On Bounded Positive Entire Solutions of Semilinear Elliptic Equations

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§ 1. Introduction

In this paper we consider the elliptic equation

\[ \Delta u + f(x, u, \nabla u) = 0, \quad x \in \mathbb{R}^n \ (n \geq 3), \]

where \( x = (x_1, \ldots, x_n) \), \( \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2 \), \( \nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n) \) and \( f: \mathbb{R}^n \times (0, \infty) \to \mathbb{R} \) is locally H"{o}lder continuous (with exponent \( \theta \in (0, 1) \)). Our objective is to obtain conditions guaranteeing the existence of positive entire solutions of (1) which have prescribed positive limits as \( |x| = (x_1^2 + \cdots + x_n^2)^{1/2} \to \infty \). By an entire solution of (1) we mean a function \( u \in C^2(\mathbb{R}^n) \) which satisfies (1) at every point of \( \mathbb{R}^n \).

Recently Naito [3] has considered the case \( f(x, u, p) = a(x)g(u) \), where \( g: (0, \infty) \to \mathbb{R} \) is positive and nondecreasing and obtained some results under the condition

\[ \int_0^\infty a^*(t) dt < \infty, \quad \text{where} \quad a^*(t) = \max_{|x| = t} |a(x)|. \]

Moreover he has obtained a possible range of the limit values at infinity of such positive entire solutions. On the other hand, Kusano and Oharu [2] have established sufficient conditions for equation (1) to have infinitely many entire solutions which are bounded from above and below by positive constants. In this paper an attempt is made to improve and strengthen considerably both of the above results.

Our method, essentially based on Naito's [3], is to construct explicitly supersolutions and subsolutions having the same prescribed limits at infinity as entire solutions of suitable Poisson equations in \( \mathbb{R}^n \).

§ 2. Main result

With regard to (1) the following conditions are assumed to hold without further mention:

(A) (Nagumo condition) For any bounded domain \( \Omega \subset \mathbb{R}^n \), there exists a continuous function \( \rho_\Omega : (0, \infty) \to [0, \infty) \) such that

\[ |f(x, u, p)| \leq \rho_\Omega(u)(1 + |p|^\theta) \]
for \((x, u, p) \in \Omega \times (0, \infty) \times \mathbb{R}^n\).

(B) There exists a nonnegative function \(F(t, u, v)\) on \([0, \infty) \times (0, \infty) \times [0, \infty)\) which is locally Hölder continuous (with exponent \(\theta\)), monotone in \(u\) and non-decreasing in \(v\), and satisfies

\[
|f(x, u, p)| \leq F(|x|, u, |p|)
\]

for \((x, u, p) \in \mathbb{R}^n \times (0, \infty) \times \mathbb{R}^n\).

In what follows we employ the notation:

\[
d(t, u, v; \lambda) = \frac{F(t, \lambda u, \lambda v)}{\lambda}.
\]

Hypotheses on the nonlinearity of \(F\) are formulated below in terms of \(d(t, u, v; \lambda)\) considered as a function of \(\lambda \in (0, \infty)\) for each fixed \((t, u, v) \in [0, \infty) \times (0, \infty) \times [0, \infty)\).

Our result is the following theorem.

**Theorem.** Suppose that one of the following conditions is satisfied:

(Cₙ) \(F\) is nondecreasing in \(u\), and for any fixed \((t, u, v)\), \(d(t, u, v; \lambda)\) is a non-decreasing function of \(\lambda \in (0, \infty)\) and satisfies \(\lim_{\lambda \to 0} d(t, u, v; \lambda) = 0\);

(Cₚ) \(F\) is nondecreasing in \(u\), and for any fixed \((t, u, v)\), \(d(t, u, v; \lambda)\) is a non-increasing function of \(\lambda \in (0, \infty)\) and satisfies \(\lim_{\lambda \to \infty} d(t, u, v; \lambda) = 0\);

(Cₗ) \(F\) is nonincreasing in \(u\), and for any fixed \((t, u, v)\), \(\lambda d(t, u, v, \lambda)\) is a non-increasing function of \(\lambda \in (0, \infty)\).

Suppose moreover that

\[
(2) \quad \int_1^\infty tF(t, c, c/t)dt < \infty
\]

for some \(c > 0\) in case (Cₙ) or (Cₚ) holds, or

\[
(3) \quad \int_1^\infty tF(t, c, ac/t)dt < \infty
\]

for some \(c > 0\) and \(\sigma > 1\) in case (Cₗ) holds.

Then, for any sufficiently small \(l > 0\) in case (Cₙ) holds, or any sufficiently large \(l > 0\) in case (Cₚ) or (Cₗ) holds, there exists a positive entire solution \(u(x)\) of (1) satisfying

\[
\lim_{|x| \to \infty} u(x) = l.
\]

**Proof.** We use the supersolution-subsolution method (see Noussair and Swanson [4]) which is by now well-known. We put \(k(t) = \max\{1, t\}\) for \(t \geq 0\).

**Case (Cₙ):** From the conditions imposed on \(F(t, u, v)\) it follows that
(4) \[ k(t)F(t, \lambda, \lambda/k(t))/\lambda \leq k(t)F(t, c, c/k(t))/c, \quad t \in [0, \infty), \]

if \(0 < \lambda \leq c\), \(c\) being the constant appearing in (2), and that for \(t \in [0, \infty)\),

\[ k(t)F(t, \lambda, \lambda/k(t))/\lambda \to 0 \quad \text{as} \quad \lambda \to 0. \]

The Lebesgue dominated convergence theorem then shows that

\[ \int_0^\infty k(t)F(t, \lambda, \lambda/k(t))dt/\lambda \to 0 \quad \text{as} \quad \lambda \to 0. \]

Therefore two constants \(\eta, \zeta > 0\) can be chosen small enough so that

\[ \int_0^\infty k(t)F(t, \eta, \eta/k(t))dt \leq \eta, \]

(5)

\[ \int_0^\infty k(t)F(t, \zeta, \zeta/k(t))dt < \zeta, \]

\[ \zeta = \eta - \frac{1}{n-2} \int_0^\infty tF(t, \eta, \eta/k(t))dt. \]

With this choice of \(\eta, \zeta\), we define functions \(z\) and \(y\) on \([0, \infty)\) by

\[ z(t) = \begin{cases} \zeta - t^{2-n} \int_0^t s^{n-3} \left( \int_s^\infty rF(r, \zeta, \zeta/k(r))dr \right)ds & \text{for} \; t > 0, \\ \zeta - \frac{1}{n-2} \int_0^\infty tF(t, \zeta, \zeta/k(t))dt & \text{for} \; t = 0. \end{cases} \]

(6)

\[ y(t) = \begin{cases} \eta - t^{2-n} \int_0^t s^{n-3} \left( \int_0^s rF(r, \eta, \eta/k(r))dr \right)ds & \text{for} \; t > 0, \\ \eta & \text{for} \; t = 0. \end{cases} \]

(7)

It is easily seen that

\[ z(t) \to \zeta \quad \text{as} \; t \to \infty, \]

\[ z'(t) = \int_0^t (s/t)^{n-1} F(s, \zeta, \zeta/k(s))ds \geq 0 \quad \text{for} \; t \geq 0, \]

\[ (t^{n-1}z'(t))' = t^{n-1}F(t, \zeta, \zeta/k(t)) \quad \text{for} \; t \geq 0. \]

From the second equation we have \(z'(0) = 0\), and \(|z'(t)| \leq \zeta/k(t)\) for \(t \geq 0\). In fact, if \(0 \leq t \leq 1\), we have

\[ |z'(t)| \leq \int_0^t F(s, \zeta, \zeta/k(s))ds \leq \int_0^\infty k(s)F(s, \zeta, \zeta/k(s))ds < \zeta, \]

and if \(t \geq 1\), we have
Then the function \( w(x) = z(|x|) \in C^{2+\delta}_loc(R^n) \) satisfies the following inequalities:

\[
0 < \xi - \frac{1}{n-2} \int_0^\infty tF(t, \xi/k(t))dt \leq w(x) \leq \xi,
\]

\[
\Delta w(x) = F(|x|, \xi/k(|x|))
\]

\[
\geq F(|x|, w(x), |\nabla w(x)|) \geq -f(x, w(x), \nabla w(x)), \quad x \in R^n.
\]

Therefore \( w(x) \) is a subsolution of (1) in \( R^n \).

In the same way we can show that \( y(t) \) satisfies the following:

\[
y(t) \rightarrow \eta - \frac{1}{n-2} \int_0^\infty tF(t, \eta/k(t))dt = \zeta \quad \text{as } t \rightarrow \infty,
\]

\[
y'(t) = -\int_0^t (s/t)^{n-1}F(s, \eta, \eta/k(s))ds \leq 0,
\]

\[
|y'(t)| \leq \eta/k(t),
\]

\[
(t^{n-1}y'(t))' = -t^{n-1}F(t, \eta, \eta/k(t)), \quad \text{for } t \geq 0.
\]

So, the function \( v(x) = y(|x|) \in C^{2+\delta}_loc(R^n) \) satisfies the following inequalities:

\[
\zeta \leq v(x) \leq \eta,
\]

\[
\Delta v(x) = -F(|x|, \eta, \eta/k(|x|))
\]

\[
\leq -F(|x|, v(x), |\nabla v(x)|) \leq -f(x, v(x), \nabla v(x)), \quad x \in R^n.
\]

Therefore \( v(x) \) is a supersolution of (1) in \( R^n \). Since \( w(x) \leq u(x) \leq v(x) \) in \( R^n \), applying the supersolution-subsolution method [4, p. 125], we see that (1) has an entire solution \( u(x) \) such that \( w(x) \leq u(x) \leq v(x) \) in \( R^n \). Since \( \lim_{|x| \rightarrow \infty} w(x) = \lim_{|x| \rightarrow \infty} v(x) = \zeta \), we conclude that \( \lim_{|x| \rightarrow \infty} u(x) = \zeta \), as desired.

**Case (C_2):** The conditions imposed on \( F(t, u, v) \) imply that (4) holds for \( \lambda \geq c \), and for each \( t \in [0, \infty) \), \( k(t)F(t, \lambda, \lambda/k(t))/\lambda \rightarrow 0 \) as \( \lambda \rightarrow \infty \). It follows that

\[
\int_0^\infty k(t)F(t, \lambda, \lambda/k(t))dt/\lambda \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty.
\]

We now choose positive constants \( \eta \) and \( \zeta \) so large that (5) is satisfied, and with these \( \eta \) and \( \zeta \), define the functions \( z \) and \( y \) by (6) and (7), respectively. The same argument as in the proof of case \( (C_1) \) yields the existence of an entire solution \( u(x) \) of (1) such that \( \lim_{|x| \rightarrow \infty} u(x) = \zeta \).
Case (C): From the conditions imposed on \( F(t, u, v) \), two constants \( \eta, \zeta > 0 \) can be chosen large enough so that

\[
\zeta - \frac{1}{n-2} \int_0^\infty tF(t, \zeta/\sigma, \zeta/k(t))dt \geq \zeta/\sigma,
\]

\[
\int_0^\infty k(t)F(t, \zeta/\sigma, \zeta/k(t))dt \leq \zeta,
\]

\[
\eta - \frac{1}{n-2} \int_0^\infty tF(t, \zeta/\sigma, \zeta/k(t))dt = \zeta.
\]

With this choice of \( \eta, \zeta \), we define the functions \( z \) and \( y \) by

\[
z(t) = \begin{cases} 
\zeta - t^{2-n} \int_0^t s^{n-3} \left( \int_s^t rF(r, \zeta/\sigma, \zeta/k(r))dr \right)ds & \text{for } t > 0, \\
\zeta - \frac{1}{n-2} \int_0^\infty tF(t, \zeta/\sigma, \zeta/k(t))dt & \text{for } t = 0.
\end{cases}
\]

\[
y(t) = \begin{cases} 
\eta - t^{2-n} \int_0^t s^{n-3} \left( \int_s^t rF(r, \zeta/\sigma, \zeta/k(r))dr \right)ds & \text{for } t > 0, \\
\eta & \text{for } t = 0.
\end{cases}
\]

Arguing as in the preceding cases, we obtain

\[
z(t) \rightarrow \zeta = \eta - \frac{1}{n-2} \int_0^\infty tF(t, \zeta/\sigma, \zeta/k(t))dt \text{ as } t \rightarrow \infty,
\]

\[
z'(t) = \int_0^t (s/t)^{n-1}F(s, \zeta/\sigma, \zeta/k(s))ds \geq 0,
\]

\[
|z'(t)| \leq \zeta/k(t),
\]

\[
(t^{n-1}z'(t))' = t^{n-1}F(t, \zeta/\sigma, \zeta/k(t)),
\]

\[
y(t) \rightarrow \zeta = \eta - \frac{1}{n-2} \int_0^\infty tF(t, \zeta/\sigma, \zeta/k(t))dt \text{ as } t \rightarrow \infty,
\]

\[
y'(t) = -\int_0^t (s/t)^{n-1}F(s, \zeta/\sigma, \zeta/k(s))ds \leq 0,
\]

\[
|y'(t)| \leq \zeta/k(t),
\]

\[
(t^{n-1}y'(t))' = -t^{n-1}F(t, \zeta/\sigma, \zeta/k(t)),
\]

Therefore the functions \( v(x) = y(|x|) \), \( w(x) = z(|x|) \) satisfy the following relations:

\[
0 < \zeta - \frac{1}{n-2} \int_0^\infty tF(t, \zeta/\sigma, \zeta/k(t))dt \leq w(x) \leq \zeta \leq v(x) \leq \eta,
\]

\[
\Delta w(x) \geq F(|x|, w(x), |\nabla w(x)|) \geq -f(x, w(x), \nabla w(x)),
\]

\[
\Delta v(x) \leq -F(|x|, v(x), |\nabla v(x)|) \leq -f(x, v(x), \nabla v(x)), \quad x \in \mathbb{R}^n,
\]

\[
\text{for } t \geq 0.
\]
lim w(x) = lim v(x) = \zeta.

Thus we can conclude that there exists a positive entire solution \( u(x) \) of (1) satisfying \( \lim_{|x| \to \infty} u(x) = \zeta \). This completes the proof.

**Remark.** Consider the equation

\[ (8) \quad \Delta u + a(x) \phi(u) = 0, \quad x \in \mathbb{R}^n, \]

where \( a(x) \phi(u) \) is locally Hölder continuous, and \( \phi : (0, \infty) \to (0, \infty) \) is nonincreasing. Let \( a^*(t) = \max_{|x| = t} |a(x)| \), and suppose that

\[ \int_0^\infty t a^*(t) dt = A^* < \infty. \]

In this case, the situation is simpler. We can choose positive constants \( \xi, \eta, \zeta \) so large that

\[ (9) \quad \eta - \frac{A^*}{n-2} \phi(\zeta) = \xi, \quad \zeta - \frac{A^*}{n-2} \phi(\xi) \geq \zeta. \]

With this choice of \( \xi, \eta, \zeta \), we define the functions \( y \) and \( z \) by

\[
y(t) = \begin{cases} \eta - \phi(\zeta) t^{2-n} \int_0^t s^{n-3} \left( \int_0^s r a^*(r) dr \right) ds & \text{for } t > 0, \\
\eta & \text{for } t = 0. \end{cases}
\]

\[
z(t) = \begin{cases} \zeta - \phi(\zeta) t^{2-n} \int_0^t s^{n-3} \left( \int_0^s r a^*(r) dr \right) ds & \text{for } t > 0, \\
\zeta - \frac{A^*}{n-2} \phi(\xi) & \text{for } t = 0. \end{cases}
\]

In the same manner as in the proof of the theorem, we can conclude that there exists a positive entire solution \( u(x) \) of (8) satisfying \( u(x) \to \zeta \) as \( |x| \to \infty \). If in particular \( \phi(u) = u^{-\gamma} \) (\( \gamma > 0 \)), then from (9) we see that (8) has a positive entire solution \( u(x) \) having the prescribed limit \( \zeta \) at infinity provided \( \zeta \geq \frac{\gamma + 1}{\gamma} (\gamma A^* / (n-2))^{1/(1+\gamma)}. \)

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References


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