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Oscillation Theory of Entire Solutions of Second Order Superlinear Elliptic Equations

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1. Introduction

In this paper we are concerned with the oscillatory behavior of entire solutions of the semilinear elliptic equation

$$(1)\qquad \Delta u + p(|x|)|u|^{\gamma}\operatorname{sgn} u = 0, \quad x \in \boldsymbol{R}^n, \quad n \geqq 3,$$

where Δ is the n-dimensional Laplacian and $|x|$ denotes the Euclidean length of $x \in \boldsymbol{R}^n$. We assume throughout that

(a) $\gamma > 1$ (namely, (1) is superlinear); and

(b) $p \in C[0, \infty) \cap C^1(0, \infty)$ and $p(t) > 0$ for $t > 0$.

By an entire solution of (1) we mean a function $u \in C^2(\boldsymbol{R}^n)$ which satisfies (1) at every point of $\boldsymbol{R}^n$. Such a solution is called oscillatory if it has a zero in any neighborhood of infinity, i.e., in any domain of the form $\{x \in \boldsymbol{R}^n : |x| > a\}$, $a > 0$. Otherwise the solution is called nonoscillatory.

Basic to our consideration is the fact that, for every $\alpha \in \boldsymbol{R}$, equation (1) has a unique radially symmetric entire solution $u_\alpha(x)$ such that $u_\alpha(0) = \alpha$ (see Theorem 1 below). Here, the radial symmetry of a function means that it depends only on $|x|$. The objective of this paper is to develop oscillation and nonoscillation criteria for these entire solutions $u_\alpha(x)$, $\alpha \in \boldsymbol{R}$. We present conditions which imply that all the $u_\alpha(x)$ are oscillatory as well as those which imply that all $u_\alpha(x)$ with $\alpha \neq 0$ are nonoscillatory. Besides, we study the asymptotic behavior as $|x| \to \infty$ of nonoscillatory entire solutions of (1).

Since our attention is focused on radially symmetric solutions, the problem under study reduces to the one-dimensional (singular) initial value problem

$$(2)\qquad (t^{n-1}y')' + t^{n-1}p(t)|y|^{\gamma}\operatorname{sgn} y = 0, \quad t > 0,$$

$$(3)\qquad y(0) = \alpha, \quad y'(0) = 0,$$

and the main results are obtained through the analysis of the problem (2)-(3) which is based on the extensive use of a Liapunov-like function as introduced in our previous paper [10]. All theorems are formulated in terms of the function

(4) $$P_{n,\gamma}(t) = t^{[n+2-\gamma(n-2)]/2}p(t), \quad t > 0.$$

For example, it is shown that the condition $P'_{n,\gamma}(t) \geqq 0$, $\not\equiv 0$ for $t>0$ guarantees the oscillation of all radially symmetric entire solutions of (1), and that if $P'_{n,\gamma}(t) \leqq 0$, $\not\equiv 0$ for $t>0$ and $\lim_{t\to\infty} P_{n,\gamma}(t)>0$, then every non-trivial radially symmetric entire solution of (1) is nonoscillatory and behaves like a constant multiple of $|x|^{(2-n)/2}$ as $|x|\to\infty$.

Semilinear elliptic equations, including (1), in the entire space $\boldsymbol{R}^n$ have been the object of intensive studies in recent years; see e.g. the papers [1, 3–16, 19]. Most of the literature on this subject, however, has been concerned with the existence and nonexistence of positive entire solutions, and very little is known about the oscillation property of entire solutions even for simple equations of the type (1). This work is an attempt at a systematic investigation of the oscillatory behavior of entire solutions of second order nonlinear elliptic equations in $\boldsymbol{R}^n$.

2. Existence of Entire Solutions

We begin by proving a basic existence theorem for radially symmetric entire solutions of (1).

Theorem 1. *For every* $\alpha \in \boldsymbol{R}$ *there exists a unique radially symmetric entire solution* $u_\alpha(x)$ *of equation* (1) *such that* $u_\alpha(0)=\alpha$.

Proof. It suffices to demonstrate the existence, for each α, of a unique solution $y_\alpha(t)$ of the problem (2)–(3). The desired entire solution of (1) is then given by $u_\alpha(x)=y_\alpha(|x|)$.

Let α be any positive number. Choose $\delta>0$ small enough so that

$$\int_0^\delta tp(t)dt \leqq \frac{(n-2)\alpha^{1-\gamma}}{2},$$

and consider the set of functions

$$Y = \{y \in C[0, \delta]: \frac{\alpha}{2} \leqq y(t) \leqq \alpha \text{ for } t \in [0, \delta]\}.$$

Clearly, Y is a bounded closed convex subset of the Banach space $C[0, \delta]$ of all continuous functions in $[0, \delta]$. Define

$$\mathscr{F}y(t) = \alpha - \frac{1}{n-2}\int_0^t \left[1 - \left(\frac{s}{t}\right)^{n-2}\right] sp(s)[y(s)]^\gamma ds, \quad t \geqq 0, \quad y \in Y.$$

It is verified without difficulty that $\mathscr{F}$ is a compact operator mapping Y into itself, and so there exists a $\hat{y} \in Y$ such that $\hat{y} = \mathscr{F}\hat{y}$ by the Schauder fixed point theorem.

Differentiation of the integral equation $\hat{y}(t)=\mathscr{F}\hat{y}(t)$ twice shows that $\hat{y}(t)$ is a (positive) solution of the equation

$$(t^{n-1}y')' + t^{n-1}p(t)|y|^{\gamma}\operatorname{sgn} y = 0, \quad 0 < t < \delta,$$

satisfying $y(0)=\alpha$ and $y'(0)=0$.

We claim that $\hat{y}(t)$ can be continued to the entire interval $(0, \infty)$. Put $\hat{z}(t)=t\hat{y}(\zeta(t))$, where $\zeta(t)=[t/(n-2)]^{1/(n-2)}$. Then $\hat{z}(t)$ is shown to be a solution of the equation

$$z'' + \tilde{p}(t)|z|^{\gamma}\operatorname{sgn} z = 0, \quad \tilde{p}(t) = t^{-3-\gamma}[\zeta(t)]^{2n-2}p(\zeta(t)), \tag{5}$$

defined in the interval $(0, \zeta^{-1}(\delta))$, where ζ^{-1} denotes the inverse function of ζ. We continue $\hat{z}(t)$ to $(0, \infty)$ and denote the continuation by $z_\alpha(t)$. This is accomplished with the aid of a theorem of Coffman and Ullrich [2, p. 390] which asserts that if $\tilde{p}(t)$ is positive, continuous and locally of bounded variation in $(0, \infty)$, then every solution of (5) exists in the whole interval $(0, \infty)$. Now define $y_\alpha(t)=z_\alpha(\zeta^{-1}(t))/\zeta^{-1}(t)$ for $t>0$. Then $y_\alpha(t)$ gives the desired continuation of $\hat{y}(t)$ to $(0, \infty)$. If $\alpha<0$ (or $\alpha=0$), then $-y_{-\alpha}(t)$ (or $y_0(t)\equiv 0$) solves the problem (2)–(3). The uniqueness of $y_\alpha(t)$ is a consequence of the superlinearity of (2). This completes the proof.

We state below a lemma which will be extensively used in studying the qualitative behavior of the entire solutions of (1) guaranteed by Theorem 1. It concerns the Liapunov-like function $V_\alpha(t)$

$$\begin{aligned} V_\alpha(t) = {} & t^{n-1}y'_\alpha(t)y_\alpha(t) + \frac{1}{n-2}t^n[y'_\alpha(t)]^2 \\ & + \frac{2}{(n-2)(\gamma+1)}t^n p(t)|y_\alpha(t)|^{\gamma+1} \end{aligned} \tag{6}$$

associated with the solution $y_\alpha(t)$ of the problem (2)–(3).

Lemma 1. *If $y_\alpha(t)$ is the solution of the problem* (2)–(3), *then the function $V_\alpha(t)$ defined by* (6) *satisfies the equations*

$$V'_\alpha(t) = \frac{2}{(n-2)(\gamma+1)}t^{(n-2)(\gamma+1)/2}P'_{n,\gamma}(t)|y_\alpha(t)|^{\gamma+1} \tag{7}$$

and

$$\begin{aligned} V_\alpha(t) = {} & \frac{2}{(n-2)(\gamma+1)}\int_0^t s^{(n-2)(\gamma+1)/2}P'_{n,\gamma}(s)ds\cdot|y_\alpha(t)|^{\gamma+1} \\ & - \frac{2}{n-2}\int_0^t\left(\int_0^s \sigma^{(n-2)(\gamma+1)/2}P'_{n,\gamma}(\sigma)d\sigma\right)|y_\alpha(s)|^{\gamma}y'_\alpha(s)\operatorname{sgn} y_\alpha(s)ds \end{aligned} \tag{8}$$

for $t>0$, *where* $P_{n,\gamma}(t)$ *denotes the function in* (4).

The verification of (7) is straightforward, and integration by parts of (7) leads immediately to (8).

Remark 1. The following indentity is useful:

$$\text{(9)} \quad \int_0^t s^{(n-2)(\gamma+1)/2} P'_{n,\gamma}(s)ds = t^n p(t) - \frac{(n-2)(\gamma+1)}{2} \int_0^t s^{n-1} p(s)ds, \quad t>0.$$

3. Nonoscillation of Entire Solutions

In this section we first give a criterion for nonoscillation of all nontrivial radially symmetric entire solutions of (1), and then investigate their asymptotic behavior as $|x|\to\infty$.

Theorem 2. *Let* $P_{n,\gamma}(t)$ *be as in* (4) *and suppose that*

$$\text{(10)} \qquad Q_{n,\gamma}(t) \equiv \int_0^t s^{(n-2)(\gamma+1)/2} P'_{n,\gamma}(s)ds \leqq 0 \qquad \textit{for} \quad t>0.$$

Then, no nontrivial radially symmetric entire solution of equation (1) *has a zero in* $\boldsymbol{R}^n$.

Proof. Suppose that there exists an $\alpha>0$ for which the solution $y_\alpha(t)$ of the problem (2)–(3) has a zero in $(0, \infty)$. Let t_α be the first zero of $y_\alpha(t)$; then $y_\alpha(t_\alpha)=0$ and $y_\alpha(t)>0$ for $t\in[0, t_\alpha)$. Since $y'_\alpha(t)\leqq 0$ for $t\in[0, t_\alpha]$, it follows from (8) and (10) that $V_\alpha(t)\leqq 0$ for $t\in[0, t_\alpha]$. On the other hand, we have $V_\alpha(t_\alpha)=t_\alpha^n[y'_\alpha(t_\alpha)]^2/(n-2)\geqq 0$ by (6). Therefore, $V_\alpha(t_\alpha)=0$, and hence $y'_\alpha(t_\alpha)=0$. The "initial condition" $y_\alpha(t_\alpha)=y'_\alpha(t_\alpha)=0$ then implies that $y_\alpha(t)\equiv 0$ for $t\geqq 0$ by uniqueness. This, however, is a contradiction, and we conclude that if $\alpha>0$, then $y_\alpha(t)$ remains positive in $[0, \infty)$. Similarly, it can be shown that $y_\alpha(t)<0$ in $[0, \infty)$ provided $\alpha<0$. This completes the proof.

Remark 2. Theorem 2 improves considerably the nonoscillation criteria

$$\gamma \geqq \frac{n+2}{n-2}, \quad P'_{n,\gamma}(t) \leqq 0 \qquad \text{for} \quad t>0$$

and

$$\gamma \geqq \frac{n+2}{n-2}, \quad \int_0^t s^n p'(s)ds \leqq 0 \qquad \text{for} \quad t>0$$

obtained in [10, Theorem 1] and [15, Theorem 4.5], respectively. A related result can be found in [3, Theorem 5.4].

Theorem 3. *Suppose that condition* (10) *holds. Let $u_\alpha(x)$ be the radially symmetric entire solution of equation* (1) *such that $u_\alpha(0)=\alpha$.*

(i) *For every $\alpha\neq 0$, $u_\alpha(x)$ satisfies*

$$|u_\alpha(x)| \leqq \left(|\alpha|^{1-\gamma} + \frac{2(\gamma-1)}{(n-2)(\gamma+1)} \int_0^{|x|} sp(s)ds\right)^{1/(1-\gamma)}, \quad x \in \boldsymbol{R}^n. \tag{11}$$

(ii) *If in addition*

$$P'_{n,\gamma}(t) \not\equiv 0 \quad \textit{for} \quad t>0, \tag{12}$$

then for every $\alpha\neq 0$, $u_\alpha(x)$ satisfies

$$\liminf_{|x|\to\infty} |x|^{(n-2)/2}|u_\alpha(x)| > 0. \tag{13}$$

Proof. (i) We need only to consider the case where $\alpha>0$. Let $y_\alpha(t)$ be the solution of (2)–(3) and define $V_\alpha(t)$ by (6). Then, $y_\alpha(t)>0$ and $y'_\alpha(t)<0$ for $t>0$, and (8) and (10) imply that $V_\alpha(t)\leqq 0$ for $t\geqq 0$. We then see from (6) that

$$t^{n-1}y'_\alpha(t)y_\alpha(t) + \frac{2}{(n-2)(\gamma+1)} t^n p(t)[y_\alpha(t)]^{\gamma+1} \leqq 0, \quad t \geqq 0,$$

or equivalently

$$y'_\alpha(t) + \frac{2}{(n-2)(\gamma+1)} tp(t)[y_\alpha(t)]^\gamma \leqq 0, \quad t \geqq 0.$$

Dividing the above by $[y_\alpha(t)]^\gamma$ and integrating over $[0, t]$, we find

$$y_\alpha(t) \leqq \left(\alpha^{1-\gamma} + \frac{2(\gamma-1)}{(n-2)(\gamma+1)} \int_0^t sp(s)ds\right)^{1/(1-\gamma)}, \quad t \geqq 0,$$

which implies (11) for $\alpha>0$.

(ii) Let α, $y_\alpha(t)$ and $V_\alpha(t)$ be as in (i). From (6) we have

$$V_\alpha(t) \geqq t^{n-1}y'_\alpha(t)y_\alpha(t), \quad t \geqq 0. \tag{14}$$

On the other hand, in view of (8) and (10) we see that

$$V_\alpha(t) \leqq -\frac{2}{n-2} \int_0^t Q_{n,\gamma}(s)[y_\alpha(s)]^\gamma y'_\alpha(s)ds, \quad t>0, \tag{15}$$

where $Q_{n,\gamma}(t)$ is as in (10). Since by (12) there is $T>0$ such that

$$\int_0^t Q_{n,\gamma}(s)[y_\alpha(s)]^\gamma y'_\alpha(s)ds \geqq \int_0^T Q_{n,\gamma}(s)[y_\alpha(s)]^\gamma y'_\alpha(s)ds \equiv \delta > 0$$

for $t\geqq T$, it follows from (15) that

(16) $$V_\alpha(t) \leqq -\frac{2\delta}{n-2} \quad \text{for} \quad t \geqq T.$$

Combining (14) with (16), we obtain

$$t^{n-1}y'_\alpha(t)y_\alpha(t) \leqq -\frac{2\delta}{n-2} \quad \text{for} \quad t \geqq T,$$

which, after integration over $[t, \tau]$, yields

$$-\frac{1}{2}[y_\alpha(t)]^2 \leqq \frac{1}{2}[y_\alpha(\tau)]^2 - \frac{1}{2}[y_\alpha(t)]^2 \leqq -\frac{2\delta}{n-2}\left(\frac{t^{2-n}}{n-2} - \frac{\tau^{2-n}}{n-2}\right)$$

for $T \leqq t < \tau$. Letting $\tau \to \infty$ in the above, we conclude that

$$t^{(n-2)/2}y_\alpha(t) \geqq \frac{2\delta^{1/2}}{n-2}, \quad t \geqq T,$$

implying that (13) is true for $\alpha > 0$. The proof of Theorem 3 is thus complete.

The following corollary, which is an immediate consequence of Theorem 3, indicates a situation in which all nonoscillatory entire solutions of (1) behave like constant multiples of $|x|^{(2-n)/2}$ as $|x| \to \infty$.

Corollary 1. *Suppose that*

(17) $$P'_{n,\gamma}(t) \leqq 0, \quad \not\equiv 0 \quad \textit{for} \quad t > 0, \quad \textit{and} \quad \lim_{t\to\infty} P_{n,\gamma}(t) > 0.$$

Then, all nontrivial radially symmetric entire solutions $u(x)$ of (1) have the same asymptotic behavior as $|x| \to \infty$:

(18) $$0 < \liminf_{|x|\to\infty} |x|^{(n-2)/2}|u(x)| \leqq \limsup_{|x|\to\infty} |x|^{(n-2)/2}|u(x)| < \infty.$$

Remark 3. Theorem 3 and Corollary 1 have points in common with [3, Theorem 5.26] and [15, Theorem 3.10]. Notice that Corollary 1 and [3, Theorem 5.26] coincide when specialized to the case $\gamma = (n+2)/(n-2)$.

Example 1. Consider the equation

(19) $$\Delta u + (1+|x|)^\beta |u|^\gamma \operatorname{sgn} u = 0, \quad x \in \boldsymbol{R}^n, \quad n \geqq 3,$$

where β and $\gamma > 1$ are constants. As before, we denote by $u_\alpha(x)$ the radially symmetric entire solution of (19) such that $u_\alpha(0) = \alpha$. If

(20) $$\gamma \geqq \max\left\{\frac{n+2}{n-2}, \quad \frac{n+2+2\beta}{n-2}\right\},$$

then the function $P_{n,\gamma}(t) = t^{[n+2-\gamma(n-2)]/2}(1+t)^\beta$ satisfies $P'_{n,\gamma}(t) \leqq 0$ for $t > 0$, and so Theorem 2 implies that none of the $u_\alpha(x)$, $\alpha \neq 0$, has a zero in $\boldsymbol{R}^n$. We take up the three special cases of (20):

$$\gamma > \frac{n+2}{n-2} \quad \text{and} \quad \beta = \frac{\gamma(n-2)-(n+2)}{2}; \tag{21}$$

$$\gamma \geqq \frac{n+2}{n-2} \quad \text{and} \quad \beta < \frac{\gamma(n-2)-(n+2)}{2}; \tag{22}$$

$$\gamma = \frac{n+2}{n-2} \quad \text{and} \quad \beta = 0. \tag{23}$$

If (21) holds, then $P'_{n,\gamma}(t) \leqq 0$, $\not\equiv 0$ for $t>0$ and $\lim_{t\to\infty} P_{n,\gamma}(t)=1$, so that by Corollary 1 all the entire solutions $u_\alpha(x)$, $\alpha \neq 0$, tend uniformly to zero as $|x|\to\infty$ exactly like constant multiples of $|x|^{(2-n)/2}$. If (22) holds, then $P'_{n,\gamma}(t) \leqq 0$, $\not\equiv 0$ for $t>0$ but $\lim_{t\to\infty} P_{n,\gamma}(t)=0$. In this case, Theorem 3 shows that each $u_\alpha(x)$, $\alpha \neq 0$, satisfies

$$\liminf_{|x|\to\infty} |x|^{(n-2)/2}|u_\alpha(x)| > 0 \quad \text{and} \quad \limsup_{|x|\to\infty} |x|^{(\beta+2)/(\gamma-1)}|u_\alpha(x)| < \infty,$$

which is not as accurate as (18) since $(\beta+2)/(\gamma-1)<(n-2)/2$ by (22). We conjecture that some of $u_\alpha(x)$ exhibit a different asymptotic behavior from (18) (see Example 2). Finally, if (23) holds, then equation (19) reduces to

$$\Delta u + |u|^{(n+2)/(n-2)} \operatorname{sgn} u = 0, \quad x \in \boldsymbol{R}^n, \quad n \geqq 3, \tag{24}$$

for which $P_{n,\gamma}(t) \equiv 1$, $t \geqq 0$. It is well known (see e.g. [7, p. 591]) that the radially symmetric entire solutions $u_\alpha(x)$ are given explicitly by

$$u_\alpha(x) = [n(n-2)]^{(n-2)/4}\left(\frac{\lambda_\alpha}{\lambda_\alpha^2+|x|^2}\right)^{(n-2)/2} \operatorname{sgn} \alpha, \tag{25}$$

where $\lambda_\alpha = [n(n-2)]^{1/2}|\alpha|^{-2/(n-2)}$. The asymptotic behavior of (25) is clearly different from (18).

Example 2. The equation

$$\Delta u + \frac{3(n-2)^2|x|^2+4n(n-2)}{16(1+|x|^2)^{3/2}} |u|^{(n+2)/(n-2)} \operatorname{sgn} u = 0, \quad x \in \boldsymbol{R}^n, \quad n \geqq 3,$$

has a positive entire solution $u(x)=(1+|x|^2)^{(2-n)/8}$, $x \in \boldsymbol{R}^n$. This example shows that, in case $P'_{n,\gamma}(t) \leqq 0$, $\not\equiv 0$ for $t>0$ and $\lim_{t\to\infty} P_{n,\gamma}(t)=0$, equation (1) may have a nonoscillatory entire solution whose order of decay as $|x|\to\infty$ is less than that of $|x|^{(2-n)/2}$.

4. Oscillation of Entire Solutions

The purpose of this section is to prove the following oscillation theorem for equation (1).

Theorem 4. *Let $P_{n,\gamma}(t)$ be as in* (4) *and suppose that*

$$P'_{n,\gamma}(t) \geqq 0, \quad \not\equiv 0 \qquad \text{for} \quad t > 0. \tag{26}$$

Then, all radially symmetric entire solutions of equation (1) *are oscillatory.*

The proof of Theorem 4 rests on the fact that under (26) every radial entire solution of (1) has at least one zero. As a matter of fact, the existence of zeros for all entire solutions of (1) is guaranteed by a much weaker condition than (26), and so we formulate this result as a theorem and prove it before Theorem 4.

Theorem 5. *Suppose that $P'_{n,\gamma}(t) \not\equiv 0$ and*

$$Q_{n,\gamma}(t) \equiv \int_0^t s^{(n-2)(\gamma+1)/2} P'_{n,\gamma}(s) ds \geqq 0 \qquad \text{for} \quad t > 0. \tag{27}$$

Then, every nontrivial radially symmetric entire solution of (1) *has a zero in $\boldsymbol{R}^n$.*

Lemma 2. *Suppose that*

$$\liminf_{t\to\infty} t^{(n-2)(\gamma-1)/2} \int_t^\infty s^{n-3} \left(\int_s^\infty \sigma^{1-\gamma(n-2)} p(\sigma) d\sigma \right) ds > 0. \tag{28}$$

If the solution $y_\alpha(t)$ of (2)–(3) *is eventually of constant sign, then the Liapunov-like function $V_\alpha(t)$ associated with $y_\alpha(t)$ satisfies*

$$\liminf_{t\to\infty} V_\alpha(t) \leqq 0. \tag{29}$$

Proof of Lemma 2. Assume that there exists a solution $y_\alpha(t)$ of (2)–(3) which is positive for $t \geqq t_0 > 0$ but does not satisfy (29). Integrating equation (2) rewritten as

$$(t^{3-n}(t^{n-2} y_\alpha(t))')' + tp(t)[y_\alpha(t)]^\gamma = 0, \quad t \geqq t_0,$$

and noting that $(t^{n-2} y_\alpha(t))' \geqq 0$, $t \geqq t_0$ (see e.g. [19]), we obtain

$$(t^{n-2} y_\alpha(t))' \geqq t^{n-3} \int_t^\infty sp(s)[y_\alpha(s)]^\gamma ds$$

$$\geqq [t^{n-2} y_\alpha(t)]^\gamma t^{n-3} \int_t^\infty s^{1-\gamma(n-2)} p(s) ds, \quad t \geqq t_0.$$

We divide the above inequality by $[t^{n-2}y_\alpha(t)]^\gamma$, integrate it over $[t, \tau]$ and let $\tau \to \infty$. Then we have

$$\frac{1}{\gamma-1}[t^{n-2}y_\alpha(t)]^{1-\gamma} \geqq \int_t^\infty s^{n-3}\left(\int_s^\infty \sigma^{1-\gamma(n-2)}p(\sigma)d\sigma\right)ds, \quad t \geqq t_0,$$

which implies

$$t^{(n-2)/2}y_\alpha(t) \leqq \left((\gamma-1)\,t^{(n-2)(\gamma-1)/2}\int_t^\infty s^{n-3}\left(\int_s^\infty \sigma^{1-\gamma(n-2)}p(\sigma)d\sigma\right)ds\right)^{1/(1-\gamma)}$$

for $t \geqq t_0$. This, combined with (28), shows that

$$t^{(n-2)/2}y_\alpha(t) \leqq C, \quad t \geqq t_0, \tag{30}$$

for some positive constant C.

Now put $w_\alpha(t) = t^{(n-2)/2}y_\alpha(t)$. Then, $w_\alpha(t)$ is bounded above and satisfies the equation

$$t^2 w''_\alpha(t) + t w'_\alpha(t) - \frac{(n-2)^2}{4} w_\alpha(t) + P_{n,\gamma}(t)[w_\alpha(t)]^\gamma = 0, \quad t \geqq t_0. \tag{31}$$

The function $V_\alpha(t)$ is expressed in terms of $w_\alpha(t)$ as follows:

$$V_\alpha(t) = -\frac{n-2}{4}[w_\alpha(t)]^2 + \frac{1}{n-2}t^2[w'_\alpha(t)]^2 + \frac{2}{(n-2)(\gamma+1)}P_{n,\gamma}(t)[w_\alpha(t)]^{\gamma+1}. \tag{32}$$

There are three possibilities for $w'_\alpha(t)$:

(i) $w'_\alpha(t) \geqq 0$ for all sufficiently large t;

(ii) $w'_\alpha(t) \leqq 0$ for all sufficiently large t;

(iii) $w'_\alpha(t)$ ultimately takes both positive and negative values.

However, case (iii) is precluded. In fact, if this case occurs, then there is a $T > t_0$ sufficiently large such that

$$w'_\alpha(T) = 0 \quad \text{and} \quad w''_\alpha(T) \geqq 0. \tag{33}$$

Since $V_\alpha(t) > 0$ for all large t, from (32) we have

$$P_{n,\gamma}(T)[w_\alpha(T)]^\gamma \geqq \frac{(n-2)^2(\gamma+1)}{8} w_\alpha(T), \tag{34}$$

and using (33) and (34) we see from (31) that where $q(t)=\int_0^t s^{n-1}p(s)ds$. Integration of (38) shows that, for each fixed $T>0$,

$$q(t) \geqq q(T)(t/T)^{(n-2)(\gamma+1)/2} \qquad \text{for} \quad t \geqq T,$$

which, combined with (37), yields

$$(39) \qquad p(t) \geqq Ct^{-[n+2-\gamma(n-2)]/2}, \quad t \geqq T,$$

where $C=(n-2)(\gamma+1)q(T)T^{-(n-2)(\gamma+1)/2}/2>0$. We observe that (39) implies (28) in Lemma 2.

Assume that, for some $\alpha>0$, the solution $y_\alpha(t)$ of (2)–(3) is positive in $[0, \infty)$, and consider the function $V_\alpha(t)$ associated with $y_\alpha(t)$. From Lemma 2 it follows that $\liminf_{t\to\infty} V_\alpha(t) \leqq 0$. On the other hand, from (8) and (27) we get

$$(40) \qquad V_\alpha(t) \geqq -\frac{2}{n-2}\int_0^t Q_{n,\gamma}(s)[y_\alpha(s)]^\gamma y'_\alpha(s)ds, \quad t>0.$$

The condition $P'_{n,\gamma}(t) \not\equiv 0$ implies the existence of a $T>0$ such that

$$\int_0^t Q_{n,\gamma}(s)[y_\alpha(s)]^\gamma y'_\alpha(s)ds \leqq \int_0^T Q_{n,\gamma}(s)[y_\alpha(s)]^\gamma y'_\alpha(s)ds \equiv -\delta < 0$$

for $t \geqq T$. Using this inequality in (40), we conclude that

$$V_\alpha(t) \geqq \frac{2\delta}{n-2} > 0 \qquad \text{for} \quad t \geqq T,$$

contradicting the conclusion of Lemma 2. Similar arguments hold for the case of negative α. It follows that condition (27) forces every entire solution $u_\alpha(x) = y_\alpha(|x|)$ of (1) to have at least one zero in $\boldsymbol{R}^n$. This completes the proof of Theorem 5.

Proof of Theorem 4. Let $\alpha \neq 0$ be any number and let $y_\alpha(t)$ be the solution of the problem (2)–(3). By Theorem 5 $y_\alpha(t)$ has a zero in $(0, \infty)$. It remains to show that $y_\alpha(t)$ has infinitely many zeros tending to ∞. Suppose to the contrary that there is an $\alpha \neq 0$ for which $y_\alpha(t)$ is eventually of constant sign. Without loss of generality we may suppose that there exists $t_\alpha>0$ such that $y_\alpha(t_\alpha)=0$ and $y_\alpha(t)>0$ for $t>t_\alpha$. The uniqueness of the solution implies that $y'_\alpha(t_\alpha)>0$, and since $V_\alpha(t)$ is nondecreasing by (7) and (26), it follows that

$$(41) \qquad V_\alpha(t) \geqq V_\alpha(t_\alpha) = t_\alpha^n[y'_\alpha(t_\alpha)]^2/(n-2) > 0 \qquad \text{for} \quad t \geqq t_\alpha.$$

On the other hand, condition (26) implies the existence of $T>0$ such that $p(t)$ satisfies (39) with $C=P_{n,\gamma}(T)>0$, so that by applying Lemma 2, we see that $\liminf_{t\to\infty} V_\alpha(t) \leqq 0$, which contradicts (41). Therefoıe, every $y_\alpha(t)$ must have a zero in any neighborhood of infinity, and hence every radially symmetric entire solution $u_\alpha(x)=y_\alpha(|x|)$ of (1) must be oscillatory. This concludes the proof of Theorem 4.

Remark 4. An oscillation theorem for equations of the form $\Delta u+f(u)=0$, $x\in \boldsymbol{R}^n$, has been given by [16, Corollary 6.7]. In the case $f(u)=|u|^\gamma \operatorname{sgn} u$, it coincides with a specialization of our Theorem 4 to the case $p(t)\equiv 1$.

Remark 5. Theorem 5 extends Theorem 2 of [10] as well as Theorem 5.13 of [3] and Theorem 4.23 of [15] for the case $\gamma=(n+2)/(n-2)$. Our Theorem 5 covers the case $\gamma\neq(n+2)/(n-2)$ also.

Example 3. Consider the equation (19) again. If

$$1<\gamma<\min\left\{\frac{n+2}{n-2},\ \frac{n+2+2\beta}{n-2}\right\}, \tag{42}$$

then Theorem 4 is applicable, and all radially symmetric entire solutions of (19) are oscillatory.

Example 4. Consider the equation

$$\Delta u+|x|^\beta|u|^\gamma \operatorname{sgn} u=0, \quad x\in \boldsymbol{R}^n, \quad n\geqq 3, \tag{43}$$

where $\beta\geqq 0$ and $\gamma>1$ are constants. As is easily seen, if $1<\gamma<(n+2+2\beta)/(n-2)$, then Theorem 4 is applicable and all radially symmetric entire solutions of (43) are oscillatory. Notice that if $\gamma\geqq(n+2+2\beta)/(n-2)$, then it follows from Theorem 2 that every nontrivial radial entire solution of (43) is nonoscillatory and has no zero in $\boldsymbol{R}^n$.

Remark 6. Theorem 4 cannot be applied to the equation

$$\Delta u+(2+\sin|x|)|u|^\gamma \operatorname{sgn} u=0, \quad x\in \boldsymbol{R}^n, \quad n\geqq 3, \quad \gamma>1; \tag{44}$$

for the function $P_{n,\gamma}(t)=t^{[n+2-\gamma(n-2)]/2}(2+\sin t)$ does not satisfy (26). However, one finds that if $1<\gamma\leqq n/(n-2)$, all (not necessarily radially symmetric) entire solutions of (44) are oscillatory, by applying to (44) a theorem of Noussair and Swanson [17, 18] which asserts that the condition

$$\int_1^\infty t^{n-1-\gamma(n-2)}p(t)dt=\infty \tag{45}$$

ensures the oscillation of all solutions of (1) defined in an *exterior* domain in $\boldsymbol{R}^n$. On the other hand, noting that $p(t)=2+\sin t$ satisfies

$$\int_0^t s^{(n-2)(\gamma+1)/2}P'_{n,\gamma}(s)ds=t^n p(t)-\frac{(n-2)(\gamma+1)}{2}\int_0^t s^{n-1}p(s)ds$$

$$\leqq 3t^n - \frac{(n-2)(\gamma+1)}{2}\int_0^t s^{n-1}ds$$

$$= \left[3 - \frac{(n-2)(\gamma+1)}{2n}\right]t^n,$$

one concludes from Theorem 2 that if $\gamma \geqq (5n+2)/(n-2)$, then every nontrivial radially symmetric entire solution of (44) is nonoscillatory in $\boldsymbol{R}^n$. No conclusion can be drawn for equation (44) with γ in $(n/(n-2), (5n+2)/(n-2))$.

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