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Partial Difference Equations Analogous to the Cauchy-Riemann Equations, II

Dedicated to Prof. J. Aczél on his 60th birthday

By

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§ 1. Introduction

Let R be the field of real numbers and C be the field of complex numbers. For a function $f: R\times R\to C$ we define the usual divided partial difference operators $\underset{x,t}{\triangle}$ and $\underset{y,t}{\triangle}$ by $(\underset{x,t}{\triangle} f)(x, y)=[f(x+t, y)-f(x, y)]/t$ and $(\underset{y,t}{\triangle} f)(x, y)=[f(x, y+t)-f(x, y)]/t$. The difference functional equation

$$(1.1)\qquad (\underset{x,t}{\triangle} f)(x, y)=-i[(\underset{y,t}{\triangle} f)(x, y)]$$

may be considered as a discrete analogue of the Cauchy-Riemann partial differential equation

$$\partial f/\partial x=-i\partial f/\partial y.$$

Equation (1.1) may be rewritten in the form

$$(1.2)\qquad f(x+t, y)-f(x, y)=-i[f(x, t+y)-f(x, y)],$$

which has a simple geometric interpretation on the plane. Throughout this note we denote by M an arbitrary monoid (not necessarily commutative) with the unit element 0. In a previous paper [1] we considered the above difference functional equation (1.2) and proved the following theorem [1, Theorem 2, p. 99]:

Theorem 1.1. *A function f: $M\times M\to C$ satisfies equation* (1.2) *for all $x, y, t\in M$ if and only if there exist an arbitrary homomorphism ϕ from M into the additive group C and an arbitrary complex constant c such that $f(x, y)=\phi(x)+i\phi(y)+c$ for all $x, y\in M$.*

In the above Theorem 1.1 we may impose the normarization $f(0, 0)=0$ so that the general solution of (1.2) is given by

$$(1.3)\qquad f(x, y)=\phi(x)+i\phi(y)$$

for all $x, y \in M$, since an arbitrary complex constant c becomes 0. Here $t+y$ in equation (1.2) instead of $y+t$ is awkward but saves the solution for noncommutative monoids. If the monoid M is commutative or at least $f(0, t+y)=f(0, y+t)$, then Theorem 1.1 remains valid with the equation

$$f(x+t, y)-f(x, y)=-i[f(x, y+t)-f(x, y)] \tag{1.4}$$

instead of equation (1.2). In view of the natural form (1.4) and equation (1.2), J. Aczél has raised the problem of determining the general solution of the difference functional equation (1.4) instead of (1.2) when M is an arbitrary monoid (not necessarily commutative) with the unit element 0.

The aim of this note is to find the general solution of (1.4) for $f\colon M\times M\to C$. In Section 2 we show that the general solution of (1.4) is also given by (1.3). Further, it will be shown in Section 3 that an arbitrary homomorphism ϕ of (1.3) is a unique homomorphism from M to C.

§ 2. The general solution of (1.4)

By splitting f into real and imaginary parts $f(x, y)=u(x, y)+iv(x, y)$ equation (1.4) implies the system

$$u(x+t, y)-u(x, y)=v(x, y+t)-v(x, y) \tag{2.1}$$

$$v(x+t, y)-v(x, y)=u(x, y)-u(x, y+t). \tag{2.2}$$

Let G be a commutative group where the equation $2a=b$ is solvable in a.

Theorem 2.1. *Two functions u, v; $M\times M\to G$ satisfy the system* (2.1) *and* (2.2) *with $u(0, 0)=0$, $v(0, 0)=0$ for all $x, y, t \in M$ if and only if there exist two homomorphisms α, β; $M\to G$ such that*

$$u(x, y)=\beta(x)-\alpha(y) \tag{2.3}$$

$$v(x, y)=\alpha(x)+\beta(y) \tag{2.4}$$

for all $x, y \in M$.

Proof. Set $y=0$ in (2.1) and (2.2) to obtain the equations

$$u(x+t, 0)-u(x, 0)=v(x, t)-v(x, 0), \qquad v(x+t, 0)-v(x, 0)=u(x, 0)-u(x, t).$$

Define new functions $\alpha, \beta\colon M\to G$ by $\alpha(x)=v(x, 0)$ and $\beta(x)=u(x, 0)$ for all $x \in M$. Then the above two equations become

$$v(x, t)=\beta(x+t)-\beta(x)+\alpha(x) \tag{2.5}$$

$$u(x, t)=\beta(x)+\alpha(x)-\alpha(x+t). \tag{2.6}$$

Next, substitute (2.5) and (2.6) back into (2.1) to obtain

$$\beta(x+t)+\alpha(x+t)-\alpha(x+t+y)-\beta(x)-\alpha(x)+\alpha(x+y)=\beta(x+y+t)-\beta(x+y),$$

which, with $x=0$, implies

$$\alpha(t+y)+\beta(y+t)=\alpha(t)+\beta(t)+\alpha(y)+\beta(y). \tag{2.7}$$

Similarly, by substituting (2.5) and (2.6) into (2.2) and setting $x=0$ in the resulting equation, we obtain

$$-\alpha(y+t)+\beta(t+y)=-\alpha(t)-\alpha(y)+\beta(t)+\beta(y). \tag{2.8}$$

Add (2.7) and (2.8) and subtract (2.8) from (2.7) to obtain two equations

$$\alpha(t+y)-\alpha(y+t)+\beta(t+y)+\beta(y+t)=2\beta(t)+2\beta(y) \tag{2.9}$$

$$\alpha(t+y)+\alpha(y+t)-\beta(t+y)+\beta(y+t)=2\alpha(t)+2\alpha(y). \tag{2.10}$$

Further, if we interchange t and y in (2.9), then

$$\alpha(y+t)-a(t+y)+\beta(y+t)+\beta(t+y)=2\beta(y)+2\beta(t). \tag{2.11}$$

Moreover, by subtracting (2.11) from (2.9) we have $2\alpha(t+y)=2\alpha(y+t)$ and $\alpha(t+y)=\alpha(y+t)$ for all $y, t \in M$. By a similar way we also obtain $\beta(t+y)=\beta(y+t)$ for all $y, t \in M$. Therefore, (2.9) and (2.10) imply $\alpha(y+t)=\alpha(y)+\alpha(t)$ and $\beta(y+t)=\beta(y)+\beta(t)$ for all $y, t \in M$. Hence, (2.3) and (2.4) follows from (2.5) and (2.6), since $\alpha, \beta: M \to C$ are homomorphisms.

Conversely, it is clear that functions defined by (2.3) and (2.4) satisfy both equations (2.1) and (2.2). This completes the proof of Theorem 2.1.

We now obtain the general solution of difference functional equation (1.4) as follows.

Theorem 2.2. *A function f: $M \times M \to C$ satisfies the equation*

$$f(x+t, y)-f(x, y)=-i[f(x, y+t)-f(x, y)] \tag{1.4}$$

for all $x, y, t \in M$ with the normalization $f(0, 0)=0$ if and only if there exists a homomorphism ϕ: $M \to C$ such that

$$f(x, y)=\phi(x)+i\phi(y) \tag{1.3}$$

for all $x, y \in M$.

Proof. Let $G=C$. Then it follows from Theorem 2.1 that $f(x, y)=u(x, y)+iv(x, y)=\beta(x)+i\alpha(x)+i(\beta(y)+i\alpha(y))=\phi(x)+i\phi(y)$ for all $x, y \in M$, where a new function ϕ: $M \to C$ is defined by $\phi(x)=\beta(x)+i\alpha(x)$ for all $x \in M$. Thus (1.3) is obtained from (2.3) and (2.4). The converse is clear.

§ 3. Uniqueness of ϕ

Theorem 3.1. *A function f: $M\times M\to C$ satisfies equation* (1.4) *for all $x, y, t\in M$ with $f(0, 0)=0$ if and only if there exists a unique homomorphism ϕ: $M\to C$ such that the representation* (1.3) *holds for all $x, y\in M$.*

Proof. The existence of the form (1.3) immediately follows from the proof of Theorem 2.2. So it only remains to show that ϕ: $M\to C$ of (1.3) is unique. In order to prove the uniqueness, it suffices to show that $\phi(x)+i\phi(y)=\psi(x)+i\psi(y)$ implies $\phi(x)=\psi(x)$ for any homomorphism ψ: $M\to C$ and for all $x\in M$. Now let $\phi(x)=\phi_1(x)+i\phi_2(x)$ and $\psi(y)=\psi_1(y)+i\psi_2(y)$ where ϕ_1, ϕ_2, ψ_1 and ψ_2 are real-valued functions for all x, $y\in M$. Then we have $\phi_1(x)-\phi_2(y)+i[\phi_1(y)+\phi_2(x)]=\psi_1(x)-\psi_2(y)+i[\psi_1(y)+\psi_2(x)]$, which implies

$$\phi_1(x)-\phi_2(y)=\psi_1(x)-\psi_2(y) \tag{3.1}$$

and $\phi_1(y)+\phi_2(x)=\psi_1(y)+\psi_2(x)$ for all $x, y\in M$. If y and x are interchanged in the second equation, then we have

$$\phi_1(x)+\phi_2(y)=\psi_1(x)+\psi_2(y). \tag{3.2}$$

By adding (3.1) and (3.2) we obtain $\phi_1(x)=\psi_1(x)$ for all $x\in M$, while by subtracting (3.2) from (3.1) $\phi_2(y)=\psi_2(y)$ for all $y\in M$. Hence, we have $\phi(x)=\psi(x)$ for all $x\in M$. Conversely, (1.3) always satisfies equation (1.4).

§ 4. Consequences

A consequence of equation (1.4), if M is a group, is that $\phi(x)=[f(x, y)+f(x, -y)]/2$, so that (most) regularity conditions imposed upon $x\to f(x, y_0)$ ($y_0\in M$ fixed) inherit to ϕ. If, for instance, $M=T$ is a topological group, then the homomorphism ϕ: $T\to C$ will be continuous.

Corollary 4.1. *A continuous function f: $T\times T\to C$ satisfies equation* (1.4) *for all $x, y, t\in T$ with $f(0, 0)=0$ if and only if there exists a unique continuous homomorphism ϕ: $T\to C$ such that* (1.3) *holds for all $x, y\in T$.*

Equation (1.4) can also be rewritten in the complex form

$$f(z+t)-f(z)=-i[f(z+it)-f(z)] \tag{4.1}$$

for all $z\in C$ and $t\in R$, where $f(z):=f(x, y)$ and f: $C\to C$. In this case the general solution of equation (4.1) is represented in the complex form $f(z)=\phi(\mathrm{Re}\, z)+i\phi(\mathrm{Im}\, z)$ for all $z\in C$.

Corollary 4.2. *If a continuous function f: $C\to C$ satisfies equation* (4.1) *with $f(0)=0$ for all $z\in C$ and $t\in R$, then f can be extended as an entire function.*

Proof. By Theorem 2.1 two functions α, β: $R \to R$ are continuous homomorphisms, since $f = u + iv$ is continuous. By well-known theorems continuous homomorphisms α and β are C^1 on R. Hence, u and v are also C^1 on $R \times R$. If we differentiate (2.1) and (2.2) with respect to t for $t=0$, then we obtain the Cauchy-Riemann equations. Hence, by $f \in C^1$ in C, f is an entire function.

References

[1] Aczél, J. and Haruki, S., Partial difference equations analogous to the Cauchy-Riemann equations, Funkcial. Ekvac., **24** (1981), 95–102.

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