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The Variation of Constants Formula and Periodicity for Linear Neutral Integro-differential Equations

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Consider the linear neutral integro-differential equations

$$\frac{d}{dt}\left(x(t)-\int_0^t D(t-s)x(s)ds\right)+D(t)x(0) =Ax(t)+\int_0^t C(t-s)x(s)ds+F(t), \tag{1}$$

$$\frac{d}{dt}\left(Z(t)-\int_0^t D(t-s)Z(s)ds\right)+D(t) =AZ(t)+\int_0^t C(t-s)Z(s)ds, \qquad Z(0)=I, \tag{2}$$

and

$$\frac{d}{dt}\left(y(t)-\int_{-\infty}^t D(t-s)y(s)ds\right) =Ay(t)+\int_{-\infty}^t C(t-s)y(s)ds+F(t), \tag{3}$$

where x, y are n-vectors, A is an $n\times n$ constant matrix, C, D are $n\times n$ matrices of continuous functions, and F: $(-\infty, \infty)\to \boldsymbol{R}^n$ is continuous.

According to [4], the solution of (2) exists and is unique, and for any $t_0\geqslant 0$, any continuous function ϕ: $[0, t_0]\to \boldsymbol{R}^n$, the solution $x(t; t_0, \phi)$ of (1) through (t_0, ϕ) exists and is unique. Moreover, if ϕ: $(-\infty, t_0]\to R^n$ is continuous and bounded, C and $D\in L^1(0, \infty)$, then the solution $y(t; t_0, \phi)$ of (3) through (t_0, ϕ) exists and is unique.

Theorem 1. *If* $x(t)=x(t; 0, x(0))$ *is the solution of* (1) *through* $(0, x(0))$, *then*

$$x(t)=Z(t)x(0)+\int_0^t Z(t-s)g(s)ds, \tag{4}$$

where $Z(t)$ *is the unique solution of* (2) *and* $g(t)$ *is the unique solution of*

$$g(t)=F(t)+\int_0^t D(t-s)g(s)ds. \tag{5}$$

Proof. Integrating the equation (2) from 0 to t, we get

$$Z(t)=I-\int_0^t D(u)du+\int_0^t D(t-s)Z(s)ds+A\int_0^t Z(u)du + \int_0^t du\int_0^u C(u-s)Z(s)ds. \tag{6}$$

For $x(t)=Z(t)x(0)+\int_0^t Z(t-s)g(s)ds$, we have by (6)

$$\begin{aligned}
& x(0)+\int_0^t D(t-s)x(s)\,ds-\int_0^t D(u)x(0)\,du+A\int_0^t x(u)du \\
&\quad +\int_0^t du\int_0^u C(u-s)X(s)\,ds+\int_0^t F(u)du \\
&=x(0)+\int_0^t F(u)du-\int_0^t D(u)x(0)\,du+\int_0^t D(t-s)Z(s)x(0)ds \\
&\quad +\int_0^t D(t-s)\int_0^s Z(s-u)g(u)duds+\int_0^t AZ(u)x(0)du \\
&\quad +\int_0^t Ads\int_0^s Z(s-u)g(u)\,du+\int_0^t du\int_0^u C(u-s)Z(s)x(0)ds \\
&\quad +\int_0^t du\int_0^u C(u-s)ds\int_0^s Z(s-v)g(v)dv \\
&=\Big(I-\int_0^t D(u)\,du+\int_0^t D(t-s)Z(s)\,ds+\int_0^t AZ(s)ds \\
&\quad +\int_0^t du\int_0^u C(u-s)Z(s)\,ds\Big)x(0)+\int_0^t F(u)du \\
&\quad +\int_0^t D(t-s)ds\int_0^s Z(s-u)g(u)\,du+A\int_0^t ds\int_0^s Z(s-u)g(u)\,du \\
&\quad +\int_0^t du\int_0^u C(u-s)ds\int_0^s Z(s-v)g(v)\,dv \\
&=Z(t)x(0)+\int_0^t F(u)du+\int_0^t\Big(\int_u^t D(t-s)Z(s-u)\,ds\Big)g(u)du \\
&\quad +A\int_0^t\Big(\int_u^t Z(s-u)ds\Big)g(u)du \\
&\quad +\int_0^t\Big(\int_v^t du\int_0^{u-v} C(u-s-v)Z(s)ds\Big)g(v)dv \\
&=Z(t)x(0)+\int_0^t\Big(g(u)-\int_0^u D(u-s)g(s)ds\Big)du \\
&\quad +\int_0^t\Big(\int_0^{t-u} D(t-u-w)Z(w)dw+\int_0^{t-u} AZ(w)\,dw\Big)g(u)du \\
&\quad +\int_0^t\Big(\int_0^{t-v}\Big(\int_0^w C(w-s)Z(s)\,ds\Big)dw\Big)g(v)dv \\
&=Z(t)x(0)+\int_0^t Z(t-u)g(u)du=x(t).
\end{aligned}$$

Then, $x(t)=Z(t)x(0)+\int_0^t Z(t-s)g(s)ds$ is the unique solution of (1) through $(0, x(0))$, the proof is completed.

Now we shall use Theorem 1 to discuss the existence of periodic solutions of (3). We always assume $F(t+T)=F(t)$.

Theorem 2. *If $C, D \in L^1(0, \infty)$, $x(t)=x(t; 0, x(0))$ is a bounded solution of* (1) *on* $[0, \infty)$, *then there exists an integer sequence $n_j \to \infty$ (as $j \to \infty$) such that $x(t+n_jT)$ converges to a solution of* (3) *on* $(-\infty, \infty)$ *and the convergence is uniform on compact subsets of* $(-\infty, \infty)$.

Proof. For $t_2 \geqslant t_1 \geqslant -nT$,

$$
\begin{aligned}
(7)\quad & x(t_2+nT)-x(t_1+nT) \\
&=\int_0^{t_2+nT} D(t_2+nT-s)x(s)\,ds-\int_0^{t_1+nT} D(t_1+nT-s)x(s)ds \\
&\quad -\int_{t_1+nT}^{t_2+nT} D(s)x(0)ds+\int_{t_1+nT}^{t_2+nT} Ax(s)ds \\
&\quad +\int_{t_1+nT}^{t_2+nT} du\int_0^u C(u-s)x(s)\,ds+\int_{t_1+nT}^{t_2+nT} F(s)ds \\
&=\int_{-nT}^{t_1} (D(t_2-s)-D(t_1-s))x(s+nT)ds+\int_{t_1}^{t_2} D(t_2-s)x(s+nT)\,ds \\
&\quad -\int_{t_1+nT}^{t_2+nT} D(s)x(0)\,ds+\int_{t_1}^{t_2} Ax(nT+s)ds \\
&\quad +\int_{t_1}^{t_2} dv\int_{-nT}^{v} C(v-s)x(s+nT)\,ds+\int_{t_1}^{t_2} F(u)du.
\end{aligned}
$$

Let $|x(t)|\leqslant M$, $|F(t)|\leqslant M$ for $t\geqslant 0$, then

$$
\begin{aligned}
&|x(t_2+nT)-x(t_1+nT)| \\
&\leqslant M\int_{-\infty}^{t_1} |D(t_2-s)-D(t_1-s)|\,ds+M\int_{t_1}^{t_2} |D(t_2-s)|\,ds+M(t_2-t_1) \\
&\quad +M\int_{t_1+nT}^{t_2+nT} |D(v)|\,dv+|A|M(t_2-t_1)+M\int_{t_1}^{t_2} dv\int_{-\infty}^{v} |C(v-s)|\,ds \\
&\leqslant M\int_{t_1+nT}^{t_2+nT} |D(v)|\,dv+M\int_0^\infty |D(t_2-t_1+s)-D(s)|\,ds \\
&\quad +M\int_{t_1}^{t_2} |D(t_2-s)|\,ds+(|A|+1+\int_0^\infty |C(s)|\,ds)(t_2-t_1)M.
\end{aligned}
$$

Since $D \in L^1[0, \infty)$, for any $\varepsilon>0$, there exist $\delta>0$, $K>0$ such that

(i) $\int_t^\infty |D(s)|\,ds<\varepsilon$, for $t\geqslant K$,

(ii) for any measurable subset J of $[0, \infty)$,

$$\int_J |D(v)|\,dv < \varepsilon$$

provided mes $J \leqslant \delta$.

Thus, if $0 \leqslant t_2 - t_1 \leqslant \delta$, then

$$|x(t_2+nT)-x(t_1+nT)| \leqslant M\varepsilon + M\int_0^K |D(t_2-t_1+s)-D(s)|\,ds + 2M\varepsilon + M\varepsilon + M\left(|A|+1+\int_0^\infty |C(s)|\,ds\right)(t_2-t_1).$$

For any $\varepsilon>0$, by the continuity of D, there exists a $\delta_1>0$ such that for $s \in [0, K]$, $0 \leqslant t_2-t_1 \leqslant \delta_1$,

$$|D(t_2-t_1+s)-D(s)| < \varepsilon/K.$$

Then, if $0 \leqslant t_2-t_1 \leqslant \min(\delta, \delta_1)$, then

$$\begin{aligned}|x(t_2+nT)-x(t_1+nT)| &\leqslant M\varepsilon+M\varepsilon+2M\varepsilon+M\varepsilon+M\left(1+|A|+\int_0^\infty |C(s)|\,ds\right)(t_2-t_1)\\ &\leqslant 5M\varepsilon+M\left(1+|A|+\int_0^\infty |C(s)|\,ds\right)(t_2-t_1).\end{aligned}$$

This implies $x(t+nT)$ are equicontinuous and uniformly bounded on $(-\infty, \infty)$, so there exists a positive integer sequence $n_j \to \infty$ (as $j \to \infty$) such that $x(t+n_jT)$ converges to some continuous function $z(t)$ and the convergence is uniform on any compact subsets of $(-\infty, \infty)$. Obviously, $z(t)$ is bounded.

Taking $t_2=t$ and $t_1=0$ in (7), we get

$$\begin{aligned}x(t+n_jT) &= x(n_jT)+A\int_0^t x(s+n_jT)\,ds+\int_0^t F(s)\,ds\\ &\quad+\int_0^t dv\int_{-n_jT}^v C(v-s)x(s+n_jT)\,ds\\ &\quad+\int_0^t D(t-s)x(s+n_jT)\,ds-\int_{n_jT}^{t+n_jT} D(v)x(0)\,dv\\ &\quad+\int_{-n_jT}^0 (D(t-s)-D(-s))x(s+n_jT)\,ds.\end{aligned} \tag{8}$$

By Lebesgue dominated convergence theorem, letting $n_j \to \infty$, in (8), we get

$$\begin{aligned}z(t) &= z(0)+A\int_0^t z(s)\,ds+\int_0^t F(s)\,ds+\int_0^t dv\int_{-\infty}^v C(v-s)z(s)\,ds\\ &\quad+\int_0^t D(t-s)z(s)\,ds+\int_{-\infty}^0 (D(t-s)-D(-s))z(s)\,ds\end{aligned}$$

$$=z(0)+A\int_0^t z(s)ds+\int_0^t F(s)ds+\int_0^t dv\int_{-\infty}^v C(v-s)z(s)ds$$
$$+\int_{-\infty}^t D(t-s)z(s)ds-\int_{-\infty}^0 D(-s)z(s)ds,$$

and then

$$\frac{d}{dt}\Big(z(t)-\int_{-\infty}^t D(t-s)z(s)ds\Big)=Az(t)+\int_{-\infty}^t C(t-s)z(s)ds+F(t).$$

This shows that $z(t)$ is a bounded solution of (3) and the proof is completed.

We now consider the related Volterra integral equations

$$g(t)=\int_0^t D(t-s)g(s)ds+F(t), \tag{9}$$

$$H(t)=I+\int_0^t D(t-s)H(s)ds, \qquad H\in R^{n\times n}, \tag{10}$$

$$g(t)=\int_{-\infty}^t D(t-s)g(s)ds+F(t). \tag{11}$$

Theorem 3. *If F, D are continuous, then*

(i) *There is one and only one solution $g(t)$ of (9) on $[0, \infty)$.*

(ii) *There is one and only one solution $H(t)$ of (10) on $[0, \infty)$.*

(iii) *The unique solution of (9) is*

$$g(t)=H(t)F(0)+\int_0^t H(t-s)F'(s)ds \tag{12}$$

provided that F' is continuous.

(iv) *If H' is continuous, then the solution of (9) is*

$$g(t)=F(t)+\int_0^t H'(t-s)F(s)ds. \tag{13}$$

Henceforth, we always assume that $F(t+T)=F(t)$ for some $T>0$.

(v) *If $g(t)$ is a solution of (9) bounded on $[0, \infty)$ with $D\in L^1[0, \infty)$, then there is a sequence of positive integers $\{n_j\}$ such that $\{g(t+n_jT)\}$ converges to a solution of (11) and the convergence is uniform on compact subsets of $(-\infty, \infty)$.*

(vi) *If H' and $D\in L^1[0, \infty)$, then the solution $g(t)$ of (9) is bounded on $[0, \infty)$ and*

$$g(t+n_jT)\longrightarrow\int_{-\infty}^t H'(t-s)F(s)ds+F(t)=g^*(t),$$

a periodic solution of (11).

(vii) *If F' is continuous with $D \in L^1 [0, \infty)$, and if there is a constant matrix K such that $H(t)-K \in L^1 [0, \infty)$ with $H(t)-K \to 0$ as $t \to \infty$, then the solution $g(t)$ of* (9) *is bounded on* $[0, \infty)$ *and*

$$g(t+n_jT) \longrightarrow \int_{-\infty}^{t} (H(t-s)-K)F'(s)ds + KF(t) = g^*(t),$$

a T-periodic solution of (11).

Proof. From [1] (see also [2] or [3]), we need only to prove the last assertion. By (12),

$$\begin{aligned} g(t) &= H(t)F(0) + \int_0^t H(t-s)F'(s)ds \\ &= H(t)F(0) + \int_0^t (H(t-s)-K)F'(s)ds + K\int_0^t F'(s)ds \\ &= (H(t)-K)F(0) + \int_0^t (H(t-s)-K)F'(s)ds + KF(t), \end{aligned}$$

and we get that $g(t)$ is bounded on $[0, \infty)$ under our assumptions. Then we have

$$\begin{aligned} g(t+n_jT) &= (H(t+n_jT)-K)F(0) \\ &\quad + \int_0^{t+n_jT} (H(t+n_jT-s)-K)F'(s)ds + KF(t+n_jT) \\ &= (H(t+n_jT)-K)F(0) + \int_{-n_jT}^{t} (H(t-s)-K)F'(s)ds + KF(t) \\ &\longrightarrow \int_{-\infty}^{t} (H(t-s)-K)F'(s)ds + KF(t) = g^*(t), \end{aligned}$$

a solution of (11). It is easy to see that

(14) $$g^*(t) = \int_0^{\infty} (H(v)-K)F'(t-v)dv + KF(t).$$

So, $g^*(t)$ is T-periodic since F' is T-periodic.

Example 1. Consider the scalar integral equations

(15) $$g(t) = \cos t + \int_0^t e^{-2(t-s)} g(s)ds,$$

(16) $$H(t) = 1 + \int_0^t e^{-2(t-s)} H(s)ds$$

and

(17) $$g(t) = \cos t + \int_{-\infty}^{t} e^{-2(t-s)} g(s)ds.$$

Differentiating (16) we have

$$H'(t)=H(t)+\int_0^t (-2)e^{-2(t-s)}H(s)ds,$$
$$H'(t)+H(t)=2,$$
$$H(t)=2+ce^{-t}.$$

But $H(0)=1$, we have $c=-1$. Therefore,

$$H(t)=2-e^{-t}$$

which is the unique solution of (16). For,

$$D(t)=e^{-2t}\in L^1[0,\infty),$$
$$H(t)-2=-e^{-t}\in L^1[0,\infty),\quad H(t)-2\longrightarrow 0 \quad \text{as} \quad t\longrightarrow\infty,$$

and $F(t)=\cos t$ is periodic, we see that all the conditions of Theorem 3 (vii) hold. Then (17) has a periodic solution

$$\begin{aligned} g^*(t)&=2F(t)+\int_{-\infty}^t (H(t-s)-2)F'(s)ds \\ &=2\cos t+\int_{-\infty}^t (-e^{-(t-s)})(-\sin s)ds \\ &=(3\cos t+\sin t)/2. \end{aligned}$$

On the other hand, $H'(t)=e^{-t}\in L^1\,[0,\infty)$ and $D(t)\in L^1\,[0,\infty)$ imply that

$$\begin{aligned} g^*(t)&=F(t)+\int_{-\infty}^t H'(t-s)F(s)ds \\ &=\cos t+\int_{-\infty}^t e^{-(t-s)}(\cos s)ds \\ &=(3\cos t+\sin t)/2, \end{aligned}$$

which is the same as before.

Finally, by (13), the unique solution $g(t)$ of (15) is

$$\begin{aligned} g(t)&=F(t)+\int_0^t H'(t-s)F(s)ds \\ &=\cos t+\int_0^t e^{-(t-s)}(\cos s)ds \\ &=(3\cos t+\sin t-e^{-t})/2. \end{aligned}$$

Then we have

$$\begin{aligned} g(t+2n_j\pi)&=(3\cos t+\sin t-e^{-(t+2n_j\pi)})/2 \\ &\longrightarrow(3\cos t+\sin t)/2=g^*(t) \quad \text{as} \quad j\longrightarrow\infty. \end{aligned}$$

Theorem 4. *Suppose that C, D, Z and* $H' \in L^1[0, \infty)$, *and that* $F(t)$ *is T-periodic. Then* (3) *has a T-periodic solution*

$$y^*(t)=\int_{-\infty}^{t} Z(t-s)g^*(s)ds \tag{18}$$

where

$$g^*(t)=\int_{-\infty}^{t} H'(t-s)F(s)ds+F(t) \tag{19}$$

is a T-periodic solution of (11).

Proof. By Theorem 3 (iv), we know that

$$g(t)=F(t)+\int_0^t H'(t-s)F(s)ds,$$

and so $g(t)$ is bounded since $H' \in L^1[0, \infty)$ and F is bounded. By Theorem 3 (v), there is a sequence of positive integers $\{n_j\}$ such that $\{g(t+n_jT)\}$ uniformly converges to a T-periodic solution $g^*(t)$ of (11) and

$$g^*(t)=\int_{-\infty}^{t} H'(t-s)F(s)ds+F(t).$$

On the other hand, let $x(t)=x(t; 0, 0)$ be the solution of (1) through (0, 0). Then we have by Theorem 1

$$x(t)=\int_0^t Z(t-s)g(s)ds,$$

which is bounded on $[0, \infty)$, and then

$$x(t+n_jT)=\int_0^{t+n_jT} Z(t+n_jT-s)g(s)ds=\int_{-n_jT}^{t} Z(t-v)g(v+n_jT)dv.$$

By Lebesque dominated convergence theorem, we get

$$x(t+n_jT) \longrightarrow \int_{-\infty}^{t} Z(t-v)g^*(v)dv=y^*(t)$$

which is a solution of (3) by Theorem 2. It is easy to see that

$$\int_{-\infty}^{t} Z(t-s)g^*(s)ds=\int_0^{\infty} Z(v)g^*(t-v)dv$$

which is T-periodic since $g^*(t)$ is T-periodic. The proof is completed.

Theorem 5. *Suppose that C, D and* $Z \in L^1[0, \infty)$, *that F is T-periodic with* F'

continuous, and that there is a constant matrix K *such that* $H(t)-K \in L^1[0, \infty)$ *with* $H(t)-K \to 0$ *as* $t \to \infty$. *Then* (3) *has a T-periodic solution*

$$y^*(t)=\int_{-\infty}^{t} Z(t-s)g^*(s)ds,$$

where

$$g^*(t)=\int_{-\infty}^{t} (H(t-s)-K)F'(s)ds+KF(t), \tag{20}$$

a T-periodic solution of (11).

The proof of this theorem is very similar to that of Theorem 4 and therefore is omitted.

Example 2. Consider the following scalar linear neutral integrodifferential equations

$$Z'(t)-\int_{0}^{t} e^{-2(t-s)}Z'(s)ds=-Z(t)+\int_{0}^{t} e^{-2(t-s)}Z(s)ds, \qquad Z(0)=1, \tag{21}$$

$$x'(t)-\int_{0}^{t} e^{-2(t-s)}x'(s)ds=-x(t)+\int_{0}^{t} e^{-2(t-s)}x(s)ds+\cos t, \tag{22}$$

and

$$y'(t)-\int_{-\infty}^{t} e^{-2(t-s)}y'(s)ds=-y(t)+\int_{-\infty}^{t} e^{-2(t-s)}y(s)ds+\cos t. \tag{23}$$

Notes that these equations can be reduced to the form of (1)–(3) with $C(t)=D(t)=e^{-2t}$ and $F(t)=\cos t$, and so one can use Theorem 4 or 5 to discuss the existence of the periodic solution of (23) directly.

It is easy to see that $Z(t)=e^{-t}$ is the unique solution of (21) with $Z(t) \in L^1[0, \infty)$.

The relevant Volterra integral equations are the same as those in Example 1, we have

$$C(t)=D(t)=e^{-2t} \in L^1[0, \infty),$$
$$H(t)=2-e^{-t}, \qquad H'(t)=e^{-t} \in L^1[0, \infty),$$
$$H(t)-2=-e^{-t} \in L^1[0, \infty), \quad H(t)-2 \to 0 \quad \text{as} \quad t \to \infty.$$

Thus, all the conditions of Theorem 4 and 5 are satisfied. Then (23) has a periodic solution

$$y^*(t)=\int_{-\infty}^{t} Z(t-s)g^*(s)ds,$$

where $g^*(t)=(3\cos t \sin t)/2$ by Example 1. Therefore,

$$y^*(t)=\int_{-\infty}^{t} e^{-(t-s)}((3\cos s+\sin s)/2)ds$$
$$=(\cos t+2\sin t)/2.$$

It is easy to verify directly that $y^*(t)$ does satisfy (23).

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