On the Dirichlet Problem with $L^1$-Boundary Data

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Introduction

This paper deals with the Dirichlet problem for the elliptic equation

\begin{equation}
Lu + \lambda u = - \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju) + \sum_{i=1}^{n} b_i(x)D_iu + (c(x) + \lambda)u = 0
\end{equation}

in $Q$,

\begin{equation}
u(x) = \phi(x) \quad \text{on} \quad \partial Q,
\end{equation}

where $Q$ is a bounded domain in $\mathbb{R}^n$, $\partial Q$ denotes its boundary and $\lambda$ is a real parameter. In recent years the Dirichlet problem with the boundary data in $L^p(\partial Q)$, $p > 1$, has been investigated by several authors. In particular V. P. Mikhailov [6], Chabrowski and Thompson [2] established the existence of a solution of (1), (2) with $\phi \in L^4(\partial Q)$. This result has been recently extended to $\phi \in L^p(\partial Q)$, $p > 1$, by Yu. A. Mikhailov [8]. The main purpose of this article is to prove the existence of a solution of (1), (2) with $\phi \in L^1(\partial Q)$.

The paper is organized as follows. The first two sections contain preliminary work. The main results here are lemmas 1 and 3. In particular, the results of these sections suggest the formulation of the Dirichlet problem adopted in this work. Section 3 is devoted to the discussion of the existence of a solution of the Dirichlet problem with $L^1$-boundary data. In section 4 we extend the above results to the Dirichlet problem for a non-homogeneous equation.

§ 1. Preliminaries

Let $Q$ be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial Q$ of class $C^2$. Let $x \in Q$, and let $r(x)$ denote the distance from $x$ to the boundary $\partial Q$.

Throughout this article we make the following assumptions

(A) There exists a positive constant $\gamma$ such that

$$\gamma^{-1} |\xi|^p \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \leq \gamma |\xi|^p$$
for all \( x \in Q \) and \( \zeta \in R_n \). Moreover the coefficients \( a_{ij} \) are of class \( C^1(Q) \), \( a_{ij} = a_{ji} \) \((i, j = 1, \ldots, n)\).

(B) \( c, b_i \) and \( D_ib_i \) belong to \( L^\infty(Q) \) \((i = 1, \ldots, n)\).

A function \( u(x) \) is said to be a weak (generalized) solution of \((1)\) if \( u \in W^{1,2}_{\text{loc}}(Q) \) and \( u \) satisfies

\[
(3) \quad \int_Q \left[ \sum_{i,j=1}^n a_{ij}(x)D_iuD_jv + \sum_{i=1}^n b_i(x)D_iv + (c(x) + \lambda)uv \right] \, dx = 0
\]

for every \( v \in W^{1,2}(Q) \) with compact support in \( Q \). Here \( W^{1,2}_{\text{loc}}(Q) \) and \( W^{1,4}(Q) \) denote the Sobolev spaces of functions on \( Q \) (for definition of these spaces, see [3]).

It follows from the regularity of the boundary \( \partial Q \) that there is a number \( \delta_0 > 0 \) such that for \( \delta \in (0, \delta_0] \) the domain \( Q_\delta = Q \cap \{ \, x; \, \min_{y \in \partial Q} |x-y| > \delta \, \} \), with the boundary \( \partial Q_\delta \), possesses the following property: to each \( x_0 \in \partial Q \) there is a unique point \( x_\delta(x_0) \in \partial Q_\delta \) such that \( x_\delta(x_0) = x_0 - \delta \nu(x_0) \), where \( \nu(x_0) \) is the outward normal to \( \partial Q \) at \( x_0 \). The above relation gives a one-to-one mapping, of class \( C^1 \), of \( \partial Q \) onto \( \partial Q_\delta \). The inverse mapping of \( x_0 \rightarrow x_\delta(x_0) \) is given by the formula \( x_\delta(x_0) = x_0 - \delta \nu(x_\delta) \), where \( \nu(x_\delta) \) is the outward normal to \( \partial Q_\delta \) at \( x_\delta \).

Let \( x_\delta \) denote an arbitrary point of \( \partial Q_\delta \). For fixed \( \delta \in (0, \delta_0] \) let

\[
A_\delta = \partial Q_\delta \cap \{ \, x; \, |x-x_\delta| < \delta \, \}, \\
B_\delta = \{ x; \, x = x_\delta + \delta \nu(x_\delta), \, x_\delta \in A_\delta \},
\]

and

\[
\frac{dS_\delta}{dS_0} = \lim_{\varepsilon \to 0} \frac{|A_\delta|}{|B_\delta|},
\]

where \( |A| \) denote the \( n-1 \) dimensional Hausdorff measure of a set \( A \). V.P. Mikhailov [6] proved that there is a positive number \( \gamma_0 \) such that

\[
(4) \quad \gamma_0^{-2} \leq \frac{dS_\delta}{dS_0} \leq \gamma_0
\]

and

\[
(5) \quad \lim_{\varepsilon \to 0} \frac{dS_\delta}{dS_0} = 1
\]

uniformly with respect to \( x_\delta \in \partial Q_\delta \).

According to Lemma 1 in [3] p. 382, the distance \( r(x) \) belongs to \( C^2(\overline{Q} - Q_\delta) \) if \( \delta_0 \) is sufficiently small. Denote by \( \rho(x) \) the extension of the function \( r(x) \) in \( \overline{Q} \) satisfying the following properties:

\[
\rho(x) = r(x) \quad \text{for} \quad x \in \overline{Q} - Q_\delta, \quad \rho \in C^2(\overline{Q}), \quad \rho(x) \geq \frac{3\delta_0}{4} \quad \text{in} \quad Q_\delta,
\]
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$t_i r(x) \leq \rho(x) \leq t_j r(x)$ in $Q$ for some positive constant $t_i$, $\partial Q_\delta = \{x; \rho(x) = \delta\}$ for $\delta \in (0, \delta_0]$ and finally $\partial Q = \{x; \rho(x) = 0\}$.

In view of Theorem 6 in [2] there exists a positive constant $\lambda_0$ such that for every $\phi \in L^2(\partial Q)$ and $\lambda \geq \lambda_0$ the Dirichlet problem (1), (2) has a unique solution $u \in W_{1,2}^1(Q)$ such that

$$\int_Q |Du(x)|^2 r(x)dx + \int_Q u(x)^2 dx + \sup_{0 < \delta < d} \int_{\partial Q_\delta} u(x)^2 dS_x$$

$$\leq C \int_{\partial Q} \phi(x)^2 dS_x,$$

where $C$ and $d$ are positive constants. Here the boundary condition is understood in the sense of $L^2$-convergence, i.e.,

$$\lim_{\delta \to 0} \int_{\partial Q} [u(x, \delta(x)) - \phi(x)]^2 dS_x = 0.$$

Fix $x \in Q$ and let $B(x, r)$ be a ball centered at $x$, with radius $r$, contained in $Q$. By Corollary 5.2 in [9] there is a positive constant $K$ such that

$$|u(y)| \leq K \left\{ r^{-n} \int_{B(x, r)} u(z)^2 dz \right\}^{1/2},$$

for $y \in B(x, r/2)$. Consequently by the Riesz representation theorem of a linear continuous functional on $L^2(\partial Q)$ we have

$$u(x) = \int_{\partial Q} K_i(x, y) \phi(y) dS_y,$$

where $K_i(x, \cdot) \in L^2(\partial Q)$. Since we may assume that $c(x) + \lambda_0 \geq 0$ on $Q$, the maximum principle yields $K_i(x, y) \geq 0$ for a.e. $x \in Q$ and $y \in \partial Q$. It follows from Fubini’s theorem that

$$\int_Q K_i(x, y) \left[ - \sum_{i,j=1}^n D_i(a_{ij}(x) D_j \psi) - \sum_{i=1}^n D_i(b_i(x) \psi) + (c(x) + \lambda) \psi \right] dx = 0,$$

for all $\psi \in C^2_\partial(Q)$. Consequently, by Theorem 9.2 in [9], for a.e. $y \in \partial Q$, $K_i(\cdot, y)$ is a weak solution of (1) belonging to $W_{1,2}^1(Q)$. Now, by Harnack’s inequality, (see [9]), for every compact subset $K \subset Q$ there exists a positive constant $M(K)$ such that

$$K_i(x, y) \leq M(K)$$

for a.e. $x \in K$ and $y \in \partial Q$.

§ 2. Energy estimate and traces

We begin by establishing an estimate involving a norm of $\phi$ in $L^1(\partial Q)$ for a solution of (1), (2) with $\phi \in L^2(\partial Q)$.
Lemma 1. Let $\phi \in L^2(\partial Q)$. Then there exist positive constants $\lambda_1$ and $d$ such that for $\lambda \geq \lambda_1$ the problem (1), (2) admits a unique solution $u \in W^{1,2}(Q)$ satisfying the estimate

\begin{equation}
\int_Q |Du|^2(u^2+1)^{-3/2}\rho dx + \lambda \int_Q |u|\rho dx + \sup_{0 < \delta < d} \int_{\partial Q_\delta} |u|dS_x \leq C \left[ \int_{\partial Q} |\phi|dS_x + \lambda \int_Q (u^2+1)^{-1/2}dx + 1 \right],
\end{equation}

where $C$ is a positive constant independent of $u$.

Proof. As we mentioned above there exists a unique solution $u$ of (1), (2) in $W^{1,2}(Q)$ provided $\lambda$ is sufficiently large, with the boundary condition understood in the sense of (7).

Let

$$v(x) = \begin{cases} u(x)(u^2+1)^{-1/2} \rho(x) & \text{for } x \in Q_\delta, \\ 0 & \text{for } x \in Q - Q_\delta, \end{cases}$$

where $0 < \delta < \delta_0$. It is clear that $v$ is an admissible test function in (3) and consequently we obtain

\begin{equation}
\int_{Q_\delta} \sum_{i,j=1}^n a_{ij}D_iD_ju(u^2+1)^{-1/2}(\rho - \delta)dx + \int_{Q_\delta} \sum_{i,j=1}^n \sum_{i=1}^n b_{ij}D_iD_ju(u^2+1)^{-1/2}(\rho - \delta)dx = 0.
\end{equation}

Now observe that the sum of the first two integrals is equal to

$$\int_{Q_\delta} \sum_{i,j=1}^n a_{ij}D_iD_ju(u^2+1)^{-1/2}(\rho - \delta)dx.$$

Hence applying (A) and the Green formula we arrive at the inequality

\begin{equation}
\gamma^{-1} \int_{Q_\delta} |Du|^2(u^2+1)^{-3/2}(\rho - \delta)dx + \int_{Q_\delta} (c + \lambda)u^2(u^2+1)^{-1/2}(\rho - \delta)dx 
\leq \int_{Q_\delta} \sum_{i,j=1}^n a_{ij}D_i\rho D_j\rho(u^2+1)^{1/2}dS_x + \int_{Q_\delta} \sum_{i,j=1}^n D_i(D_iD_j\rho)(u^2+1)^{1/2}dx 
- \int_{Q_\delta} \sum_{i=1}^n D_i(b_i(\rho - \delta))(u^2+1)^{1/2} dx.
\end{equation}
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Since $L^3$-convergence yields $L^1$-convergence, letting $\delta \to 0$ in (11), we obtain

\begin{equation}
\int_Q |Du|^3(u^2+1)^{-3/2}\rho dx + \lambda \int_Q u^2(u^2+1)^{-1/2}\rho dx 
\leq C_1 \left( \int_{\partial Q} |\phi| dS_x + \int_Q (u^2+1)^{1/2} dx + 1 \right),
\end{equation}

where $C_1$ is a positive constant independent of $u$. On the other hand again by the use of the Green formula we can deduce from (10) the following estimate

\begin{equation}
\int_{Q_\delta} (u^2+1)^{1/2} dS_x 
\leq C_2 \left[ \int_{Q_\delta} |Du|^3(u^2+1)^{-3/2}(\rho-\delta) dx + \lambda \int_{Q_\delta} u^2(u^2+1)^{-1/2}(\rho-\delta) dx + \int_{Q_\delta} (u^2+1)^{1/2} dx \right],
\end{equation}

Since

\begin{equation}
\int_Q u^2(u^2+1)^{-1/2}\rho dx = \int_Q (u^2+1)^{1/2}\rho dx - \int_Q (u^2+1)^{-1/2}\rho dx,
\end{equation}

combining (12) and (13) we obtain

\begin{equation}
\int_Q |Du|^3(u^2+1)^{-3/2}\rho dx + \lambda \int_Q (u^2+1)^{1/2}\rho dx + \int_{\partial Q_\delta} (u^2+1)^{1/2} dS_x 
\leq C_3 \left[ \int_{Q_\delta} |\phi| dS_x + \int_Q (u^2+1)^{1/2} dx + 1 \right] + \lambda \int_Q (u^2+1)^{-1/2}\rho dx,
\end{equation}

where $C_3$ is a positive constant. Now notice that for every $0<d \leq \delta_0$, we have

\begin{equation}
\int_Q (u^2+1)^{1/2} dx = \int_{Q_0+Q_\delta} (u^2+1)^{1/2} dx + \int_{Q_\delta} (u^2+1)^{1/2} dx 
\leq d \sup_{0<\delta<d} \int_{Q_\delta} (u^2+1)^{1/2} dS_x + \frac{1}{d} \int_Q (u^2+1)^{1/2}\rho dx,
\end{equation}

consequently (9) follows from (14) provided $\lambda_1$ is sufficiently large and $d$ sufficiently small.

To proceed further we need

**Lemma 2.** Suppose that $u \in W^{1,\infty}_{loc}(Q)$ and that $\int_Q |Du(x)|^3[u(x)^2+1]^{-3/2}r(x) dx < \infty$. Then we have for $\delta \in (0, \delta_0/2]$

\begin{equation}
\int_Q [u(x)^2+1]^{1/2} dx \leq K \left\{ \int_{Q_{\delta_0}} [u(x)^2+1]^{1/2} dx + \delta_0 \int_{\partial Q_{\delta_0}} [u(x)^2+1]^{1/2} dS_x 
+ \delta_0 \int_{Q_{\delta_0}-Q_{\delta_0}} |Du(x)|^3[u(x)^2+1]^{-3/2}(\rho(x)-\delta) dx \right\},
\end{equation}
where $K$ is a positive constant independent of $\delta$.

**Proof.** Let $\delta \in (0, \frac{\delta_0}{2}]$ and put

$$
\int_{Q_2} (u^2 + 1)^{1/2} dx = \int_{Q_3 - Q_{\delta_0}} (u^2 + 1)^{1/2} dx + \int_{Q_{\delta_0}} (u^2 + 1)^{1/2} dx.
$$

We now note that

$$
\int_{Q_3 - Q_{\delta_0}} (u^2 + 1)^{1/2} dx = \int_0^{\delta_0} \int_{Q_\delta} (u(x_t(x_0))^2 + 1)^{1/2} dS_{\delta_t} dS_{\delta_0} dt
$$

$$
\leq \gamma_0^2 \int_0^{\delta_0} \Delta u(x_t(x_0))^2 + 1)^{1/2} dS_{\delta_0} dt.
$$

As $\int_{Q_\delta} (u(x_t(x)) + 1)^{1/2} dS_x$ is absolutely continuous on $[\delta, \delta_0]$ integrating by parts we get

$$
\int_{Q_3 - Q_{\delta_0}} (u^2 + 1)^{1/2} dx \leq \gamma_0^2 \int_{Q_\delta} (u(x_t(x_0))^2 + 1)^{1/2} dS_{\delta_0}
$$

$$
+ \gamma_0^2 \int_0^{\delta_0} \int_{Q_\delta} |Du(x_t(x_0))||u(x_t(x_0))|(u(x_t(x_0))^2 + 1)^{-1/2} \left| \frac{\partial}{\partial t} x_t(x_0) \right| dS_{\delta_0} dt
$$

$$
\leq \gamma_0^2 \int_0^{\delta_0} \int_{Q_\delta} (u^2 + 1)^{1/2} dS + \gamma_0^2 \int_{Q_3 - Q_{\delta_0}} (u^2 + 1)^{1/2} dx
$$

$$
+ \gamma_0^2 \int_0^{\delta_0} \int_{Q_\delta} |Du(x_t(x_0))|(u(x_t(x_0))^2 + 1)^{-1/2} (\rho(x) - \delta) dx
$$

$$
\leq \gamma_0^2 \int_0^{\delta_0} \int_{Q_\delta} (u^2 + 1)^{1/2} dS + \beta \gamma_0^2 \int_{Q_3 - Q_{\delta_0}} (u^2 + 1)^{1/2} dx
$$

$$
+ \gamma_0^2 \beta \int_0^{\delta_0} \int_{Q_\delta} |Du(x_t(x_0))|(u(x_t(x_0))^2 + 1)^{-3/2} (\rho - \delta) dx
$$

where we have used Young's inequality in the final step. Now choosing $\beta = \gamma_0^{-4}/2$ the result follows.

**Lemma 3.** Let $u$ be a solution of (1) belonging to $W^{1,2}_{loc}(Q)$. Then the following conditions are equivalent

1. $\int_{Q_\delta} |u(x)| dS_x$ is bounded on $(0, \delta_0]$,
2. $\int_{Q} |Du(x)|^2 (u(x)^2 + 1)^{-3/2} r(x) dx < \infty$,
3. $\int_{Q} |u(x_t(x))| dS_x$ is continuous on $[0, \delta_0]$. 


Proof. The proof is similar to that of Theorem 1 in [2] and therefore we only give an outline. It follows from (11) that

$$
\int_{\partial Q_\delta} |D_u|^3 (u^2 + 1)^{-3/2} (\rho - \delta) \, dx 
\leq C \left[ \int_{\partial Q_\delta} |u| \, dS_x + \int_{Q_\delta} |u| \, dx + \lambda \int_{Q_\delta} |u| (\rho - \delta) \, dx + 1 \right],
$$

where a positive constant $C$ is independent of $\delta$. On the other hand (I) yields $u \in L^1(Q)$ and "I⇒II" follows by the Monotone Convergence Theorem.

To prove "II⇒III" note that Lemma 2 implies

$$
\int_{\partial Q_\delta} \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho (u^2 + 1)^{1/2} \, dS_x \leq C
$$

for $\delta \in (0, \delta_0/2]$, where a positive constant $C$ is independent of $\delta$. First we prove the continuity of

$$
\int_{\partial Q_\delta} \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho (u^2 + 1)^{1/2} \, dS_x \quad \text{at } \delta = 0.
$$

It follows from the proof of Lemma 1 that

$$
\int_{\partial Q_\delta} \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho (u^2 + 1)^{1/2} \, dS_x 
= - \int_{\partial Q_\delta} \sum_{i,j=1}^n D_i(a_{ij} D_j \rho)(u^2 + 1)^{1/2} \, dx 
- \int_{Q_\delta} \sum_{i=1}^n D_i(b_i (\rho - \delta))(u^2 + 1)^{1/2} \, dx + \int_{Q_\delta} (c + \lambda) u^2 (u^2 + 1)^{-1/2} \, dx 
+ \int_{Q_\delta} \sum_{i,j=1}^n a_{ij} D_i u D_j u (u^2 + 1)^{-3/2} (\rho - \delta) \, dx.
$$

Thus

$$
\lim_{\delta \to 0} \int_{\partial Q_\delta} \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho (u^2 + 1)^{1/2} \, dS_x
$$

exists by the Dominated Convergence Theorem. Since $\sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho$ is continuous on $\bar{Q}$, it follows that $\int_{\partial Q_\delta} |u| \, dS_x$ is continuous at $\delta = 0$.

It is clear that "II⇒III" follows from the relationship

$$
\int_{\partial Q_\delta} |u(x)| \, dS_x - \int_{\partial Q} |u(x_\delta(x))| \, dS_x = \int_{\partial Q} |u(x_\delta(x))| \left[ \frac{dS_\delta}{dS_0} - 1 \right] \, dS,
$$

since $dS_\delta/dS_0 \to 1$ uniformly as $\delta \to 0$. 
Lemma 4. Suppose that one of the conditions (I), (II) or (III) holds. Then there exists a Borel signed measure μ on ∂Q, with |μ| (∂Q) < ∞, such that

(15) \[ \lim_{\delta \to 0} \int_{\partial Q} u(x_{\delta}(x))g(x)dS_{x} = \int_{\partial Q} g(x)\mu(dx) \quad \text{for every } g \in C(\partial Q). \]

It is clear that (15) holds for certain subsequence \( \delta_{n} \to 0 \). Now the proof of the existence of the limit (15) is identical to that of Theorem 3 in [2] and therefore is omitted.

§ 3. The Dirichlet problem

In view of Lemma 4 it is natural to formulate the Dirichlet problem in the following manner.

Let \( \phi \in L^{1}(\partial Q) \). A weak solution \( u \in W^{1,2}_{\text{loc}}(Q) \) of (1) is a solution of the Dirichlet problem with the boundary condition (2) if

(16) \[ \lim_{\delta \to 0} \int_{\partial Q} u(x_{\delta}(x))g(x)dS_{x} = \int_{\partial Q} g(x)\phi(x)dS_{x} \]

for every \( g \in C(\partial Q) \).

Now we are in a position to prove the main result of this paper.

Theorem 1. Let \( \lambda \geq \lambda_{i} \). Then for every \( \phi \in L^{1}(\partial Q) \) there exists a unique solution \( u \in W^{1,2}_{\text{loc}}(Q) \) of the Dirichlet problem (1), (2).

Proof. Let \( \{\phi_{m}\} \) be a sequence of functions in \( L^{2}(\partial Q) \) converging in \( L^{2}(\partial Q) \) to the function \( \phi \). Let \( u_{m} \) be a solution of the Dirichlet problem

\[ Lu - \lambda u = 0 \quad \text{in } Q, \]
\[ u = \phi_{m} \quad \text{on } \partial Q, \]

in \( W^{1,2}_{\text{loc}}(Q) \) with the boundary condition understood in the sense of (7). Here we may assume that \( \lambda_{i} \) is sufficiently large that Theorem 6 in [2] on the existence of solution in \( W^{1,2}_{\text{loc}}(Q) \) is applicable. On the other hand

(17) \[ u_{m}(x) = \int_{\partial Q} K_{i}(x, y)\phi_{m}(y)dS_{y} \quad \text{in } Q, \]

for \( m = 1, 2, \ldots \). By (8) for every compact subset \( K \) of \( Q \) there is a positive constant \( M(K) \) such that

(18) \[ |u_{m}(x)| \leq M(K) \int_{\partial Q} |\phi_{m}(y)|dS_{y} \]

for all \( x \in K, m = 1, 2, \ldots \). Le:
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\[ M_i = \sup_{m > 1} M(K) \int_{\partial Q} |\phi_m(y)| dS_y, \]
then it follows from (9) that there exists a positive constant $C$ such that

\[ (M_i^2 + 1)^{-\alpha} \int_{\partial Q} |Du_m|^\alpha \mu(dx) + \int_{\partial Q} |u_m| \mu(dx) \leq C \]

for all $m$. Consequently in view of (18) and (19) we may assume that there exists a function $u \in L^1(Q)$ with $D_i u \in L^1(Q)$ $(i = 1, \ldots, n)$ such that

\[ \lim_{m \to \infty} u_m = u \quad \text{weak in} \] 

and

\[ \lim_{m \to \infty} D_i u_m = D_i u \quad (i = 1, \ldots, n) \text{ weakly} \]
in $L^1(K)$ for every compact subset $K$ of $Q$. It is obvious that $u$ is a weak solution of (1). By (20) $u_m$ converges uniformly to $u$ on every compact subset of $Q$. Hence

\[ \int_{\partial Q} |Du|^\alpha (u^2 + 1)^{-\alpha/2} \mu(dx) < \infty. \]

It remains to prove that $u$ satisfies the boundary condition (2) in the sense of (16). It follows from Lemma 4 that there exists a Borel signed measure $\mu$ on $\partial Q$, with $|\mu(\partial Q)| < \infty$, such that

\[ \lim_{\varepsilon \to 0} \int_{\partial Q} u(x_\varepsilon(x)) g(x) dS_x = \int_{\partial Q} g(x) \mu(dx) \]

for every $g \in C(\partial Q)$. Note that integrating by parts we obtain $a$

\[ \int_{\partial Q} \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho \phi_m dS_x = - \int_{Q} u_m \sum_{i,j=1}^n D_i(a_{ij} D_j \rho \phi) dx \]

\[ - \int_{Q} u_m \sum_{i,j=1}^n D_i(a_{ij} D_j \rho \phi) dx - \int_{Q} u_m \sum_{i=1}^n D_i(b_i \rho \phi) dx + \int_{Q} (c + \lambda)u_m \rho \phi dx \]

$m = 1, 2, \ldots$ and

\[ \int_{\partial Q} \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho \phi \mu(dx) = - \int_{Q} u \sum_{i,j=1}^n D_i(a_{ij} D_j \phi \rho) dx \]

\[ - \int_{Q} u \sum_{i,j=1}^n D_i(a_{ij} D_j \phi \rho) dx - \int_{Q} u \sum_{i=1}^n D_i(b_i \phi \rho) dx + \int_{Q} (c + \lambda)u \phi \rho dx \]

for all $\phi \in C^2(\overline{Q})$. Since $\lim_{m \to \infty} \phi_m = \phi$ in $L^1(\partial Q)$, (20) implies that $\int_{\partial Q} \phi \mu(dx) = \int_{\partial Q} \phi \phi dS_x$ for all $\phi \in C^2(\overline{Q})$ and by the Weierstrass theorem this identity holds for all
\( \Phi \in C(\partial Q) \). This identity together with (24) shows that \( u \) satisfies the boundary condition (16).

Under the assumptions of Lemma 1 we have the following estimate for a solution \( u \) of the problem (1), (2)

\[
\int_\Omega |u(x)| \, dx \leq \text{Const} \left[ \int_{\partial \Omega} |\phi(x)| \, dS_x + 1 \right].
\]

This estimate can be slightly improved as follows

**Theorem 2.** Let \( \phi \in L^1(\partial Q) \). Then there exists a positive constant \( \lambda_0 \) such that for every \( \lambda \geq \lambda_0 \) there exists a unique solution \( u \in W^{1,2}_0(\Omega) \) to the problem (1), (2) satisfying the estimate

\[
\int_\Omega |u(x)| \, dx \leq C \int_{\partial \Omega} |\phi(x)| \, dS_x + 1,
\]

where \( C \) is a positive constant independent of \( u \).

**Proof.** First we assume that \( \phi \in L^2(\partial Q) \) and put

\[
v(x) = \begin{cases} u(x)(u(x) + \epsilon)^{-1/2}(\rho(x) - \delta) & \text{in } Q, \\ 0 & \text{in } Q - Q, \end{cases}
\]

where \( \epsilon > 0 \) and \( 0 < \delta < \delta_0 \), then

\[
\int_{Q} \left[ \sum_{i,j=1}^{n} a_{ij} D_i D_j u (u^2 + \epsilon)^{-1/2}(\rho - \delta) - \sum_{i,j=1}^{n} a_{ij} D_i D_j u \cdot u (u^2 + \epsilon)^{-3/2}(\rho - \delta) \\
+ \sum_{i=1}^{n} b_i D_i u \cdot u (u^2 + \epsilon)^{-1/2} \rho + \sum_{i=1}^{n} b_i D_i u \cdot u (u^2 + \epsilon)^{-1/2}(\rho - \delta) \\
+ \int_{Q} (c + \delta) u^2 (u^2 + \epsilon)^{-1/2}(\rho - \delta) \right] dx = 0.
\]

By Green's formula we deduce

\[
R^{-1} \epsilon \int_{Q_\delta} |Du|^2(u^2 + \epsilon)^{-3/2}(\rho - \delta) \, dx + \int_{Q_\delta} (c + \delta) u^2 (u^2 + \epsilon)^{-1/2}(\rho - \delta) \, dx \\ \leq C_1 \left[ \int_{\partial Q_\delta} (u^2 + \epsilon)^{1/2} dS_x + \int_{Q_\delta} (u^2 + \epsilon)^{1/2} \, dx \right],
\]

where \( C_1 \) is a positive constant independent of \( \epsilon \) and \( \delta \). Letting \( \epsilon \to 0 \) and \( \delta \to 0 \) we obtain

\[
\int_\Omega \lambda |u| \, dx \leq C_2 \left[ \int_{\partial \Omega} |\phi| \, dS_x + \int_\Omega |u| \, dx \right],
\]

where \( C_2 \) is a positive constant. Similarly there exists a positive constant \( C_3 \) such that
On the Dirichlet Problem with $L^1$-Boundary Data

(25) \[ \sup_{\theta<\delta<d} \int_{\partial Q_\delta} |u| dS_x \leq C_3 \left[ \lambda \int_Q |u| \rho dx + \int_Q |u| dx \right], \]

where $0 < d \leq \delta_0$. Combining (26) and (27) we get

\[ \sup_{\theta<\delta<d} \int_{\partial Q_\delta} |u| dS_x + \lambda \int_Q |u| \rho dx \leq C_4 \left[ \int_{\hat{a}Q} |\phi| dS_x + \int_Q |u| dx \right], \]

where $C_4$ is a positive constant. Applying the argument of the final step of the proof of Lemma 1 the result easily follows provided $\lambda$ is sufficiently large and $d$ sufficiently small and $\phi \in L^2(\partial Q)$. To complete the proof we take a sequence $\{\phi_m\}$ in $L^2(\partial Q)$ converging in $L^1(\partial Q)$ to $\phi$. By what we have already proved the corresponding sequence of solutions $\{u_m\}$ to the problem (1), (2) (with $\phi = \phi_m$) satisfies the estimate (23) and moreover

\[ \int_Q |u_p - u_q| dx \leq C \int_{\partial Q} |\phi_p - \phi_q| dS_x \]

for $p, q = 1, 2, \ldots$. Applying Theorem 1 the result easily follows.

§ 4. Non-homogeneous equation

In this section we assume that $c \geq 0$ on $Q$. In view of Theorem 9.1 in [9] for any signed Borel measure $\mu$ with bounded variation on $Q$ there exists a unique solution $u$ of the equation

(1') \[ Lu = \mu \]

in $\tilde{W}^{1,p}(Q)$ for every $p < n/(n-1)$.

**Theorem 3.** Let $\phi \in L^1(\partial Q)$ and let $\mu$ be a Borel signed measure of bounded variation on $Q$. Then there exists a unique solution $u$ of (1') (2) in $W^{1,2}_{0}(Q) + \tilde{W}^{1,p}(Q)$ for any $p < n/(n-1)$.

**Proof.** It follows from Theorem 1 that if $\lambda_0$ is sufficiently large then the Dirichlet problem

\[ Lu_0 + \lambda_0 u_0 = 0 \quad \text{in} \ Q, \]

\[ u_0 = \phi \quad \text{on} \ \partial Q, \]

has a unique solution $u_0 \in W^{1,2}_{0}(Q)$ satisfying the estimates (9) and (23). On the other hand the Dirichlet problem

(26) \[ Lw = \lambda_0 u_0 + \mu \quad \text{in} \ Q, \]

(27) \[ w = 0 \quad \text{on} \ \partial Q, \]
has a unique solution in \( \dot{W}^{1,p}(Q) \), \( p < n/(n-1) \) (see [9], Theorem 9.1). The function \( w + u_0 \) is a unique solution of (1), (2) in \( W^{1,\infty}_{\text{loc}}(Q) + \dot{W}^{1,p}(Q) \) for any \( p < n/(n-1) \).

Remark. Given a signed Borel measure \( \nu \) of bounded variation on \( \partial Q \), one can construct a sequence \( \{\phi_m\} \) in \( L^1(\partial Q) \) such that \( \lim_{m \to \infty} \int_{\partial Q} \phi_m(x) g(x) d\nu = \int_{\partial Q} g(x) \nu(dx) \) for every \( g \in C(\partial Q) \) and \( \sup_{m \geq 1} \int_{\partial Q} |\phi_m(x)| dS_x < \infty \) (see [4], p. 296–297).

Therefore we can solve the Dirichlet problem for (1') with the boundary condition

\[
(28) \quad u = \nu \quad \text{on} \quad \partial Q
\]

where \( \mu \) and \( \nu \) are signed Borel measures of bounded variation on \( Q \) and \( \partial Q \) respectively. Indeed, we first solve the Dirichlet problem

\[
Lu_0 + \lambda_0 u_0 = 0 \quad \text{in} \quad Q, \quad u_0 = \nu \quad \text{on} \quad \partial Q,
\]

provided \( \lambda_0 \) is sufficiently large. A solution \( u_0 \) is obtained as a limit of solutions \( u_m \) of the Dirichlet problem

\[
Lu_m + \lambda_0 u_m = 0 \quad \text{in} \quad Q, \quad u_m = \phi_m \quad \text{on} \quad \partial Q,
\]

(see the proof of Theorem 1). The function \( u_0 + w \), where \( w \) is a solution of (26), (27), is a solution in \( W^{1,\infty}_{\text{loc}}(Q) + \dot{W}^{1,p}(Q) \), \( p < n/(n-1) \), of (1'), (28). Here the boundary condition is understood in the sense of a weak convergence (see (16)).

References


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