Global Strong Solutions of Coupled
Klein-Gordon-Schrödinger Equations

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§ 1. Introduction

We consider the following system of equations

\[
\begin{align*}
{id\psi/dt - A_1\psi} & = -\psi\partial f(|\psi|^2, \phi)\partial|\psi|^2, \\
{d^2\phi/dt^2 + A_2\phi} & = f(|\psi|^2, \phi) + \phi\partial f(|\psi|^2, \phi)\partial\phi,
\end{align*}
\]

where \(A_1\) and \(A_2\) denote second order elliptic operators over a bounded or unbounded domain in \(\mathbb{R}^3\) and \(f\) is a real \(C^1\)-function of two variables \(|\psi|^2\) and \(\phi\). We use the customary notations for the partial derivative of \(f(y, z)\). If \(A_1 = -\Delta, A_2 = -\Delta + I\) and \(f(y, z) = y\), (1.1) and (1.2) are the so called Klein-Gordon-Schrödinger (K-G-S) equations with Yukawa coupling in which \(\psi\) describes complex scalar neucleon field and \(\phi\) real scalar meson field.

The first study for the K-G-S equations was done by I. Fukuda-M. Tsutumi [8]. They considered the initial-boundary value problem for the K-G-S equations under the initial conditions \(\psi(x, 0) = \psi_0(x) \in H_0^{3/2}(\Omega) \cap H_0^{3/2}(\Omega), \phi(x, 0) = \phi_0(x) \in H_0^{1/2}(\Omega) \cap H_0^{1/2}(\Omega), \phi_0(x, 0) = \phi_1(x) \in H_0^{1/2}(\Omega)\) and the boundary conditions \(\psi(x, t) = \psi(x, t) = 0\) for \(x \in \partial\Omega\) and \(t \in \mathbb{R}\). Here \(\Omega\) is a bounded domain in \(\mathbb{R}^3\) and \(\partial\Omega\) is smooth boundary of \(\Omega\). They proved the global existence of strong solutions by Galerkin's method. But, as the initial condition on \(\psi_0(x)\) is unnatural, it should be changed into the natural condition such as \(\phi_0(x) \in H_0^{3/2}(\Omega) \cap H_0^{3/2}(\Omega)\).

The second study was done by J.B. Baillon-J.M. Chadam [2]. They proved the global existence of strong solutions to the initial value problem for and K-G-S equations \(\psi_0(x) \in H^{3/2}(\mathbb{R}^3), \phi_0(x) \in H^{3/2}(\mathbb{R}^3)\) and \(\phi_1(x) \in H^{1/2}(\mathbb{R}^3)\). They obtained the result by using \(L^p - L^q\) estimates of the elementary solution of the linear Schrödinger equation (see also A. Bachelot [1]). Hence, it does not seem that their method is directly applicable to a domain which is not equal to \(\mathbb{R}^3\) and to more general \(A_1\).

We will prove the global existence of strong solutions of more general systems of equations over an arbitrary smooth domain in \(\mathbb{R}^3\), under the same boundary conditions as [8] and the same initial conditions as [2] by using the time derivative of (1.1) (see section 3). Moreover, if we use the estimates of nonlinearity in fractional
order Besov spaces which were developed by P. Brenner-W. v. Wahl [4], and the
inequality of H. Brezis-T. Gallouet-S. Wainger [5], [6], we can treat more general
nonlinear terms in the case of the initial value problem for (1.1) and (1.2) (see section
4).

§ 2. Preliminaries and Estimates of nonlinear terms

Let \( \Omega \) be a bounded or unbounded domain in \( \mathbb{R}^n \) with smooth compact boundary \( \partial \Omega \). Let \( s \in \mathbb{R}^+ \) and \( 1 \leq p < \infty \). For simplicity, we denote the space of complex valued functions and real valued functions by the same symbol. Let \( H^{s,p}(\mathbb{R}^n) \), \( H^{s,p}(\Omega) \) and \( H^0_{0,p}(\Omega) \) be the usual Sobolev spaces of fractional order \( s \) of all \( L^p \) functions.

Let \( 1 < p, q < \infty \) and \( s=[s]+\sigma \) where \([s]\) denotes the largest integer less than \( s \) and \( 0<\sigma<1 \). The Besov space \( B^{s,q}_p(\mathbb{R}^n) \) is the completion of \( S(\mathbb{R}^n) \) in the norm:

\[
\|u\|_{s,q,p} = \|u\|_{L_p(\mathbb{R}^n)} + \left( \int_0^{\infty} t^{-\frac{\sigma q}{q}} \sup_{|k| \leq t} \sum_{|a| \leq k} \|D^a(u-u_k)\|_{L_p(\mathbb{R}^n)} \cdot \frac{dt}{t} \right)^{\frac{1}{q}},
\]

where \( u_k(x) = u(x+k) \) (see [3]).

Positive constants will be denoted by \( C \) and will change from line to line. The norms of \( L^p(\Omega) \) and \( H^{s,p}(\Omega) \) are denoted by \( \| \cdot \|_p \) and \( \| \cdot \|_{s,p} \), respectively. We simply denote the norms of \( L^p(\mathbb{R}^n) \) and \( H^{s,p}(\mathbb{R}^n) \) by the same symbols as \( L^p(\Omega) \) and \( H^{s,p}(\Omega) \), respectively.

We now state some useful Lemmas which are used in this paper.

**Lemma 1** ([3]). Let \( s, t \in \mathbb{R}^+ \) and \( 1 < p \leq q < \infty \). Then

\[
B^{s,t}_q(\mathbb{R}^n) \subset B^{s,q}_p(\mathbb{R}^n)
\]

provided that \((1/q) \geq (1/p) - (s-t)/3 \), and \( s \geq t \).

**Lemma 2** ([3]). Let \( s \in \mathbb{R}^+ \). Then

\[
B^{s,t}_q(\mathbb{R}^n) = H^{s,t}(\mathbb{R}^n).
\]

**Lemma 3** ([7]). Let \( 1 \leq q, r \leq \infty \), and let \( j, m \) be any integer satisfying \( 0 \leq j < m \). Then

\[
\sum_{|\beta|=j} \|D^\beta u\|_p \leq M \sum_{|\beta|=m} \|D^\beta u\|_p \|u\|_{q}^{-\sigma},
\]

for all \( u \in H^{m,q}(\mathbb{R}^n) \) (or \( u \in H^{m,q}(\Omega) \)), where \( (1/p) = (j/3) + \alpha((1/r)-(m/3)) + (1-\alpha)/q \) for all \( \alpha \) in the interval \( (jm) \leq \alpha \leq 1 \), and \( M \) is a constant depending only on \( m, j, q, r, \alpha \), with the following exception: If \( m-j-(3/j)r \) is a nonnegative integer, then the above inequality is asserted for \( \alpha = j/m \).
Next, we give some estimates of the nonlinear term in fractional order Besov spaces, which are needed if we consider the initial value problem for (1.1) and (1.2).

**Lemma 4.** Let $u \in H^{1,2}(\mathbb{R}^3)$, $v \in H^{3/2,2}(\mathbb{R}^3)$ and $f(y, z)$ satisfy the following conditions

\begin{align*}
(2.1) \quad & |f_y(y, z)z|, |f(y, z)| \leq C|y|^{r/2}|z|^{s}, \\
(2.2) \quad & |f_{yy}(y, z)z|, |f_y(y, z)| \leq C(1 + |y| + |z|), \\
(2.3) \quad & |f_{yy}(y, z)y^{1/2}z|, |f_y(y, z)y^{1/2}| \leq C(1 + |y|^{1/2} + |z|),
\end{align*}

where $y \in \mathbb{R}^r$, $z \in \mathbb{R}$, $0 \leq r$, $s$ and $1 \leq r + s \leq 3$. Then we have

\begin{align*}
(2.4) \quad & \|f(|u|^2, v)\|_{1/2,2} + \|f_y(|u|^2, v)v\|_{1/2,2} \\
& \quad \leq C(1 + \|u\|_{1,2}^2 + \|v\|_{1,2}^2)(\|u\|_{1,2} + \|v\|_{3/2,2}).
\end{align*}

**Proof.** By the definition of Besov space and Lemma 2

\begin{align*}
(2.5) \quad & \|f(|u|^2, v)\|_{1/2,2} \leq C\|f(|u|^2, v)\|_2 \\
& \quad + C\left(\int_0^{\infty} t^{-1} \sup_{|h| \leq t} \|f(|u_h|^2, v_h) - f(|u|^2, v)\|_2 \frac{dt}{t}\right)^{1/2}.
\end{align*}

The mean-value theorem, (2.2) and (2.3) give

\begin{align*}
|f(|u_h|^2, v_h) - f(|u|^2, v)| \leq C(1 + |u_h|^2 + |v_h|^2 + |v|)|v_h - v| \\
& \quad + C(1 + |u_h| + |u| + |v|^2)|u_h - u|.
\end{align*}

Hölder's inequality shows

\begin{align*}
(2.6) \quad & \|f(|u_h|^2, v_h) - f(|u|^2, v)\|_2 \leq C(1 + \|u_h\|_6^2 + \|v_h\|_6^2)\|v_h - v\|_6 \\
& \quad + C(1 + \|u_h\|_6 + \|v_h\|_6)\|u_h - u\|_6.
\end{align*}

By (2.1), (2.6) and Lemma 1,

\begin{align*}
(2.7) \quad & \|f(|u|^2, v)\|_{1/2,2} \leq C(1 + \|u\|_{1,2}^2 + \|v\|_{1,2}^2)(\|u\|_{1,2} + \|v\|_{1,2}) \\
& \quad + C(1 + \|u\|_{1,2}^2 + \|v\|_{1,2}^2)\|v\|_{1/2,2}^2 \\
& \quad + C(1 + \|u\|_{1,2} + \|v\|_{1,2})\|u\|_{1/2,2}.
\end{align*}

From Lemma 1 and Lemma 2,

\begin{align*}
(2.8) \quad & \|v\|_{1/2,2,2} \leq C\|v\|_{1/2,2,2}, \quad \|u\|_{1/2,2,2} \leq C\|u\|_{1,2}.
\end{align*}

It is easy to see that (2.7) and (2.8) imply

\begin{align*}
(2.9) \quad & \|f(|u|^2, v)\|_{1/2,2} \leq C(1 + \|u\|_{1,2}^2 + \|v\|_{1,2}^2)(\|u\|_{1,2} + \|v\|_{1/2,2}^2).
\end{align*}
$\|f_v(|u|^2, v)v\|_{1/2}$ is estimated by the right hand side of (2.9) in the same way as in the proof of (2.9), and so we omit it.

**Lemma 5.** Let $w \in L^2(\mathbb{R}^3)$, $u, v \in L^\infty(\mathbb{R}^3)$ and $f(y, z)$ satisfy (2.3) and the conditions

\begin{equation}
|f_{y}(y, z)z| \leq C(1 + |z|^2),
\end{equation}

for all $y \in \mathbb{R}^3$, $z \in \mathbb{R}$. Then we have

\begin{align}
&||f_v(|u|^2, v)v||_{1} \leq C(1 + |v|_2^2), \label{2.11} \\
&||f_{y,y}(|u|^2, v)|u|^2vw||_{2} \leq C(1 + |w|_2^2), \label{2.12}
\end{align}

**Proof.** (2.11) and (2.12) follow from (2.3) and (2.10) easily. Therefore, we omit it.

Finally, we give the Brezis-Gallouet inequality [5] (see also [6]).

**Lemma 6.** Let $u \in H^{3/2,2}(\mathbb{R}^3) \cap H^{2,2}(\mathbb{R}^3)$. Then we have

\begin{equation}
||u||_{\infty} \leq C(1 + |u|_{3/2,2} \vee \overline{1}_{\mathrm{g}}(1 + ||u||_{2,2})), \label{2.13}
\end{equation}

§ 3. **Initial-Boundary-Value problem for (1.1) and (1.2) in $L^2(\Omega)$**

It this section we assume that $a_{i,j}^{k}(x)$ and $a^{k}(x)$ are sufficiently smooth real valued functions on $\mathbb{R}^3$ satisfying the following conditions: Every derivative of them is bounded on $\mathbb{R}^3$ and

\begin{align}
C|\xi|^p \leq \sum_{i,j=1}^{3} a_{i,j}^{k} |\xi_i \xi_j|^r \leq C^{-1} |\xi|^r, \quad a_{i,j}^{k} = a_{j,i}^{k},
\end{align}

where $\xi \in \mathbb{R}^3$, $1 \leq i, j \leq 3$ and $k = 1, 2$.

We define the operators $A_1$ and $A_2$ by

\begin{equation}
A_k u = - \sum_{i,j=1}^{3} \partial_i(a_{i,j}^{k}(\partial_i u)) + a^{k} u
\end{equation}

for $u \in D(A_k) = H_0^{3/2}(\Omega) \cap H^{3/2}(\Omega), \quad k = 1, 2.$

Then $A_1$ and $A_2$ are selfadjoint operators in $L^2(\Omega)$. For our purposes we may assume without loss of generality that

\begin{equation}
(A_k u, \bar{u}) \geq C ||u||_{3/2}^2, \quad k = 1, 2.
\end{equation}

Furthermore we suppose that $f(y, z) = g(z)y$, where $g(z) \in C^0(\mathbb{R}^3)$ satisfies the following properties:

\begin{align}
|g(z)| \leq C(1 + |z|^{1/2}), \quad |g'(z)| \leq C.
\end{align}
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We will consider the initial boundary value problem for (1.1) and (1.2) with the operators $A_1$ and $A_2$ defined by (3.1) satisfying (3.2), and the nonlinear term satisfying (3.3). The following local existence theorem is proved in the standard manner (see [9]).

**Theorem 1.** Let $\psi_0 \in D(A_1)$, $\phi_0 \in D(A_2)$ and $\phi_1 \in D(A_2^{1/2})$. Then there exist the unique strong solutions (1.1) and (1.2) in some time interval such that

$$\psi \in C([0, T]; D(A_1)) \cap C^1([0, T]; L^2(\Omega)),
\phi \in C([0, T]; D(A_2)) \cap C^1([0, T]; D(A_2^{1/2})) \cap C^2([0, T]; L^2(\Omega)).$$

Our purpose in this paper is to show that the above solutions can be extended to $T=\infty$. This can be proved if a priori estimates of $||\psi(t)||_{2,2}$, $||\phi(t)||_{2,2}$, $||\phi_t(t)||_{1,2}$ and $||\phi_{tt}(t)||_2$ are obtained. We first state a priori estimates of $||\psi(t)||_{1,2}$, $||\phi(t)||_{1,2}$.

**Lemma 7.** Let $\psi_0$ be complex, $\phi_0$ and $\phi_1$ be real. We assume that the hypotheses of Theorem 1 are satisfied. Then the solutions of (1.1) and (1.2) mentioned in Theorem 1 satisfy the inequality

$$||\psi(t)||_{1,2} + ||\phi(t)||_{1,2} + ||\phi_t(t)||_2 \leq C.$$  

**Proof.** (1.1) give

$$||\psi(t)||_{1,2} = ||\psi_0||^2.$$  

We use (1.1) and (1.2) to obtain

$$(A_1\psi, \overline{\psi}) + (A_2\phi, \phi) + (A_2\phi, \phi) + (\phi_{tt}, \phi_t) = d(g(\phi)|\psi|^2, \phi)/dt$$

from which we have

$$d(\text{Re} (A_1\psi, \overline{\psi}) + (A_2\phi, \phi) + (\phi_{tt}, \phi_t)/2 = d(g(\phi)|\psi|^2, \phi)/dt.$$  

By virtue of (3.1), (3.3), (3.5), (3.6) and Lemma 3, we get (3.4) (see also [2], [8]).

**Theorem 2.** We assume that the hypotheses of Lemma 7 are satisfied. Then there exist the unique strong solutions of (1.1) and (1.2) such that

$$\psi \in C([0, \infty); D(A_1)) \cap C^1([0, \infty); L^2(\Omega)),
\phi \in C([0; \infty); D(A_2)) \cap C^1([0, \infty); D(A_2^{1/2})) \cap C^2([0, \infty); L^2(\Omega)).$$

**Proof.** On differentiating (1.1) with respect to $t$, we get

$$idw/dt - A_1w = -(w\phi + \psi\phi, g(\phi)) - \psi\phi_tg_\psi(\phi),$$
where \( w = d \psi/dt \). On multiplying (3.7) by \( \overline{w} \), taking imaginary part and integrating by parts, we get

\[
\frac{d}{dt} \|w(t)\|_{2}^{2} = -\mathrm{Im} \int_{\Omega} (\psi \phi_{t} \overline{g}(\phi) + \psi \phi \phi_{t} \overline{g}(\phi)) \overline{w}(t) \, dx.
\]

From Hölder's inequality, Lemma 3, Lemma 7, (3.3) and (3.8), we have

\[
\|\psi_{t}(t)\|_{2} \leq C + \sup_{0 \leq t \leq \tau} \|\psi_{tt}(t)\|_{L^{1,2}}.
\]

The integral equation corresponding to (1.2), Lemma 3, Lemma 7 give

\[
\|\phi_{t}(t)\|_{2,2} \leq C + \sup_{0 \leq t \leq \tau} \|\phi_{tt}(t)\|_{H^{2,2}}.
\]

Theorem 2 follows from Theorem 1, (3.12) and (3.13).
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where \( a^e_{i,j} \) and \( a^e \) are the same as those in section 3, and let (3.1) be satisfied. Then we have the relation.

\[
(4.2) \quad C\|u\|_{2p,2} \leq \|A^e u\|_2 \leq C^{-1} \|u\|_{2p,2} \quad \text{for } u \in D(A^e) = H^{2p,2} (\mathbb{R}^d), \quad \rho \geq 0.
\]

Moreover we suppose that \( f(y,z) \) satisfies the hypotheses of Lemma 4 and Lemma 5.

As the above relation is not true generally in the case that we take into consideration boundary values, our method stated below is not applicable to the initial boundary value problem for (1.1) and (1.2) stated in section 3.

We will consider the initial value problem for (1.1) and (1.2) under the above conditions. We give our main result in this section.

**Theorem 3.** Let \( \psi \in H^{k, \gamma} (\mathbb{R}^d) \) be complex, \( \phi_0 \in H^{k, \gamma} (\mathbb{R}^d) \) and \( \phi_1 \in H^{k, \gamma} (\mathbb{R}^d) \) be real. Then there exist the unique strong solutions of (1.1) and (1.2) such that

\[
\psi \in C([0; \infty); H^{k, \gamma} (\mathbb{R}^d)) \cap C^1 ([0, \infty); L^2 (\mathbb{R}^d)),
\]

\[
\phi \in C([0, \infty); H^{k, \gamma} (\mathbb{R}^d)) \cap C^1 ([0, \infty); H^{1, \gamma} (\mathbb{R}^d)) \cap C^1 ([0, \infty); L^2 (\mathbb{R}^d)).
\]

**Proof.** In the same way as in the proof of Theorem 2, we have to obtain a priori estimates of local solutions stated in Theorem 1 in order to prove Theorem 3. In the same procedure as in the proof of Lemma 7, we can easily get

\[
(4.3) \quad \|\psi(t)\|_{1,2} \leq \|\phi(t)\|_{1,2} \leq \|\phi_1(t)\|_2 \leq C \quad \text{for any } t \in [0, T]
\]

We apply Lemma 4 and (4.2) to the integral equation corresponding to (1.2) to obtain

\[
\|\phi(t)\|_{2,2} \leq C + C \int_0^t (1 + \|\psi(s)\|_{1,2} + \|\phi(s)\|_{1,2}) (\|\phi(s)\|_{1,2} + \|\psi(s)\|_{1,2}) ds
\]

for any \( t \in [0, T] \).

(4.3) and Gronwall’s inequality give

\[
(4.4) \quad \sup_{0 \leq t \leq T} \|\phi(t)\|_{2,2} \leq C(T).
\]

By (1.1), (4.2), (4.3) and Lemma 3 we see

\[
(4.5) \quad \|\psi(t)\|_2 \leq C \|\psi(t)\|_{2,2} + C,
\]

\[
(4.6) \quad \|\psi(t)\|_{2,2} \leq C \|\psi(t)\|_2 + C.
\]

Differentiating (1.1) with respect to \( t \), we get

\[
(4.7) \quad idw/dt \cdot A_x w = -f(\psi, \psi \phi)(\psi \phi \psi w + \psi^2 \phi w) - f(\psi, \phi \psi \phi \psi w - f(\psi, \psi \phi \psi w + \psi \phi \phi w),
\]
where $w = d\psi/dt$. By the integral equation corresponding to (4.7) and Lemma 5, we have
\begin{equation}
\|\psi_1(t)\|_\infty \leq C + C \int_0^t (1 + \|\phi(s)\|_\infty) \|\psi_1(s)\|_\infty ds + C \int_0^t (1 + \|\psi(s)\|_\infty + \|\phi(s)\|_\infty) \|\phi_1(s)\|_\infty ds
\end{equation}
for any $t \in [0, T]$.

The integral equation corresponding to (1.2) and (4.3) gives
\begin{equation}
\|\phi(t)\|_{2,2} \leq C + C \int_0^t (1 + \|\phi(s)\|_{2,2} + \|\phi(s)\|_{1,2}) ds
\end{equation}
for any $t \in [0, T]$.

Lemma 6, (4.3), (4.4), (4.5), (4.6) and (4.8) show
\begin{equation}
\|\psi(t)\|_{2,2} \leq C + C(T) \int_0^t (1 + \log (1 + \|\phi(s)\|_{2,2}))(\|\psi(s)\|_{2,2} + \|\phi(s)\|_{2,2}) ds
\end{equation}
for any $t \in [0, T]$.

We denote by $f(t)$ the right hand side in (4.9) and by $g(t)$ the right hand side in (4.10). By a simple calculation, we find
\begin{equation}
dG(t)/dt \leq C(T)G(t)(1 + \log(1 + G(t)))
\end{equation}
for any $t \in [0, T]$, where $G(t) = f(t) + g(t)$. Consequently
\[\sup_{0 \leq t \leq T} G(t) \leq C(T)\]
Therefore, we have
\begin{equation}
\sup_{0 \leq t \leq T} \|\psi_1(t)\|_\infty, \sup_{0 \leq t \leq T} \|\psi(t)\|_{2,2}, \sup_{0 \leq t \leq T} \|\phi(t)\|_{2,2} \leq C(T).
\end{equation}
From (1.2) and (4.12), we can easily get
\begin{equation}
\sup_{0 \leq t \leq T} \|\phi_{tt}(t)\|_{2,2}, \sup_{0 \leq t \leq T} \|\phi(t)\|_{1,2} \leq C(T).
\end{equation}
This completes the proof.

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