

L^2 -Solutions for Nonlinear Schrödinger Equations and Nonlinear Groups

By

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§ 1. Introduction and main results

We consider the unique global existence of solutions in a weaker class than $H^1(\mathbf{R}^n)$ and some properties of the solution operator for the following nonlinear Schrödinger equation:

$$(1.1) \quad i \frac{\partial u}{\partial t} = -\Delta u + \lambda |u|^{p-1} u, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^n,$$

$$(1.2) \quad u(t_0, x) = u_0(x),$$

where $t_0 \in \mathbf{R}$ and $\lambda \in \mathbf{R}$. By $\alpha(n)$ we denote ∞ if $n=1$ or $n=2$ and $(n+2)/(n-2)$ if $n \geq 3$. There are many papers concerning the global existence of solutions for Problem (1.1)–(1.2) (see, e.g., [1]–[2], [4]–[7] and [9]). In [1] Baillon, Cazenave and Figueira show that if $1 \leq n \leq 3$, $1 < p < \alpha(n)$ and $\lambda > 0$, Problem (1.1)–(1.2) has a unique global strong solution $u(t) \in C(\mathbf{R}; H^2(\mathbf{R}^n)) \cap C^1(\mathbf{R}; L^2(\mathbf{R}^n))$ for any $u_0 \in H^2(\mathbf{R}^n)$. In [2] Ginibre and Velo show that if $1 < p < \alpha(n)$ and $\lambda > 0$ or if $1 < p < 1 + 4/n$ and $\lambda < 0$, Problem (1.1)–(1.2) has a unique global weak solution $u(t) \in C(\mathbf{R}; H^1(\mathbf{R}^n))$ for any $u_0 \in H^1(\mathbf{R}^n)$. In [6] Strauss shows that if $\lambda > 0$ and $p > 1$, Problem (1.1)–(1.2) has at least one global weak solution $u(t) \in L^\infty(\mathbf{R}; H^1(\mathbf{R}^n) \cap L^{p+1}(\mathbf{R}^n))$ for any $u_0 \in H^1(\mathbf{R}^n) \cap L^{p+1}(\mathbf{R}^n)$ (see also [5]). In [10] M. Tsutsumi and N. Hayashi discuss the unique global existence of classical solutions for (1.1)–(1.2) with $\lambda > 0$ (see also Pecher and von Wahl [4]). Furthermore, in [9] M. Tsutsumi discusses the unique global solution in $\mathcal{S}(\mathbf{R}^n)$ or in the weighted Sobolev space for (1.1)–(1.2) with $\lambda > 0$. In almost all of previous papers the solution of (1.1)–(1.2) has been constructed in a space not larger than the energy space, that is, $H^1(\mathbf{R}^n)$, because the proofs in almost all of previous papers are based on the energy conservation law. However, in [7] Strauss constructs the wave operators from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$ for the equation (1.1) with $p=1+4/n$ (see [7, Theorem 5]). His results are almost equivalent to the construction in $L^2(\mathbf{R}^n)$ of unique local solutions for (1.1)–(1.2) with $p=1+4/n$. In this paper we prove that when $1 < p < 1 + 4/n$, we can construct the unique global solution of (1.1)–(1.2) for u_0 in $L^2(\mathbf{R}^n)$ (but possibly not in $H^1(\mathbf{R}^n)$). Such a solution is

called an “ L^2 -solution”. Furthermore, we show that when $1 < p < 1 + 4/n$, the solution operator of the evolution equation (1.1) constitutes a strongly continuous nonlinear operator group in $L^2(\mathbf{R}^n)$. Our proof is based on the L^2 -norm conservation law and the dispersive effect of solutions (see, e.g., Lemma 2.2).

Our main theorem in this paper is the following.

Theorem 1.1. *Assume that $1 < p < 1 + 4/n$. Then, for any $u_0 \in L^2(\mathbf{R}^n)$ and any $t_0 \in \mathbf{R}$ there exists a unique global solution $u(t)$ of (1.1)–(1.2) such that*

$$(1.3) \quad u(t) \in C(\mathbf{R}; L^2(\mathbf{R}^n)) \cap L^r_{\text{loc}}(\mathbf{R}; L^{p+1}(\mathbf{R}^n)),$$

$$(1.4) \quad u(t) = U(t-t_0)u_0 - i \int_{t_0}^t U(t-\tau)f(u(\tau))d\tau, \quad t \in \mathbf{R},$$

$$(1.5) \quad \|u(t)\|_{L^2(\mathbf{R}^n)} = \|u_0\|_{L^2(\mathbf{R}^n)}, \quad t \in \mathbf{R},$$

where $r = \frac{4(p+1)}{n(p-1)}$, $U(t) = e^{i\Delta t}$, $f(z) = \lambda|z|^{p-1}z$ ($z \in \mathbf{C}$) and the integral in (1.4) is the Bochner integral in $H^{-1}(\mathbf{R}^n)$. Furthermore, let u_{0j} , $j=1, 2, \dots$, and u_0 be such that $u_{0j}, u_0 \in L^2(\mathbf{R}^n)$ and $u_{0j} \rightarrow u_0$ in $L^2(\mathbf{R}^n)$ ($j \rightarrow \infty$). Let $u_j(t)$ and $u(t)$ be the solutions of (1.1) with $u_j(t_0) = u_{0j}$ and $u(t_0) = u_0$, respectively. Then, for each $t \in \mathbf{R}$

$$(1.6) \quad u_j(t) \longrightarrow u(t) \quad \text{in } L^2(\mathbf{R}^n) \quad (j \rightarrow \infty).$$

By Theorem 1.1 we can define the solution operator of the evolution equation (1.1) as a mapping from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$, when $1 < p < 1 + 4/n$. We denote it by $S(t)$. The following result is an immediate consequence of Theorem 1.1.

Corollary 1.2. *Assume that $1 < p < 1 + 4/n$. Then, $\{S(t); -\infty < t < +\infty\}$ is a strongly continuous nonlinear operator group in $L^2(\mathbf{R}^n)$. That is, $S(t)$ is a homeomorphism from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$ for each $t \in \mathbf{R}$, and*

$$(1.7) \quad S(t+s) = S(t)S(s), \quad t, s \in \mathbf{R},$$

$$(1.8) \quad S(0) = I,$$

$$(1.9) \quad S(h)v \longrightarrow v \quad \text{in } L^2(\mathbf{R}^n) \quad (h \rightarrow 0), \quad v \in L^2(\mathbf{R}^n),$$

where I is the identity operator from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$.

Our plan in this paper is as follows. In Section 2 we summarize several lemmas needed for the proof of Theorem 1.1. In Section 3 we give a proof of Theorem 1.1.

We conclude this section with several notations given. We abbreviate $L^q(\mathbf{R}^n)$ and $H^m(\mathbf{R}^n)$ to L^q and H^m , respectively. (\cdot, \cdot) denotes the scalar product in L^2 . For a closed interval I in \mathbf{R} and a Hilbert space H we denote the set of all weakly

continuous functions from I to H by $C_w(I; H)$. Let $U(t) = e^{iAt}$ be the evolution operator of the free Schrödinger equation. We put $f(z) = \lambda|z|^{p-1}z$ for $z \in \mathbb{C}$. Let $h(x)$ be an even and positive function in $C_0^\infty(\mathbb{R}^n)$ with $\|h\|_{L^1} = 1$. We put $h_j(x) = j^n h(jx)$ for each positive integer j . $*$ denotes the convolution with respect to spatial variables. In the course of calculations below various constants will be simply denoted by C . In particular, $C = C(*, \dots, *)$ will denote a constant depending only on the quantities appearing in parentheses.

§2. Lemmas.

In this section we summarize several results needed for the proof of Theorem 1.1.

For $U(t)$ we have the following two lemmas.

Lemma 2.1. *Let q and r be positive numbers such that $1/q + 1/r = 1$ and $2 \leq q \leq \infty$. For any $t \neq 0$, $U(t)$ is a bounded operator from L^r to L^q satisfying*

$$(2.1) \quad \|U(t)v\|_{L^q} \leq (4\pi|t|)^{n/q-n/2} \|v\|_{L^r}, \quad v \in L^r, \quad t \neq 0,$$

and for any $t \neq 0$, the map $t \rightarrow U(t)$ is strongly continuous. For $q=2$, $U(t)$ is unitary and strongly continuous for all $t \in \mathbb{R}$.

Lemma 2.2. *Let q and r be positive numbers such that $1 \leq q-1 < \alpha(n)$ and $(n/2 - n/q)r = 2$. Then,*

$$(2.2) \quad \|U(\cdot)v\|_{L^r(\mathbb{R}; L^q)} \leq C \|v\|_{L^2},$$

where $C = C(n, q)$.

Lemma 2.1 is well known (see, e.g., [2, Lemma 1.2]). For Lemma 2.2, see Strichartz [8, Corollary 1 in §3] and Ginibre and Velo [3, Proposition 7].

Furthermore, we need the following two lemmas.

Lemma 2.3. *Let I be an open interval in \mathbb{R} . Let $1 < q, r < \infty$ and $a, b > 0$. We put*

$$M = \{v(t) \in L^\infty(I; L^2) \cap L^r(I; L^q); \\ \|v(t)\|_{L^2} \leq a, \quad \text{a.e. } t \in I, \\ \|v\|_{L^r(I; L^q)} \leq b\}.$$

Then M is a closed subset in $L^r(I; L^q)$.

Proof. Let $\{v_j(t)\} \subset M$ and $v(t) \in L^r(I; L^q)$ be such that $\|v_j - v\|_{L^r(I; L^q)} \rightarrow 0$ ($j \rightarrow \infty$). Then, since $\{v_j(t)\}$ is bounded in $L^\infty(I; L^2)$, we can choose a subsequence $\{v_{j'}(t)\} \subset \{v_j(t)\}$ and $w(t) \in L^\infty(I; L^2)$ such that

$$(2.3) \quad v_{j'}(t) \longrightarrow w(t) \quad \text{*}-\text{weakly in } L^\infty(I; L^2).$$

Since by the definitions of $\{v_{j'}(t)\}$ and $v(t)$ and (2.3) we have

$$(2.4) \quad v_{j'}(t) \longrightarrow v(t) \quad \text{in } \mathcal{D}'(I \times \mathbf{R}^n) \quad (j' \rightarrow \infty),$$

$$(2.5) \quad v_{j'}(t) \longrightarrow w(t) \quad \text{in } \mathcal{D}'(I \times \mathbf{R}^n) \quad (j' \rightarrow \infty),$$

we conclude by the uniqueness of limit that $v(t) = w(t) \in L^\infty(I; L^2) \cap L^r(I; L^q)$. Moreover, we have by (2.3) and the definitions of $\{v_{j'}(t)\}$ and $v(t)$

$$(2.6) \quad \|v\|_{L^\infty(I; L^2)} \leq \liminf_{j' \rightarrow \infty} \|v_{j'}\|_{L^\infty(I; L^2)} \leq a,$$

$$(2.7) \quad \|v\|_{L^r(I; L^q)} = \lim_{j' \rightarrow \infty} \|v_{j'}\|_{L^r(I; L^q)} \leq b,$$

which imply that $v(t) \in M$. This completes the proof of Lemma 2.3. Q. E. D.

Lemma 2.4. *Let T_1 and T_2 be constants with $T_1 < T_2$. Assume that $v(t) \in C([T_1, T_2]; H^{-1})$ and for some $K > 0$*

$$(2.8) \quad \|v(t)\|_{L^2} \leq K, \quad \text{a.e. } t \in [T_1, T_2].$$

Then, $v(t) \in C_w([T_1, T_2]; L^2)$ and (2.8) holds for all $t \in [T_1, T_2]$.

Proof. We first prove that (2.8) holds for all $t \in [T_1, T_2]$. For that purpose, we assume otherwise and derive the contradiction. We assume that (2.8) does not hold for some $t_0 \in [T_1, T_2]$. Then, by (2.8) we can choose $\{t_n\} \subset [T_1, T_2]$ such that $t_n \rightarrow t_0$ ($n \rightarrow \infty$) and (2.8) holds for all t_n . Therefore, we can choose a subsequence $\{t_{n'}\} \subset \{t_n\}$ and $w \in L^2$ such that

$$(2.9) \quad v(t_{n'}) \longrightarrow w \quad \text{weakly in } L^2 \quad (n' \rightarrow \infty).$$

On the other hand, since $v(t) \in C([T_1, T_2]; H^{-1})$, we have

$$(2.10) \quad v(t_{n'}) \longrightarrow v(t_0) \quad \text{in } H^{-1} \quad (n' \rightarrow \infty).$$

(2.9) and (2.10) give us

$$(2.11) \quad v(t_{n'}) \longrightarrow w \quad \text{in } \mathcal{D}'(\mathbf{R}^n) \quad (n' \rightarrow \infty),$$

$$(2.12) \quad v(t_{n'}) \longrightarrow v(t_0) \quad \text{in } \mathcal{D}'(\mathbf{R}^n) \quad (n' \rightarrow \infty).$$

By (2.11–12) and the uniqueness of limit we obtain $v(t_0) = w \in L^2$. Moreover,

$$(2.13) \quad \|v(t_0)\|_{L^2} \leq \liminf_{n' \rightarrow \infty} \|v(t_{n'})\|_{L^2} \leq K,$$

which contradicts the assumption of t_0 . Accordingly, we conclude that (2.8) holds for all $t \in [T_1, T_2]$.

We next prove that $v(t) \in C_w([T_1, T_2]; L^2)$. For any $t_0 \in [T_1, T_2]$, let $\{t_n\}$

be an arbitrary sequence $\subset [T_1, T_2]$ such that $t_n \rightarrow t_0$ ($n \rightarrow \infty$). Since $v(t) \in C([T_1, T_2]; H^{-1})$, we have

$$(2.14) \quad (v(t_n), \psi) \longrightarrow (v(t_0), \psi) \quad (n \rightarrow \infty)$$

for $\psi \in H^1$. By the boundedness in L^2 of $\{v(t_n)\}$ and (2.14) we obtain

$$(2.15) \quad v(t_n) \longrightarrow v(t_0) \quad \text{weakly in } L^2 \quad (n \rightarrow \infty),$$

which implies the weak continuity in L^2 of $v(t)$ at $t = t_0$. Since t_0 is an arbitrary point in $[T_1, T_2]$, we conclude that $v(t) \in C_w([T_1, T_2]; L^2)$. Q. E. D.

Finally we describe the results of Ginibre and Velo [2] concerning the unique global existence of weak solutions for (1.1)–(1.2). We first formulate this problem precisely. We consider the integral equation

$$(2.16) \quad u(t) = U(t-t_0)u_0 - i \int_{t_0}^t U(t-\tau)f(u(\tau))d\tau,$$

as the integral version of the initial value problem (1.1)–(1.2). For (2.16) we have the following proposition (see, e.g., [2, Theorem 3.1]).

Proposition 2.5. *Assume that $1 < p < 1 + 4/n$. Then, for any $t_0 \in \mathbf{R}$ and any $u_0 \in H^1$ there exists a unique solution $u(t)$ of (2.16) such that*

$$(2.17) \quad u(t) \in C(\mathbf{R}; H^1),$$

$$(2.18) \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t \in \mathbf{R}.$$

§3. Proof of Theorem 1.1.

In this section we give a proof of Theorem 1.1. By I_t and \bar{I}_t we denote an open interval $(t_0 - t, t_0 + t)$ and a closed interval $[t_0 - t, t_0 + t]$, respectively, for $t \geq 0$. Let $r = \frac{4(p+1)}{n(p-1)}$ throughout this section.

We first prove the following result concerning the unique local existence of L^2 -solutions for (1.1)–(1.2).

Lemma 3.1. *Assume that $1 < p < 1 + 4/n$. Then, for any $t_0 \in \mathbf{R}$ and any $u_0 \in L^2$ there exists a $T = T(p, n, \lambda, \|u_0\|_{L^2}) > 0$ such that Problem (1.1)–(1.2) has a unique local solution $u(t)$:*

$$(3.1) \quad u(t) \in C(\bar{I}_T; L^2) \cap L^r(I_T; L^{p+1}),$$

$$(3.2) \quad u(t) = U(t-t_0)u_0 - i \int_{t_0}^t U(t-\tau)f(u(\tau))d\tau, \quad t \in \bar{I}_T,$$

where the integral in (3.2) is the Bochner integral in H^{-1} . Furthermore, the solution $u(t)$ satisfies

$$(3.3) \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t \in \bar{I}_T.$$

Proof. We consider the following integral equation:

$$(3.4) \quad u_j(t) = U(t-t_0)h_j * u_0 - i \int_{t_0}^t U(t-\tau)f(u_j(\tau))d\tau, \quad j = 1, 2, \dots.$$

By Proposition 2.5 we have the unique global solution $u_j(t)$ in $C(\mathbf{R}; H^1)$ of (3.4) for each j . Let $\rho = \|u_0\|_{L^2}$. By δ we denote the constant appearing in (2.2) with $q = p+1$ and $r = \frac{4(p+1)}{n(p-1)}$. We note that δ depends only on n and p . We put

$$(3.5) \quad M = \{v(t) \in L^\infty(I_T; L^2) \cap L^r(I_T; L^{p+1}); \\ \|v\|_{L^\infty(I_T; L^2)} \leq \rho, \|v\|_{L^r(I_T; L^{p+1})} \leq 2\delta\rho\},$$

where T is a small positive constant to be determined later. We note that by Lemma 2.3 M is closed in $L^r(I_T; L^{p+1})$.

We first show that if T is sufficiently small, then

$$(3.6) \quad u_j(t) \in M \quad \text{for all } j.$$

By (2.18) we have

$$(3.7) \quad \|u_j(t)\|_{L^2} = \|h_j * u_0\|_{L^2} \leq \|u_0\|_{L^2}, \quad t \in \bar{I}_T,$$

for all j . We define $\tilde{u}_j^s(t)$ by

$$(3.8) \quad \tilde{u}_j^s(t) = \begin{cases} u_j(t), & t \in I_s, \\ 0, & t \notin I_s, \end{cases}$$

for $s \geq 0$. By (3.4) and (2.1-2) we have for s with $0 \leq s \leq T$

$$(3.9) \quad \|u\|_{L^r(I_s; L^{p+1})} \leq \delta\rho \\ + C \left\| \int_{t_0}^t |t-\tau|^{-n/2+n/(p+1)} \|f(u_j(\tau))\|_{L^{1+1/p}} d\tau \right\|_{L^r(I_s)} \\ \leq \delta\rho + C \left\| \int_{-\infty}^{+\infty} |t-\tau|^{-n/2+n/(p+1)} \|\tilde{u}_j^s(\tau)\|_{L^{p+1}}^p d\tau \right\|_{L^r(\mathbf{R})} \\ \leq \delta\rho + C \|\tilde{u}_j^s\|_{L^{q_1}(\mathbf{R}; L^{p+1})}^p \\ \leq \delta\rho + C \|u_j\|_{L^{q_1}(I_s; L^{p+1})}^p, \quad j = 1, 2, \dots,$$

where $q_1 = \frac{4p(p+1)}{n+4-(n-4)p}$. Here we have used the generalized Young inequality

(see [12, §4 in Chapter IX]) at the third inequality. We note that $1 < q_1 < r$ for $1 < p < 1 + 4/n$. On the other hand, we have by Hölder's inequality

$$(3.10) \quad \|u_j\|_{L^{q_1}(I_s; L^{p+1})} \leq \left(\int_{t_0-s}^{t_0+s} d\tau \right)^{1/q_2} \|u_j\|_{L^r(I_s; L^{p+1})} \\ \leq CT^{1/q_2} \|u_j\|_{L^r(I_s; L^{p+1})}, \quad 0 \leq s \leq T, \quad j = 1, 2, \dots,$$

where $q_2 = \frac{4p}{n+4-np}$ and $C = C(n, p)$. (3.9) and (3.10) give us

$$(3.11) \quad \|u_j\|_{L^r(I_s; L^{p+1})} \leq \delta\rho + C_0 T^{p/q_2} \|u_j\|_{L^r(I_s; L^{p+1})}^p, \\ 0 \leq s \leq T, \quad j = 1, 2, \dots,$$

where $C_0 = C_0(n, p, \lambda)$. Now we choose $T > 0$ so small that there exists a positive number y satisfying $C_0 T^{p/q_2} y^p + \delta\rho - y < 0$ and $0 < y \leq 2\delta\rho$. For that purpose, it is sufficient to choose $T > 0$ so that

$$(3.12) \quad T < (2C_0(2\delta\rho)^{p-1})^{-q_2/p}.$$

Then we put

$$(3.13) \quad y_0 = \min \{2\delta\rho \geq y > 0; C_0 T^{p/q_2} y^p + \delta\rho - y = 0\}.$$

Putting $X_j(s) = \|u_j\|_{L^r(I_s; L^{p+1})}$, we have by (3.11)

$$(3.14) \quad X_j(s) \leq \delta\rho + C_0 T^{p/q_2} X_j(s)^p, \quad 0 \leq s \leq T,$$

$$(3.15) \quad X_j(0) = 0,$$

for all j . If T is chosen so small that (3.12) holds, then by (3.13) and (3.14–15) we obtain

$$(3.16) \quad X_j(s) \leq y_0 \leq 2\delta\rho, \quad 0 \leq s \leq T, \quad j = 1, 2, \dots.$$

From (3.16) and Fatou's lemma it follows that

$$(3.17) \quad \|u_j\|_{L^r(I_T; L^{p+1})} \leq 2\delta\rho, \quad j = 1, 2, \dots.$$

By (3.7) and (3.17) we obtain (3.6), if T is chosen so small that (3.12) holds. From now on we assume that T is chosen so small that (3.12) holds.

For any j and k , in the same way as (3.9–11) we have by (3.4), Lemmas 2.1, 2.2 and the generalized Young inequality

$$(3.18) \quad \|u_j - u_k\|_{L^r(I_T; L^{p+1})} \leq K(j, k) \\ + C \int_{t_0}^t |t - \tau|^{-n/2+n/(p+1)} \|f(u_j(\tau)) - f(u_k(\tau))\|_{L^{1+1/p}} d\tau \|_{L^r(I_T)}$$

$$\begin{aligned}
&\leq K(j, k) + C \left\| \int_{t_0}^t |t-\tau|^{-n/2+n/(p+1)} \right. \\
&\quad \times (\|u_j(\tau)\|_{L^{p+1}}^{p-1} + \|u_k(\tau)\|_{L^{p+1}}^{p-1}) \|u_j(\tau) - u_k(\tau)\|_{L^{p+1}} d\tau \Big\|_{L^r(I_T)} \\
&\leq K(j, k) + C(\|u_j\|_{L^{q_1}(I_T; L^{p+1})}^{p-1} + \|u_k\|_{L^{q_1}(I_T; L^{p+1})}^{p-1}) \\
&\quad \times \|u_j - u_k\|_{L^{q_1}(I_T; L^{p+1})} \\
&\leq K(j, k) + \bar{C}_0 T^{p/q_2} (\|u_j\|_{L^r(I_T; L^{p+1})}^{p-1} + \|u_k\|_{L^r(I_T; L^{p+1})}^{p-1}) \\
&\quad \times \|u_j - u_k\|_{L^r(I_T; L^{p+1})} \\
&\leq K(j, k) + \bar{C}_0 T^{p/q_2} \cdot 2(2\delta\rho)^{p-1} \|u_j - u_k\|_{L^r(I_T; L^{p+1})},
\end{aligned}$$

where $K(j, k) = \delta \|h_j * u_0 - h_k * u_0\|_{L^2}$, $q_1 = \frac{4p(p+1)}{n+4-(n-4)p}$, $q_2 = \frac{4p}{n+4-np}$ and $\bar{C}_0 = \bar{C}_0(n, p, \lambda)$. We have used the generalized Young inequality at the third inequality and have used Hölder's inequality with $1/q_2 + q_1/r = 1$ at the last inequality but one. If we choose T so small in (3.18) that

$$(3.19) \quad \bar{C}_0 T^{p/q_2} \cdot 2(2\delta\rho)^{p-1} \leq \frac{1}{2},$$

then we have by (3.18)

$$(3.20) \quad \|u_j - u_k\|_{L^r(I_T; L^{p+1})} \leq 2K(j, k)$$

for all j and k . Since $K(j, k) \rightarrow 0$ ($j, k \rightarrow \infty$), we have by (3.20)

$$(3.21) \quad \|u_j - u_k\|_{L^r(I_T; L^{p+1})} \longrightarrow 0 \quad (j, k \rightarrow \infty),$$

if T is chosen so small that (3.19) holds. Furthermore, we have by (3.4), (3.21), the Sobolev imbedding theorem and Hölder's inequality

$$\begin{aligned}
(3.22) \quad & |(u_j(t) - u_k(t), \psi)| \\
& \leq \|h_j * u_0 - h_k * u_0\|_{L^2} \|\psi\|_{L^2} \\
& \quad + \left| \int_{t_0}^t (U(t-\tau) \{f(u_j(\tau)) - f(u_k(\tau))\}, \psi) d\tau \right| \\
& \leq \|h_j * u_0 - h_k * u_0\|_{L^2} \|\psi\|_{L^2} \\
& \quad + \int_{t_0-T}^{t_0+T} \|f(u_j(\tau)) - f(u_k(\tau))\|_{L^{1+1/p}} \|U(\tau-t)\psi\|_{L^{p+1}} d\tau \\
& \leq \|h_j * u_0 - h_k * u_0\|_{L^2} \|\psi\|_{L^2} \\
& \quad + C \|\psi\|_{H^1} \int_{t_0-T}^{t_0+T} (\|u_j(\tau)\|_{L^{p+1}}^{p-1} + \|u_k(\tau)\|_{L^{p+1}}^{p-1}) \\
& \quad \times \|u_j(\tau) - u_k(\tau)\|_{L^{p+1}} d\tau
\end{aligned}$$

$$\begin{aligned} &\leq \|h_j * u_0 - h_k * u_0\|_{L^2} \|\psi\|_{L^2} \\ &\quad + CT^{q_3} \|\psi\|_{H^1} \cdot 2(2\delta\rho)^{p-1} \|u_j - u_k\|_{L^r(I_T; L^{p+1})} \\ &\quad \longrightarrow 0 \quad (j, k \rightarrow \infty) \quad \text{uniformly in } t \in \bar{I}_T, \end{aligned}$$

for $\psi \in H^1$, where $q_3 = \frac{4 + (n+4)p - np^2}{4(p+1)}$. We note that $q_3 > 0$ for $1 < p < 1 + 4/n$. Since by Proposition 2.5 $u_j(t) \in C(\bar{I}_T; H^{-1})$ for any j , (3.22) implies that $\{u_j(t)\}$ is the Cauchy sequence in $C(\bar{I}_T; H^{-1})$.

Therefore, by (3.6), (3.21), (3.22) and Lemma 2.3 we obtain the solution $u(t)$ of (1.1)–(1.2) such that

$$(3.23) \quad u(t) \in L^\infty(I_T; L^2) \cap L^r(I_T; L^{p+1}) \cap C(\bar{I}_T; H^{-1}),$$

$$(3.24) \quad u(t) = U(t-t_0)u_0 - i \int_{t_0}^t U(t-\tau)f(u(\tau))d\tau, \quad t \in \bar{I}_T,$$

$$(3.25) \quad \|u(t)\|_{L^2} \leq \|u_0\|_{L^2}, \quad \text{a.e. } t \in I_T$$

$$(3.26) \quad u_j(t) \longrightarrow u(t) \quad \text{in } L^r(I_T; L^{p+1}) \quad \text{and in } C(\bar{I}_T; H^{-1}) \quad (j \rightarrow \infty),$$

where T is a positive constant with (3.12) and (3.19) and the integral in (3.24) is the Bochner integral in H^{-1} . (3.23), (3.25) and Lemma 2.4 imply that

$$(3.27) \quad u(t) \in C_w(\bar{I}_T; L^2)$$

and that for all $t \in \bar{I}_T$ (3.25) holds.

We next show that the solution satisfying (3.23) and (3.24) is unique. Let $u(t)$ and $v(t)$ be two solutions satisfying (3.23) and (3.24) with the same initial datum. We put $t_1 = \sup \{t \in [0, T]; u(s) = v(s) \text{ on } \bar{I}_t\}$. If $t_1 = T$, then $u(t) = v(t)$ on \bar{I}_T , which is the desired result. If $t_1 < T$, in the same way as (3.18) we have by (3.24) and the assumption of t_1

$$\begin{aligned} (3.28) \quad &\|u - v\|_{L^r(I_{t_1, t_2}; L^{p+1})} = \|u - v\|_{L^r(I_{t_2}; L^{p+1})} \\ &\leq C(t_1 - t_2)^{p/q_2} (\|u\|_{L^r(I_{t_1, t_2}; L^{p+1})}^{p-1} + \|v\|_{L^r(I_{t_1, t_2}; L^{p+1})}^{p-1}) \\ &\quad \times \|u - v\|_{L^r(I_{t_1, t_2}; L^{p+1})}, \quad t_2 \in (t_1, T], \end{aligned}$$

where $q_2 = \frac{4p}{n+4-np}$, $C = C(n, q, p)$ and $I_{t_1, t_2} = (-t_2, -t_1) \cup (t_1, t_2)$. We can choose t_2 such that $t_2 > t_1$ and the coefficient of $\|u - v\|_{L^r(I_{t_1, t_2}; L^{p+1})}$ is smaller than $1/2$ in the right hand side of (3.28). Then,

$$(3.29) \quad \|u - v\|_{L^r(I_{t_1, t_2}; L^{p+1})} \leq 0,$$

which implies that $u(t) = v(t)$ on \bar{I}_{t_1, t_2} . This contradicts the assumption of t_1 .

Therefore, $u(t) = v(t)$ for $t \in \bar{I}_T$.

Thus, for any $t_1 \in \bar{I}_T$ we can uniquely solve (1.1)–(1.2) in the time interval $[t_1 - T, t_1 + T]$ with the initial time t_0 and the initial datum u_0 replaced by t_1 and $u(t_1)$, respectively, where T is the same as in the case of the initial time t_0 and the initial datum u_0 . Therefore, reversing the roles of 0 and t , we obtain the reverse inequality to (3.25) for all $t \in \bar{I}_T$. Accordingly, we obtain (3.3). (3.3) and (3.27) give us

$$(3.30) \quad u(t) \in C(\bar{I}_T; L^2).$$

This completes the proof of Lemma 3.1.

Q. E. D.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. The unique global existence of L^2 -solutions for (1.1)–(1.2) follows directly from Lemma 3.1, which shows the unique local solvability in L^2 of (1.1)–(1.2) and the a priori bound of the L^2 -norm of L^2 -solutions.

It remains only to prove the continuous dependence of L^2 -solutions on the initial data. Let u_{0j} , $j=1, 2, \dots$, and u_0 be such that $u_{0j}, u_0 \in L^2$ and $u_{0j} \rightarrow u_0$ in L^2 ($j \rightarrow \infty$). Let $u_j(t)$ and $u(t)$ be the global L^2 -solutions of (1.1) with $u_j(t_0) = u_{0j}$ and $u(t_0) = u_0$, respectively. We put $\rho = \sup \{\|u_0\|_{L^2}, \|u_{0j}\|_{L^2}, j=1, 2, \dots\}$. For this ρ , let $T > 0$ be defined as in (3.12) and (3.19). Then, by using the same argument as in the proof of Lemma 3.1 we have

$$(3.31) \quad u_j(t) \longrightarrow u(t) \quad \text{in } C(\bar{I}_t; H^{-1}) \quad (j \rightarrow \infty)$$

(see, e.g., (3.22) and (3.26)). Particularly, we have

$$(3.32) \quad (u_j(t), \psi) \longrightarrow (u(t), \psi), \quad t \in \bar{I}_t$$

for $\psi \in H^1$. By (3.32) and the boundedness in L^2 of $\{u_j(t)\}$ we obtain

$$(3.33) \quad u_j(t) \longrightarrow u(t) \quad \text{weakly in } L^2 \quad (j \rightarrow \infty), \quad t \in \bar{I}_T.$$

On the other hand, we have by (1.5)

$$(3.34) \quad \|u_j(t)\|_{L^2} \longrightarrow \|u(t)\|_{L^2} \quad (j \rightarrow \infty), \quad t \in \mathbf{R}.$$

Therefore, (3.33) and (3.34) give us

$$(3.35) \quad u_j(t) \longrightarrow u(t) \quad \text{in } L^2 \quad (j \rightarrow \infty)$$

for each $t \in \bar{I}_T$. The length of T is determined only by n , p , λ and ρ (see (3.12) and (3.19)). By the L^2 -norm conservation law (see (1.5) and (3.3)) we see that $\sup \{\|u(t)\|_{L^2}, \|u_j(t)\|_{L^2}, j=1, 2, \dots\}$ is constant for $t \in \mathbf{R}$. Accordingly, we use the above argument with the initial time t_0 and the initial data $u_0, u_{0j}, j=1, 2, \dots$,

replaced by $t_0 + T$ and $u(t_0 + T)$, $u_j(t_0 + T)$, $j=1, 2, \dots$, or by $t_0 - T$ and $u(t_0 - T)$, $u_j(t_0 - T)$, $j=1, 2, \dots$, respectively, to obtain (3.35) for each $t \in \bar{I}_{2T}$. Repeating this procedure, we obtain (1.6) for each $t \in \mathbf{R}$. This completes the proof of Theorem 1.1. Q. E. D.

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