Funkcialaj Ekvacioj, 30 (1987) 9–17

Existence and Uniqueness of Solutions of Neutral Delay-Differential Equations with State Dependent Delays*

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1. Introduction

Let there be given real numbers $\gamma \leq a < b$, a function f: $[a, b] \times \boldsymbol{R}^3 \rightarrow \boldsymbol{R}$, initial function g: $[\gamma, a] \rightarrow \boldsymbol{R}$, and delay functions α, β: $[a, b] \times \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $\gamma \leq \alpha(t, y) \leq t$, $\gamma \leq \beta(t, y) \leq t$. Here, $\boldsymbol{R}$ denotes the set of real numbers. We consider the initial-value problem for delay-differential equations of neutral type

$$
(1) \qquad \begin{aligned} y'(t) &= F(t, y, y'), && t \in [a, b], \\ y(t) &= g(t), && t \in [\gamma, a], \end{aligned}
$$

where for any functions y, z: $[\gamma, b] \rightarrow \boldsymbol{R}$, F is defined by

$$
F(t, y, z) := f(t, y(t), y(\alpha(t, y(t))), z(\beta(t, y(t)))).
$$

Denote by $\mathrm{Lip}_1 [t_1, t_2]$ the space of real-valued Lipschitz continuous functions on $[t_1, t_2]$ and by $C^{1,1}[t_1, t_2]$ the space of functions whose first derivative belongs to $\mathrm{Lip}_1 [t_1, t_2]$. We impose the following conditions on the functions f, g, α, and β which define problem (1):

(i) $g \in C^{1,1}[\gamma, a]$;

(ii) $F(a, g, g') = g'_-(a)$ ($g'_-(a)$ denotes the left hand derivative of g at a);

(iii) $|f(t_1, y_1, u_1, z_1) - f(t_2, y_2, u_2, z_2)| \leq L_1(|t_1 - t_2| + |y_1 - y_2| + |u_1 - u_2|) + L_2|z_1 - z_2|$, $L_1, L_2 \geq 0$, $t_1, t_2 \in [a, b]$, $y_1, y_2, u_1, u_2, z_1, z_2 \in \boldsymbol{R}$;

(iv) $|\alpha(t_1, y_1) - \alpha(t_2, y_2)| \leq A_1|t_1 - t_2| + A_2|y_1 - y_2|$, $A_1, A_2 \geq 0$, $t_1, t_2 \in [a, b]$, $y_1, y_2 \in \boldsymbol{R}$.

(v) $|\beta(t_1, y_1) - \beta(t_2, y_2)| \leq B_1|t_1 - t_2| + B_2|y_1 - y_2|$, $B_1, B_2 \geq 0$, $t_1, t_2 \in [a, b]$, $y_1, y_2 \in \boldsymbol{R}$.

Additional conditions on some Lipschitz constants appearing above will be given in the formulation of our theorems.

Equations of type (1) arise as a model for a two-body problem of classical electrodynamics and were studied extensively by Driver [2–4]. He proved the

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*) This research was supported in part by the National Science Foundation under grant DMS-8401013.

existence and uniqueness results for the case where $\gamma \leq \beta(t, y) < t$ (see [2, 3]). The case where f is linear in the last argument was studied by Hale and Cruz [6]. Grimm [5] proved an existence result for (1) with $\gamma = a$ assuming that f is bounded by M, $L_2 < 1$, and $B_1 + B_2 M \leq 1$ by use of the Schauder fixed point theorem. He also proved a uniqueness result in case β is independent of y by the Banach contraction principle. In this paper we relaxed this very restrictive condition. We proved an existence and uniqueness result for (1) with β depending on both arguments t and y under the condition $L_2(1 + B_1 + B_2 G) < 1$, where $G = G(f, g)$. It is also shown that the last inequality can be replaced by $L_2(B_1 + B_2 G^*) < 1$, $G^* = G^*(f, g)$, if $\gamma \leq \beta(t, y) \leq t - \delta$ for some $\delta > 0$.

2. Properties of modified euler sequences

Let $J = \{h\colon h = (b-a)/n,\ n \geq n_0\}$, where n_0 is a positive integer, and for $h \in J$ put $t_i = a + ih$, $i = 0, 1, \ldots, n$. Define the modified Euler sequences for (1) by

$$\begin{aligned} y_h(t_i + rh) &= y_h(t_i) + rhz_h(t_i), \qquad && r \in (0, 1], \\ z_h(t_i + rh) &= (1-r)z_h(t_i) + rz_h(t_{i+1}), && r \in (0, 1], \\ z_h(t_{i+1}) &= F(t_{i+1}, y_h, z_h), \end{aligned} \tag{2}$$

$i = 0, 1, \cdots, n-1$, where $y_h(t) = g(t)$ and $z_h(t) = g'(t)$ for $t \in [\gamma, a]$. Note that (2) differs from the Euler method considered by Castleton and Grimm [1], where the approximation to the derivative of the solution is a piecewise constant function rather than piecewise linear. Let $C[t_1, t_2]$ denote the space of continuous functions from $[t_1, t_2]$ into $\boldsymbol{R}$. For any $\phi \in C[\gamma, b]$ and $[c, d] \subset [\gamma, b]$ put $\|\phi\|_{[c,d]} := \sup\{|\phi(t)|\colon t \in [c, d]\}$. Let L_g and $L_{g'}$ be constants such that

$$\begin{aligned} |g(t) - g(\tau)| &\leq L_g|t - \tau|, \\ |g'(t) - g'(\tau)| &\leq L_{g'}|t - \tau|, \end{aligned}$$

for $t, \tau \in [\gamma, a]$. The existence of such constants follows from (i). Define

$$\begin{aligned} M &:= \sup\{|F(t, 0, 0)|\colon t \in [a, b]\}, \\ C_1 &:= (\|g'\|_{[\gamma,a]} + M)/(1 - L_2), \\ C_2 &:= 2L_1/(1 - L_2), \\ Y &:= (\|g\|_{[\gamma,a]} + C_1/C_2)\exp((b-a)C_2), \\ Z &:= C_1 + C_2 Y, \\ G &:= \max\{L_g, Z\}. \end{aligned}$$

We have the following:

Lemma 1. *Assume that* (i)–(v) *hold and that* $L_2(1+B_1+B_2G)<1$. *Then the sequences* $\{y_h\}_{h\in J}$ *and* $\{z_h\}_{h\in J}$ *are relatively compact in* $C[\gamma, b]$.

Proof. It is clear that y_h and z_h are continuous. We will show that $\{y_h\}_{h\in J}$ and $\{z_h\}_{h\in J}$ are uniformly bounded and uniformly Lipschitz-continuous. From (2) it follows that

$$\|y_h\|_{[\gamma,t_{i+1}]} \le \|y_h\|_{[\gamma,t_i]} + h\|z_h\|_{[\gamma,t_i]}, \tag{3}$$

$i=0, 1,\cdots, n-1$. We also have

$$|z_h(t_i)| \le |F(t_i, y_h, z_h)-F(t_i, 0, 0)| + |F(t_i, 0, 0)|$$

(for $i=0$ this follows from (ii)). Hence, in view of (iii) we obtain

$$\|z_h\|_{[\gamma,t_i]} \le \|z_h\|_{[\gamma,a]} + M + 2L_1\|y_h\|_{[\gamma,t_i]} + L_2\|z_h\|_{[\gamma,t_i]}.$$

Noting that $L_2<1$ we get

$$\|z_h\|_{[\gamma,t_i]} \le C_1 + C_2\|y_h\|_{[\gamma,t_i]}, \tag{4}$$

$i=0, 1,\cdots, n$, where C_1 and C_2 are defined above. Substitution of (4) into (3) yields

$$\|y_h\|_{[\gamma,t_{i+1}]} \le (1+hC_2)\,\|y_h\|_{[\gamma,t_i]} + hC_1,$$

and by induction it follows that

$$\|y_h\|_{[\gamma,t_i]} \le (1+hC_2)^i\|y_h\|_{[\gamma,a]} + \frac{((1+hC_2)^i-1)C_1}{C_2},$$

$i=0, 1,\cdots, n$. Consequently, $\|y_h\|_{[\gamma,b]}\le Y$ and by (4) $\|z_h\|_{[\gamma,b]}\le Z$, which proves that $\{y_h\}_{h\in J}$ and $\{z_h\}_{h\in J}$ are bounded uniformly in h. It is easy to check that y_h are uniformly Lipschitz-continuous on $[\gamma, b]$ with a constant G. Denote by D a constant such that $D\ge L_{g'}$ and

$$L_1(1+G(1+A_1+A_2G)) + L_2D(B_1+B_2G) \le D(1-L_2). \tag{5}$$

Note that existence of this constant is guaranteed by the condition $L_2(1+B_1+B_2G)<1$. We will prove that the z_h, $h\in J$, are uniformly Lipschitz-continuous on $[\gamma, b]$ with the constant D, i.e.

$$|z_h(t)-z_h(\tau)| \le D|t-\tau|, \tag{6}$$

for t, $\tau\in[\gamma, b]$. By (i) this condition is satisfied for t, $\tau\in[\gamma, a]$. Assume that (6) holds for t, $\tau\in[\gamma, t_i]$. Then

$$|z_h(t_{i+1})-z_h(t_i)| \le L_1[h+|y_h(t_{i+1})-y_h(t_i)|$$

$$+|y_h(\alpha(t_{i+1}, y_h(t_{i+1}))) - y_h(\alpha(t_i, y_h(t_{i+1})))|$$
$$+|y_h(\alpha(t_i, y_h(t_{i+1}))) - y_1(\alpha(t_i, y_h(t_i)))|]$$
$$+ L_2[|z_h(\beta(t_{i+1}, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_{i+1})))|$$
$$+|z_h(\beta(t_i, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_i)))|]$$
$$\leq L_1(1+G(1+A_1+A_2G))h + L_2DB_2Gh$$
$$+ L_2|z_h(\beta(t_{i+1}, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_{i+1})))|.$$

Two cases are possible: either $\beta(t_{i+1}, y_h(t_{i+1})) \leq t_i$ or $\beta(t_{i+1}, y_h(t_{i+1})) \in (t_i, t_{i+1}]$. In the first case

$$|z_h(\beta(t_{i+1}, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_{i+1})))| \leq DB_1h$$

and

$$|z_h(t_{i+1}) - z_h(t_i)| \leq L_1(1+G(1+A_1+A_2G))h + L_2D(B_1+B_2G)h \leq D(1-L_2)h \leq Dh$$

in view of (5). In the second case

$$z_h(\beta(t_{i+1}, y_h(t_{i+1}))) = (1-r)z_h(t_i) + rz_h(t_{i+1}),$$

where $r = (\beta(t_{i+1}, y_h(t_{i+1})) - t_i)/h$, and

$$|z_h(\beta(t_{i+1}, y_h(t_{i+1}))) - z_h(\beta(t_i, y_h(t_{i+1})))|$$
$$\leq |z_h(t_i) - z_h(\beta(t_i, y_h(t_{i+1})))| + r|z_h(t_{i+1}) - z_h(t_i)|$$
$$\leq D|\beta(t_{i+1}, y_h(t_{i+1})) - \beta(t_i, y_h(t_{i+1}))| + |z_h(t_{i+1}) - z_h(t_i)|$$
$$\leq DB_1h + |z_h(t_{i+1}) - z_h(t_i)|.$$

Hence,

$$(1-L_2)|z_h(t_{i+1}) - z_h(t_i)| \leq L_1(1+G(1+A_1+A_2G))h + L_2D(B_1+B_2G)h \leq (1-L_2)D.$$

The last inequality follows from (5). Because z_h is piecewise continuous on $[a, b]$, this proves (6) for $t, \tau \in [\gamma, t_{i+1}]$. Thus $\{y_h\}_{h\in J}$ and $\{z_h\}_{h\in J}$ are uniformly bounded and uniformly Lipschitz-continuous. In view of the Ascoli-Arzela theorem they are relatively compact in $C[\gamma, b]$, which completes the proof of Lemma 1.

Since $L_2(1+B_1+B_2G)<1$, L_2 is necessarily less than 1. This condition can be relaxed if $\gamma \leq \beta(t, y) \leq t-\delta$ for some $\delta>0$. Denote by K the smallest integer such that $K \geq 2(b-a)/\delta$ and put $S(\mu) = \sum_{j=0}^{\mu} L_2^j$. Define

$$C_1^* := S(K)\,\|g'\|_{[\gamma,a]} + S(K-1)M,$$
$$C_2^* := 2L_1 S(K-1),$$
$$Y^* := (\|g\|_{[\gamma,a]} + C_1^*/C_2^*)\exp((b-a)C_2^*),$$
$$Z^* := C_1^* + C_2^* Y^*,$$
$$G^* := \max\{L_g, Z^*\}.$$

We have the following:

Lemma 2. *Assume that* (i)–(v) *hold,* $\gamma \le \beta(t, y) \le t-\delta$, $t \in [a, b]$, *for some* $\delta > 0$ *and that* $L_2(B_1 + B_2 G^*) < 1$. *Then the sequences* $\{y_h\}_{h\in J}$ *and* $\{z_h\}_{h\in J}$ *are relatively compact in* $C[\gamma, b]$.

Proof. As in the proof of Lemma 1, we obtain

$$\|z_h\|_{[\gamma,t_i]} \le \|z_h\|_{[\gamma,a]} + M + 2L_1\|y_h\|_{[\gamma,t_i]} + L_2\|z_h\|_{[\gamma,t_i-\delta]},$$

$i = 0, 1, \cdots, n$. To estimate $\|z_h\|_{[\gamma,t_i]}$ by $\|y_h\|_{[\gamma,t_i]}$ we use arguments similar to those in [7, 8]. Put $i(0) = i$ and denote by $i(v+1)$ the smallest integer such that $t_{i(v)} - \delta \le t_{i(v+1)}$, $v = 0, 1, \cdots$. Note that if $h \le \delta$ then $i(v+1) < i(v)$, $i = 0, 1, \cdots, K_i - 1$, and $i(v) = 0$ for $v \ge K_i$, where K_i is the smallest integer such that $K_i[\delta/h] \ge i$. Here $[\delta/h]$ denotes the integer part of δ/h. For $h \le \delta$ we have $K[\delta/h] \ge 2(b-a)[\delta/h]/\delta \ge (b-a)h \ge i$. Since K_i is the smallest integer with this property it follows that $K_i \le K$, i.e. K_i can be bounded by a constant independent of h. As in [7, 8], after $v \le K_i$ iterations we obtain

$$\|z_h\|_{[\gamma,t_i]} \le S(v-1)(\|g'\|_{[\gamma,a]} + M) + 2S(v-1)L_1\|y_h\|_{[\gamma,t_i]} + L_2^v\|z_h\|_{[\gamma,t_{i(v)}]}.$$

Putting $v = K_i$ in this inequality gives

$$\|z_h\|_{[\gamma,t_i]} \le S(K_i)\,\|g'\|_{[\gamma,a]} + S(K_i-1)M + 2S(K_i-1)L_1\|y_h\|_{[\gamma,t_i]}$$
$$\le C_1^* + C_2^*\|y_h\|_{[\gamma,t_i]},$$

which is the desired estimate for $\|z_h\|_{[\gamma,t_i]}$. From this point, proceeding as in the proof of Lemma 1 we get $\|y_h\|_{[\gamma,b]} \le Y^*$, $\|z_h\|_{[\gamma,b]} \le Z^*$ and $|y_h(t) - y_h(\tau)| \le G^*|t-\tau|$ for $t, \tau \in [\gamma, b]$. Denote by $D^* \ge L_{g'}$ a constant such that

$$L_1(1 + G^*(1 + A_1 + A_2 G^*)) + L_2 D^*(B_1 + B_2 G^*) \le D^*. \tag{7}$$

The existence of such a constant is guaranteed by the condition $L_2(B_1 + B_2 G^*) < 1$. Assume that

$$|z_h(t) - z_h(\tau)| \le D^*|t-\tau|$$

for $t, \tau \in [\gamma, t_i]$. Then, if $h \le \delta$, in view of (7) we have

$$|z_h(t_{i+1}) - z_h(t_i)| \le L_1(1+G^*(1+A_1+A_2G^*))h + L_2D^*(B_1+B_2G^*) \le D^*.$$

This proves that z_h, $h \in J$, are uniformly Lipschitz-continuous, and the Lemma follows.

3. Existence and uniqueness

For any $\phi \in C[\gamma, b]$ and $h \in J$ define ϕ^h and $\bar{\phi}^h$ by

$$\phi^h(t) := \begin{cases} \phi(t), & t \in [\gamma, a), \\ \phi(t_i), & t \in [t_i, t_{i+1}); \end{cases}$$

and

$$\bar{\phi}^h(t) := \begin{cases} \phi(t), & t \in [\gamma, a), \\ \dfrac{t_{i+1}-t_i}{h}\phi(t_i) + \dfrac{t-t_i}{h}\phi(t_{i+1}), & t \in [t_i, t_{i+1}), \end{cases}$$

$i=0, 1, \cdots, n$. We will use this notation in the proof of the following theorem.

Theorem 1. *Assume that* (i)–(v) *hold and that* $L_2(1+B_1+B_2G)<1$, *where* G *is defined as above. Then the initial-value problem* (1) *has a solution* $y \in C^{1,1}[\gamma, b]$. *This solution is unique in the space of continuously differentiable functions* $C^1[\gamma, b]$.

Proof. We will first show existence. It follows from Lemma 1 that $\{y_h\}_{h\in J}$ and $\{z_h\}_{h\in J}$ defined by (2) are relatively compact in $C[\gamma, b]$. Denote by $\{z_h\}_{h\in J'}$, $J' \subset J$, $\inf J'=0$, any convergent subsequence of $\{z_h\}_{h\in J}$ with limit z, i.e. $\|z_h - z\|_{[\gamma,b]} \to 0$ as $h \to 0$, $h \in J'$. It is clear that $z \in \mathrm{Lip}_1[\gamma, b]$. We will show that $\{y_h\}_{h\in J'}$ converges to the function y defined by

$$y(t) = \begin{cases} g(t), & t \in [\gamma, a], \\ g(a) + \displaystyle\int_a^t z(s)ds, & t \in [a, b], \end{cases}$$

and that y is a solution of our problem (1). Note that for $t \in [a, b]$

$$z_h(t) = \bar{z}_h^h(t) = \bar{z}^h(t) + o(1)$$

and

$$y_h(t) = g(a) + \int_a^t z_h^h(s)ds = g(a) + \int_a^t z^h(s)ds + o(1).$$

Define

$$\bar{y}_h(t) = g(a) + \int_a^t z_h(s)ds.$$

Clearly, $\bar{y}_h \to y$ as $h \to 0$, $h \in J'$. We have also

$$|y_h(t) - \bar{y}_h(t)| \leq \int_a^t |z^h(s) - \bar{z}^h(s)|ds + o(1).$$

Hence, $\|y_h - \bar{y}_h\|_{[\gamma,b]} \to 0$ as $h \to 0$, $h \in J'$. Consequently

$$\|y_h - y\|_{[\gamma,b]} \leq \|y_h - \bar{y}_h\|_{[\gamma,b]} + \|\bar{y}_h - y\|_{[\gamma,b]},$$

which proves that $y_h \to y$ as $h \to 0$, $h \in J'$. For $t \in [a, b]$ we have

$$\begin{aligned} &|y_h(t) - \int_a^t F(s, y, y')ds - g(a)| \\ (8) \qquad &\leq \int_a^t |F^h(s, y_h, z_h) - F(s, y, y')|ds \\ &\leq \gamma_1(y, h) + \gamma_2(y, h), \end{aligned}$$

where

$$\gamma_1(y, h) := \int_a^t |F^h(s, y_h, z_h) - F^h(s, y, y')|ds,$$

$$\gamma_2(y, u) := \int_a^t |F^h(s, y, y') - F(s, y, y')|ds.$$

To estimate $\gamma_1(y, h)$ note that

$$\gamma_1(y, h) = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} |F(t_j, y_n, z_h) - F(t_j, y, y')|ds + \int_{t_i}^t |F(t_i, y_h, z_h) - F(t_i, y, y')|ds,$$

for $t \in (t_i, t_{i+1}]$. As in the proof of Lemma 1 we obtain

$$|F(t_j, y_h, z_h) - F(t_j, y, y')| \leq (L_1(2 + GA_2) + L_2DB_2)\|y_h - y\|_{[\gamma,b]} + L_2\|z_h - y'\|_{[\gamma,b]}.$$

From this inequality it follows that $\gamma_1(y, h) \to 0$ as $h \to 0$, $h \in J'$. It is also clear that $\gamma_2(y, h) \to 0$ as $h \to 0$, $h \in J'$, because the function $s \to F(s, y, y')$ is continuous (in particular Riemann integrable). Thus, in view of (8), y is a solution of (1).

Now we will show uniqueness. Assume that there is another solution $\tilde{y} \in C^1[\gamma, b]$ of (1) and put $u := y - \tilde{y}$. Then $u \equiv 0$ on $[\gamma, a]$ and

$$\begin{aligned}|u'(t)| &\le |F(t, y, y')-F(t, \tilde{y}, \tilde{y}')| \le L_1[|u(t)| \\ &+|y(\alpha(t, y(t)))-y(\alpha(t, \tilde{y}(t)))|+|y(\alpha(t, \tilde{y}(t)))-\tilde{y}(\alpha(t, \tilde{y}(t)))|] \\ &+L_2[|y'(\beta(t, y(t)))-y'(\beta(t, \tilde{y}(t)))|+|y'(\beta(t, \tilde{y}(t)))-\tilde{y}'(\beta(t, \tilde{y}(t)))|] \\ &\le (L_1(2+GA_2)+L_2DB_2)\,\|u\|_{[a,t]} + L_2\|u'\|_{[a,t]}.\end{aligned}$$

Hence,

$$\|u'\|_{[a,b]} \le W\|u\|_{[a,b]},$$

where

$$W=(L_1(2+GA_2)+L_2DB_2)/(1-L_2).$$

We also have

$$\begin{aligned}|u(t)| &\le \int_a^t |F(s, y, y')-F(s, \tilde{y}, \tilde{y}')|ds \\ &\le (L_1(2+GA_2)+L_2DB_2)\int_a^t \|u\|_{[a,s]}ds + L_2\int_a^t \|u'\|_{[a,s]}ds.\end{aligned}$$

Hence,

$$\|u\|_{[a,t]} \le E\int_a^t \|u\|_{[a,s]}ds,$$

where

$$E=(L_1(2+GA_2)+L_2DB_2) + L_2W.$$

By induction it follows that

$$\|u\|_{[a,t]} \le \frac{(E(t-a))^k}{k!}\|u\|_{[a,b]}$$

for any $t\in[a, b]$ and any integer $k\ge 0$ (compare also Lemma 2 in [9]). Consequently, $u\equiv 0$ and $y\equiv\tilde{y}$ which is our claim.

If $\gamma\le\beta(t, y)\le t-\delta$ for some $\delta>0$ we can relax the condition that $L_2<1$, and we have the following:

Theorem 2. *Assume that* (i)–(v) *hold,* $\gamma\le\beta(t, y)\le t-\delta, \delta>0,$ *for* $t\in[a, b]$, *and that* $L_2(B_1+B_2G^*)<1$, *where* G^* *is defined as above. Then problem* (1) *has a solution* $y\in C^{1,1}[\gamma, b]$. *This solution is unique in* $C^1[\gamma, b]$.

Proof. It follows from Lemma 2 that the modified Euler sequences $\{y_h\}_{h\in J}$ and $\{z_h\}_{h\in J}$ are relatively compact in $C[\gamma, b]$. From this point the proof is similar to that of Theorem 1.

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