilon_n$, where $\varepsilon_n=O(n^{-2})$ as $n\to\infty$ or $\sum_{n=0}^{\infty}(n\varepsilon_n)^2<+\infty$ according as $p(x)\in SBV^1$ or $p(x)\in SH^1$. Hence it follows that

$$A_1 := \sum_{n=0}^{\infty} |\lambda_n(p(\cdot)-\alpha_0,\ h)-\lambda_n(p_N,\ h_N)| = \sum_{n=N+1}^{\infty} |\varepsilon_n|,$$

$$A_2 := \sum_{n=0}^{\infty} (1+n)^{-1}|\lambda_n(p(\cdot)-\alpha_0,\ h)-\lambda_n(p_N,\ h_N)| = \sum_{n=N+1}^{\infty} (1+n)^{-1}|\varepsilon_n|,$$

and that, if $p(x)\in SBV^1$, then $A_1=O(\sum_{n=N+1}^{\infty} n^{-2})=O(N^{-1})$, $A_2=O(\sum_{n=N+1}^{\infty} n^{-3})=O(N^{-2})$, whereas, if $p(x)\in SH^1$, then $A_1=O(\sum_{N+1}^{\infty} n^{-1}\cdot n|\varepsilon_n|)\leqq O(\{\sum_{n=N+1}^{\infty} n^{-2}\}^{1/2}\cdot\{\sum_{n=N+1}^{\infty}(n|\varepsilon_n|)^2\}^{1/2})=o(N^{-1/2})$, $A_2=O(\sum_{n=N+1}^{\infty} n^{-2}\cdot n|\varepsilon_n|)\leqq O(\{\sum_{n=N+1}^{\infty} n^{-4}\}^{1/2}\cdot\{\sum_{n=N+1}^{\infty}(n|\varepsilon_n|)^2\}^{1/2})=o(N^{-3/2})$.

Now applying Theorem B, we obtain

$$\|p(\cdot)-\alpha_0-p_N(\cdot)\|_\infty = O(N^{-1}) \quad (\text{resp.} = o(N^{-1/2})),$$

$$|h-h_N| = O(N^{-2}) \quad (\text{resp.} =o(N^{-3/2}) \quad \text{as} \quad N \longrightarrow \infty,$$

if $p(x)\in SBV^1$ (resp. $p(x)\in SH^1$). Hence the theorem is proved.

§8. Proof of Theorem 2.3.

Let us put $\lambda_n=\lambda_n(p,\ h)$, $\tilde{\lambda}_n=\lambda_n(\tilde{p},\ \tilde{h})$, for brevity. The formula (4.4) implies that under the assumption of the assertion 1) the transformation kernel $K(x,\ t)$ vanishes identically, and hence $\langle p,\ h\rangle=\langle\tilde{p},\ \tilde{h}\rangle$, while under the assumption of assertion 2) $K(x,\ t)$ takes the following form:

$$K(x,\ t) = \tilde{d}_N(x,\ \lambda_N)c_0(t,\ \lambda_N)/\|c_0(\cdot,\ \lambda_N)\|^2. \tag{8.1}$$

Thus if we define the function $K^*(x,\ t)$ by

$$K^*(x,\ t) := -\ \tilde{d}_N(t,\ \lambda_N)c_0(x,\ \lambda_N)/\|c_0(\cdot,\ \lambda_N)\|^2, \tag{8.2}$$

then immediately we observe that $K^*(x, t)$ is the solution of the hyperbolic problem:

$$\begin{cases} (1) \quad K_{xx}^* - K_{tt}^* = \{p(x)-\tilde{p}(t)\}K^*, \\ (2) \quad K^*(x,\ x) = (h-\tilde{h}) + (1/2)\int_0^x \{p(t)-\tilde{p}(t)\}dt, \\ (3) \quad K_t^*(x,\ t)|_{t=0} = \{\tilde{d}_N'(0,\ \lambda_N)/\tilde{d}_N(0,\ \lambda_N)\}K^*(x,\ 0), \end{cases} \tag{8.3}$$

if $\tilde{d}_N(0,\ \lambda_N)=0$, then (3) should be replaced by

$$(3') \quad K^*(x,\ 0) = 0.$$

Lemma 8.1.

1) $\tilde{h}=h$ *if and only if* $\tilde{d}_N(0, \lambda_N)=0$.

2) *If* $\tilde{h}\neq h$, *then*

$$\tilde{d}'_N(0, \lambda_N)/\tilde{d}_N(0, \lambda_N) = \tilde{h} + (\tilde{\lambda}_N-\lambda_N)/(\tilde{h}-h). \tag{8.4}$$

Proof. Since $K(0, 0)=\tilde{h}-h$, putting $x=t=0$ in (8.1), we have

$$\tilde{h} - h = \tilde{d}_N(0, \lambda_N)/\|c_0(\cdot, \lambda_N)\|^2, \tag{8.5}$$

from which the assertion 1) follows. On the other hand, from the definition of $\tilde{d}_N(x, \lambda)$, we see that

$$\begin{aligned}\tilde{d}'_N(0, \lambda_N) &= \tilde{h} + (-1)^N\tilde{c}'_0(1, \lambda_N)\\ &= \tilde{h}\{1+(-1)^{N+1}\tilde{c}_0(1, \lambda_N)\} + (-1)^N\{\tilde{c}'_0(1, \lambda_N)+\tilde{h}\tilde{c}_0(1, \lambda_N)\}\\ &= \tilde{h}\tilde{d}_N(0, \lambda_N) + (-1)^N\tilde{w}(\lambda_N).\end{aligned}$$

Since $\tilde{\lambda}_n=\lambda_n$ or $\tilde{\lambda}_n\neq\lambda_n$ according as $n\neq N$ or $n=N$, it follows $\tilde{w}(\lambda)/(\lambda-\tilde{\lambda}_N)= w(\lambda)/(\lambda-\lambda_N)$, and hence

$$\begin{aligned}(-1)^N\tilde{w}(\lambda_N) &= (\tilde{\lambda}_N-\lambda_N)\cdot(-1)^N\dot{w}(\lambda_N)\\ &= (\tilde{\lambda}_N-\lambda_N)\,\|c_0(\cdot, \lambda_N)\|^2\\ &= \tilde{d}_N(0, \lambda_N)\,\{(\tilde{\lambda}_N-\lambda_N)/(\tilde{h}-h)\}.\end{aligned}$$

Here the last equality follows from (8.5). Hence we obtain

$$\tilde{d}'_N(0, \lambda_N) = \{\tilde{h}+(\tilde{\lambda}_N-\lambda_N)/(\tilde{h}-h)\},$$

which proves the assertion 2).

Now, if $\tilde{h}\neq h$, then the formula (2) of (8.3) should be rewritten as follows:

$$K^*(x, x) = \left(h+\frac{\tilde{\lambda}_N-\lambda_N}{\tilde{h}-h}\right) - \left(\tilde{h}+\frac{\tilde{\lambda}_N-\lambda_N}{\tilde{h}-h}\right) + \int_0^x \{p(t)-\tilde{p}(t)\}\,dt. \tag{$\bar{2}$}$$

Let us define the functions $\varphi(x, \lambda)$ and $\tilde{\varphi}(x, \lambda)$ by the solutions of the following differential equations:

$$-\varphi'' + p(x)\varphi = \lambda\varphi, \quad -\tilde{\varphi}'' + \tilde{p}(x)\tilde{\varphi} = \lambda\tilde{\varphi}, \tag{8.6}$$

which satisfy the initial conditions:

$$\begin{aligned}&\varphi(0) = 1, \quad \varphi'(0) = j := h + (\tilde{\lambda}_N-\lambda_N)/(\tilde{h}-h),\\ &\tilde{\varphi}(0) = 1, \quad \tilde{\varphi}'(0) = \tilde{j} := \tilde{h} + (\tilde{\lambda}_N-\lambda_N)/(\tilde{h}-h),\end{aligned} \tag{8.7}$$

where, if $\tilde{h}=h$, then the initial conditions should be replaced by

(8.7') $\varphi(0)=1,\quad \varphi'(0)=0,\quad \tilde{\varphi}(0)=1,\quad \tilde{\varphi}'(0)=0,$

and the j and $\tilde{j}$ be regarded as infinity ∞. Then (8.3) and Lemma 8.1 imply that $K^*(x, t)$ is the transformation kernel of $\tilde{\varphi}(x, \lambda)$ into $\varphi(x, \lambda)$, i.e.,

(8.8) $$\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) + \int_0^x K^*(x, t)\tilde{\varphi}(t, \lambda)dt.$$

On the other hand, by Lemma 8.1. and the identity $\tilde{d}_N(1-x, \lambda_N) = (-1)^{N+1}\tilde{d}_N(x, \lambda_N)$, there exists a non-negative integer M such that

(8.9) $$\begin{cases} M - N \equiv 1 \text{ or } 0 \pmod 2 \text{ according as } h \neq \tilde{h} \text{ or } h = \tilde{h}, \\ \lambda_N = \lambda_M(\tilde{p}, \tilde{j}), \\ \tilde{d}_N(x, \lambda_N) \text{ is an eigenfunction of } \langle \tilde{p}, \tilde{j} \rangle \text{ corresponding to} \\ \text{the eigenvalue } \lambda_M(\tilde{p}, \tilde{h}), \end{cases}$$

where we note that $\tilde{d}_N(x, \lambda_N)$ does not vanish identically, since $\tilde{\lambda}_N \neq \lambda_N$. In the following we shall write as $\lambda_n^* := \lambda_n(p, j)$, $\tilde{\lambda}_n^* := \lambda_n(\tilde{p}, \tilde{j})$.

By virtue of (8.8), we observe that

$$\varphi'(1, \lambda) + j\varphi(1, \lambda) = \tilde{\varphi}'(1, \lambda) + \left(j - \frac{c_0(1, \lambda_N)\tilde{d}_N(1, \lambda_N)}{\|c_0(\cdot, \lambda_N)\|^2} \right) \tilde{\varphi}(1, \lambda)$$
$$+ c_N \int_0^1 \tilde{d}_N(t, \lambda_N)\tilde{\varphi}(t, \lambda)dt,$$

where $c_N := -\{c_0'(1, \lambda_N) + jc_0(1, \lambda_N)\}/\|c_0(\cdot, \lambda_N)\|^2$. Since it follows from (8.5) that $j - c_0(1, \lambda_N)\tilde{d}_N(1, \lambda_N)/\|c_0(\cdot, \lambda_N)\|^2 = j + \tilde{d}_N(0, \lambda_N)/\|c_0(\cdot, \lambda_N)\|^2 = j + (\tilde{h} - h) = \tilde{j}$, so that

(8.10) $$\varphi'(1, \lambda) + j\varphi(1, \lambda) = \tilde{\varphi}'(1, \lambda) + \tilde{j}\tilde{\varphi}(1, \lambda)$$
$$+ c_N \int_0^1 \tilde{d}_N(t, \lambda_M^*)\tilde{\varphi}(t, \lambda)dt$$

holds. Now, putting $\lambda = \tilde{\lambda}_m^*$ $(m \neq M)$ in (8.10), then (8.9) and the orthogonality of the eigenfunctions yield $\varphi'(1, \tilde{\lambda}_m^*) + j\varphi(1, \tilde{\lambda}_m^*) = 0$, consequently it follows that

(8.11) $\{\tilde{\lambda}_m^*\}_{m \neq M}$ are also the eigenvalues of $\langle p, j \rangle$.

Exchanging the roles of $\langle p, h \rangle$ and $\langle \tilde{p}, \tilde{h} \rangle$, we can also show that there exists a non-negative integer M' such that

(8.12) $$M' - N \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pmod 2 \text{ according as } h \begin{pmatrix} \neq \\ = \end{pmatrix} h, \quad \tilde{\lambda}_N = \lambda_{M'}^*.$$

(8.13) $\{\lambda_m^*\}_{m\neq M'}$ are also the eigenvalues of $\langle\tilde{p}, \tilde{j}\rangle$.

Now we shall show that $M=M'$ holds. To this end we may assume $M\geqq M'$. With the asymptotic distributions of λ_m^* and $\tilde{\lambda}_m^*$ took into account, (8.11) and (8.13) imply that, for any sufficiently large m, $\lambda_m^*=\tilde{\lambda}_m^*$ holds, and subsequently it follows that $\lambda_m^*=\tilde{\lambda}_m^*$ for $m\geqq M+1$ or $m\leqq M'-1$. Hence, if $M\neq M'$, then it must follow from (8.11) and (8.13) that either $\lambda_M^*=\tilde{\lambda}_M^*$ or $\lambda_M^*=\tilde{\lambda}_{M-1}^*$ hold. In the former case, further we must have $\lambda_m^*=\tilde{\lambda}_m^*$ for $m=M',\cdots, M-1$, and then for all indeces. By the assertion 1), this implies $\langle p, j\rangle=\langle\tilde{p}, \tilde{h}\rangle$, and then $\langle p, h\rangle=\langle\tilde{p}, h\rangle$, which contradicts the assumption $\lambda_N\neq\tilde{\lambda}_N$. In the latter case, since $\varphi(1-x, \lambda_M^*)=(-1)^M\cdot\varphi(x, \lambda_M^*)$ and $\tilde{\varphi}(1-x, \tilde{\lambda}_M^*)=(-1)^{M-1}\tilde{\varphi}(x, \tilde{\lambda}_M^*)$ hold, the replacement of x (resp. t) by $1-x$ (resp. $1-t$) and the substitution of $\lambda_M^*=\tilde{\lambda}_{M-1}^*$ into λ in the formula (8.8) yield the following formula,

$$(8.14)\qquad (-1)^M\varphi(x, \lambda_M^*) = (-1)^{M-1}\left\{\tilde{\varphi}(x, \tilde{\lambda}_{M-1}^*)+\int_x^1 K^*(x, t)\tilde{\varphi}(t, \tilde{\lambda}_{M-1}^*)dt\right\},$$

here the relation $K^*(1-x, 1-t)=-K^*(x, t)$ is used. Hence we calculate (8.8)+$(-1)^{M-1}$(8.14) to obtain:

$$(8.15)\qquad 0 = 2\tilde{\varphi}(x, \tilde{\lambda}_{M-1}^*) + \int_0^1 K^*(x, t)\tilde{\varphi}(t, \tilde{\lambda}_{M-1}^*)dt.$$

The second term on the right-hand side of (8.15) is zero, which follows from the explicit form (8.2) of $K^*(x, t)$ and the orthogonality of $\tilde{d}_N(\cdot, \lambda_N)$ and $\tilde{\varphi}(\cdot, \tilde{\lambda}_{M-1}^*)$ in $L^2(0, 1)$ (see (8.9)). Hence $\tilde{\varphi}(\cdot, \tilde{\lambda}_{M-1}^*)$ vanishes identically, which contradicts the definition of $\tilde{\varphi}(\cdot, \lambda)$. Hence we must have

$$(8.16)\qquad M = M',\quad \lambda_m^* = \tilde{\lambda}_m^*\qquad \text{for}\quad m\neq M.$$

To establish the Theorem 2.3, it only remains to show that the number M is equal to $N-1$, N or $N+1$ according as $(\tilde{h}-h)(\tilde{\lambda}_N-\lambda_N)$ is positive, zero or negative.

In the case where $(\tilde{h}-h)(\tilde{\lambda}_N-\lambda_N)$ is positive, j is finite and $j>h$. Since the quadratic forms assocoated with the B.V.P.'s $\langle p, h\rangle$ and $\langle p, j\rangle$ are given by $H_{p,h}[\cdot]$ and $H_{p,j}[\cdot]$ respectively, where

$$H_{p,k}[u] := \int_0^1 \{|u'(x)|^2+p(x)|u(x)|^2\}dx + k\{|u(0)|^2+|u(1)|^2\},$$

with $k=h, j$, it follows that $H_{p,j}>H_{p,h}$, and consequently $\lambda_M^*>\lambda_M$. Hence we have $\lambda_{N+1}=\tilde{\lambda}_{N+1}>\tilde{\lambda}_N=\lambda_M^*>\lambda_M$, i.e. $N\geqq M$. On the other hand, since $\varphi(x, \lambda_M^*)$ and $c_0(x, \lambda_{N-1})$ satisfies the differential equations $\varphi''+\{\lambda_M^*-p(x)\}\varphi=0$ and $c_0''+\{\lambda_{N-1}-p(x)\}c_0=0$ respectively and the latter has $N-1$ zeros in $0<x<1$, the inequality $\lambda_M^*=\tilde{\lambda}_N>\lambda_{N-1}$ and Sturm's comparison theorem imply that the M-th eigenfunction $\varphi(x, \lambda_M^*)$ of $\langle p, j\rangle$ has at least $N-2$ zeros in $0<x<1$, and hence $M\geqq N-2$. Thus, taking $N-M\equiv 1 \pmod 2$ in to account, we obtain $M=N-1$. In the case where $(\tilde{\lambda}_N-\lambda_N)(\tilde{h}-h)$ is negative, we can also obtain $M=N+1$ in a similar manner.

Finally we consider the case $(\tilde{\lambda}_N-\lambda_N)(\tilde{h}-h)=0$, i.e. $h=\tilde{h}$. In this case $j=\tilde{j}=\infty$. Without loss of generality we may assume that $\lambda_N<\tilde{\lambda}_N$ $(=\lambda_M^*)$. This assumption and Sturm's comparision theorem imply that $\varphi(x, \lambda_M^*)$ has at least $N-1$ zeros in $0<x<1$, and hence $M\geqq N-1$, since $c_0(x, \lambda_N)$ (resp. $\varphi(x, \lambda_M^*)$) has N (resp. M) zeros there. On the other hand, since $d_N(x, \tilde{\lambda}_N)$ is the M-th eigenfunction of $\langle p, \infty\rangle$, $d_N(x, \tilde{\lambda}_N)$ has $M+2$ zeros in $0\leqq x\leqq 1$ including the both end points. Hence, by the comparision theorem, $c_0(x, \lambda_{N+1})$ must have at least $M+1$ zeros in $0<x<1$, so that $N\geqq M$. Thus, taking $M\equiv N \pmod 2$ into account, we obtain $M=N$. Hence the theorem is established.

§ 9. Notes and comments

The arguements so far developed can be extended to the general case where the symmetricity of the potential and the boundary condition need not be assumed.

Let us consider the B.V.P.'s:

$$\text{(9.1)} \qquad \langle p, h, H\rangle: \quad \begin{cases} -y'' + p(x)y = \lambda y \quad (0<x<1), \\ y'(0) - hy(0) = y'(1) + Hy(1) = 0, \end{cases}$$

$$\text{(9.2)} \qquad \langle p, \infty, \infty\rangle: \quad \begin{cases} -y'' + p(x)y = \lambda y \quad (0<x<1), \\ y(0) = y(1) = 0. \end{cases}$$

The B.V.P.'s $\langle p, h, \infty\rangle$ and $\langle p, \infty, H\rangle$ may be also considered. The eigenvalues of (9.1) and (9.2) will be denoted by $\{\lambda_n=\lambda_n(p, h, H)\}_{n=0}^{\infty}$ and $\{\mu_n=\lambda_n(p, \infty, \infty)\}_{n=1}^{\infty}$ respectively. Let the solutions $c_\nu(x, \lambda)$, $s_\nu(x, \lambda)$ $(\nu=0, 1)$ of the differential equation $-y''+p(x)y=\lambda y$ be defined by requiring the initial conditions:

$$\text{(9.3)} \qquad \begin{aligned} &c_0(0, \lambda) = c_1(1, \lambda) = 1, \quad c_0'(0, \lambda) = h, \quad c_1'(1, \lambda) = -H, \\ &s_0(0, \lambda) = s_1(1, \lambda) = 0, \quad s_0'(0, \lambda) = 1, \quad s_1'(1, \lambda) = -1. \end{aligned}$$

Since the multiplicity of each eigenvalue is one, there exist real numbers $k_n=k_n(p, h, H)$ and $j_n=k_n(p, \infty, \infty)$ such that the relations:

$$\text{(9.4)} \qquad \begin{aligned} &c_0(x, \lambda_n) = (-1)^n k_n c_1(x, \lambda_n), \quad (n\geqq 0), \\ &s_0(x, \mu_n) = (-1)^{n+1} j_n s_1(x, \mu_n) \quad (n\geqq 1). \end{aligned}$$

The numbers k_n, j_n are shown to be positive. The sequence of pairs $\{(\lambda_n(p, h, H), k_n(p, h, H))\}_{n=0}^{\infty}$(resp. $\{(\lambda_n(p, \infty, \infty), k_n(p, \infty, \infty))\}_{n=1}^{\infty}$) will be called the spectral characteristic of $\langle p, h, H\rangle$ (resp. $\langle p, \infty, \infty\rangle$), as is well known, which is equivalent to what is usually called so, that is, the sequence of the pairs of eigenvalues and normalizing constants. Note that the spatial symmetric problem corresponds to the case where $k_n=1$ (or $j_n=1$) for every n's.

Let us consider other B.V.P.'s $\langle \tilde{p}, \tilde{h}, \tilde{H}\rangle$ and $\langle \tilde{p}, \infty, \infty\rangle$, introduce the deformation kernels $K(x, t)$, $L(x, t)$, $\tilde{K}(x, t)$ e.t.c., and then define the kernels $F(x, t)$ and $G(x, t)$ in a similar manner as in (3.11) and (3.12). Now the functions $d_n(x, \lambda)$ and $r_n(x, \lambda)$ defined by (21.), (2.2) are clearly replaced by

$$\begin{aligned} d_n(x, \lambda) &:= c_0(x, \lambda) + (-1)^{n+1} k_n c_1(x, \lambda), \\ r_n(x, \lambda) &:= s_0(x, \lambda) + (-1)^n j_n s_1(x, \lambda). \end{aligned} \tag{9.5}$$

We can deduce an explicit form of the kernel $F(x, t)$ and $G(x, t)$ and then prove an extended versions of Theorem A, B, C. Note that an extension of Theorem A can also obtained by combinig Theorem A with the results by E. L. Issacson, H. P. McKean, E. Trubowitz, [11] and E. L. Issacson, E. Trubowitz [12], where they studied the isospectral manifold $M(p, h, H)=\{\langle \tilde{p}, \tilde{h}, \tilde{H}\rangle; \lambda_n(\tilde{p}, \tilde{h}, \tilde{H})=\lambda_n(p, h, H)$ for every $n\}$ and obtained a formula to express any element in $M(p, h, H)$ in terms of the given base point in $M(p, h, H)$, moreover, they showed that each isospectral manifold contains one and only one spatially symmetric B.V.P.

Furthermore our method has some applications to the periodic problem. First, if two periodic potentials $p(x)$ and $\tilde{p}(x)$ are finite gap potentials with the same periodic eigenvalues, then the kernel $G(x, t)$ reduces to the degenerate one, and then it gives rise to the so-called FIT formula in A. Finkel, E. Issacson, E. Trubowitz [5]. Second, we can give an alternative proof of the regularity theorems that, if the width of the n-th gap decreases with oreder $O(n^{-k-2})$, then the potential is in C^k-class (H. P. McKean, E. Trubowitz [15]), and that, if it decreases exponentially, then the potential is real nalytic (E. Trubowitz [25]). Finally we can prove the approximation theorem that any periodic potential can be approximated by finite gap potentials.

In the forthcoming article, we shall report on these matters.

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