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The Baire Category Method in Existence Problems for a Class of Multivalued Differential Equations with Nonconvex Right Hand Side

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§ 1. Introduction

In this paper we present some results on the existence of solutions to the Cauchy problem

$$\dot{x} \in F(t, x), \quad x(t_0)=x_0, \quad \left(\cdot = \frac{d}{dt}\right) \tag{1.1}$$

for multivalued differential equations in a Banach space X. When X is infinite dimensional, most existence theorems for (1.1) have been obtained under some compactness hypothesis on F ([2] [4] [5] [18] [22]). In this note the existence of solutions to the Cauchy problem (1.1) will be established (see Theorem 2.4) under a somehow opposite assumption on F. In fact we shall suppose (among other things) that F be such that the closed convex hull of $F(t, x)$ have non empty interior.

Let us briefly describe the method of the proof. We associate with (1.1) the Cauchy problem

$$\dot{x} \in \overline{\text{co}}\, F(t, x), \qquad x(t_0)=x_0 \tag{1.2}$$

and consider a well defined nonempty set $\mathcal{M}$ of solutions of (1.2), which is complete under the metric of the uniform convergence. Then we show that the set $\mathcal{M}_F$ of all $x \in \mathcal{M}$ which are solutions of (1.1) is residual in $\mathcal{M}$, that is its complement $\mathcal{M} \backslash \mathcal{M}_F$ is of the Baire first category in $\mathcal{M}$. Since $\mathcal{M}$ is complete, it follows that $\mathcal{M}_F$ is dense in $\mathcal{M}$. Hence $\mathcal{M}_F$ is nonempty and the Cauchy problem (1.1) has solutions.

The above approach has already been used in [3] [6] to study the structure of the solution set of (1.1). In [6] the existence follows as a corollary; incidentally we observe that in [6], because of the regularity of F, the existence could be proved directly, without appeal to the Baire category method.

Evidently, (1.1) has solution if F admits a continuous selection f, for which the Cauchy problem $\dot{x}=f(t, x)$, $x(t_0)=x_0$ have solutions. This is just what occurs in [6]. On the contrary, under the hypotheses of Theorem 2.4, F might have no continuous selection at all [14, Example 31] or, if F admits a continuous selection f, the Cauchy problem $\dot{x}=f(t, x)$, $x(t_0)=x_0$, might fail to have solutions. By Godunov's theorem [12] this is possible when X is infinite dimensional.

To focus this point, we construct (see Theorem 2.5) a singular multifunction F given by $F(t, x)=\bigcup_{i=1}^{\infty} f_i(t, x)$ (where the functions f_i are continuous and, at each point, assume mutually different values) satisfying the assumptions of our existence theorem; yet the set of all continuous selections of F is exactly $\{f_i \mid i=1, 2, \cdots\}$ and, for each such f_i, the Cauchy problem $\dot{x}=f_i(t, x)$, $x(t_0)=x_0$ has no solution.

We wish to point out that, under the hypotheses of Theorem 2.4, the existence of solutions to the Cauchy problem (1.1) is a new result, only when X is infinite dimensional. If x is finite dimensional, the existence of solutions to (1.1) has been proved by Filippov [11] (see also [1] [10] [17] [19]) under more general hypotheses on F.

§ 2. Notations and main results

Throughout this paper X denotes a real (infinite dimensional) reflexive Banach space. Additional hypotheses on X will be stated where necessary. Denote by $\mathscr{K}$ (resp. $\mathscr{B}$) the space of all nonempty subsets of X which are closed and bounded (resp. closed bounded and convex with nonempty interior). $\mathscr{K}$ (in particular $\mathscr{B}$) is endowed with the Hausdorff distance

$$h(A, B)=\inf\{r>0 \mid A\subset B+rS,\ B\subset A+rS\},$$

where $A, B \in \mathscr{K}$ and $S=\{x \in X \mid |x|<1\}$. Also, set $e(u, A)=\inf\{r>0 \mid u \in A+rS\}$, where $u \in X$ and $A\subset X$ is nonempty.

Let us introduce some notations of frequent use. Let A be a subset of a normed space Y. We denote by $\bar{A}$, $\operatorname{int} A$, ∂A, $\operatorname{co} A$, $\overline{\operatorname{co}} A$, respectively, the closure, the interior, the boundary, the convex hull, the closed convex hull of A. If A is nonempty, diam A satnds for the diameter of A. By $S(y_0, d)$ (resp. $\bar{S}(y_0, d)$), we mean the open (resp. closed) ball in Y with center y_0 and radius $d>0$. For notational convenience we put $S=S(0, 1)$, $\bar{S}=S(0, 1)$. For any (Lebesgue) measurable set $J\subset \boldsymbol{R}$, we denote by χ_J the characteristic function of J. In $\boldsymbol{R}\times X$ we introduce the norm $|(t, x)|=\max\{|t|, |x|\}$. With such norm $\boldsymbol{R}\times X$ is a Banach space.

Let D be a nonempty open set contained in $\boldsymbol{R}\times X$. Let $F: D\to\mathscr{K}$ be a multifunction satisfying the hypotheses:

(i) F is Hausdorff continuous and such that $\overline{\operatorname{co}} F(t, x) \in \mathscr{B}$, for each $(t, x) \in D$;

(ii) there exists a Hausdorff continuous multifunction $U: D\to\mathscr{B}$ such that $\partial U(t, x)\cap \overline{\operatorname{co}} F(t, x)=F(t, x)$, for each $(t, x) \in D$.

Fix $(t_0, x_0) \in D$ and consider the Cauchy problem (1.1). Since we are interested in local solutions of (1.1), and F is Hausdorff continuous, we assume without loss of generality that F satisfies also the following hypothesis:

(iii) there exist positive constants a, R, and M such that $h(F(t, x), 0)<M$ for each $(t, x) \in D_2$ where $D_2=\{t \in \boldsymbol{R} \,|\, |t-t_0|<2a\}\times\{x \in X \,|\, |x-x_0|<2R\}$.

Since $U: D\to\mathscr{B}^2$ Hausdorff continuous implies that $\partial U: D\to\mathscr{K}$ is Hausdorff continuous (see [7]), the assumptions (i)–(iii) are certainly satisfied if we take $F(t, x)=\partial U(t, x)$, $(t, x) \in D$. This special case has been considered in [6].

By a *solution* of (1.1) (resp. (1.2)) we mean a function $x: I\to X$, where $I=[t_0-T, t_0+T]$, $T>0$, which is Lipschitzean (hence x admits derivative a.e., since X is reflexive), and satisfies (1.1) (resp. (1.2)) a.e.. By a *polygonal solution* of (1.2) we mean a solution of (1.2) which has the following property: there exists a countable family $\{I_q\}$ of nonempty pairwise disjoint open intervals $I_q\subset I$ such that in $m(I\setminus\bigcup_q I_q)=0$ (m denotes the Lebesgue measure in $\boldsymbol{R}$) and, moreover, $\dot{x}$ is constant on each interval I_q and satisfies $\dot{x}(t) \in \overline{\mathrm{co}}\, F(t, x(t))$ for every $t \in \bigcup_q I_q$.

Let F satisfies (i)–(iii). For notational convenience we introduce the multifunction $G: D\to\mathscr{B}$ defined by

$$G(t, x)=\overline{\mathrm{co}}\, F(t, x) \qquad \text{for each } (t, x) \in D.$$

Observe that from (ii) it follows $F(t, x)\subset\partial U(t, x)$ and so $G(t, x)\subset U(t, x)$, $(t, x) \in D$. Evidently G is Hausdorff continuous and satisfies $h(G(t, x), 0)<M$ for each $(t, x) \in D_2$. Then, as in [6, proof of Proposition 2.1], it can be shown that (1.2) has polygonal solutions which are defined on $I=[t_0-T, t_0+T]$ where $0<T<\min\{a, R/M\}$.

First of all we shall construct a nonempty set $\mathscr{M}$ of solutions of (1.2) which is complete under the metric of the uniform convergence. Next we shall show that the set $\mathscr{M}_F$ consisting of all $x \in \mathscr{M}$ which are solutions of (1.1) is residual in $\mathscr{M}$. Since $\mathscr{M}$ is complete, it follows that $\mathscr{M}_F$ is dense in $\mathscr{M}$, thus $\mathscr{M}_F$ is nonempty and (1.1) has solutions.

Let $f: D\to X$ be a continuous function such that $f(t, x) \in \mathrm{int}\, G(t, x)$, $(t, x) \in D$. The existence of such an f is ensured, for example, by [7, Remark 3.10].

For each $(t, x) \in D$ and $0\leq r\leq 1$ we set

$$G_r(t, x)=f(t, x)+r[G(t, x)-f(t, x)].$$

If $0\leq r\leq 1$ we have $G_r(t, x) \in \mathscr{B}$. Moreover, $(t, x)\mapsto G_r(t, x)$ is Hausdorff continuous from D to $\mathscr{B}$. Also observe that $0\leq r'<r''\leq 1$ implies $G_{r'}(t, x)\subset G_{r''}(t, x) \in G(t, x)$, $(t, x) \in D$.

Let $\{r_k\}$ be a strictly increasing sequence of numbers $0<r_k<1$ $(k=0, 1, 2, \cdots)$ converging to 1. For each $k \in N$, let $\mathscr{V}_{r_k}$ be the set of all polygonal solutions to the Cauchy problem

$$\text{(2.1)} \qquad \dot{x} \in \mathrm{int}\, G_{r_k}(t, x), \qquad x(t_0)=x_0,$$

which are defined on $I=[t_0-T, t_0+T]$ (where $0<T<\min\{a, R/M\}$). Clearly each $x \in \mathscr{V}_{r_k}$ is solution to (1.2) and we have $\mathscr{V}_{r_1}\subset\mathscr{V}_{r_2}\subset\cdots$. By [6, Proof of Proposition 2.1] for each $k \in N$ the Cauchy problem $\dot{x} \in G_{r_{k-1}}(t, x)$, $x(t_0)=x_0$ has a polygonal solution x defined on I. Since $G_{r_{k-1}}(t, x)\subset \text{int}\, G_{r_k}(t, x)$, $(t, x) \in D$, it follows that $x \in \mathscr{V}_{r_k}$, that is for each $k \in N$ the set $\mathscr{V}_{r_k}$ is nonempty. Define

$$\mathscr{M}=\overline{\bigcup_{k=1}^{\infty} \mathscr{V}_{r_k}},$$

where the closure of the set on the right is taken in $C(I, X)$ (with the metric of the uniform convergence). As in [6, Proposition 2.1], under our assumptions (X reflexive and $G\colon D\to\mathscr{B}$ Hausdorff continuous), it follows that each $x \in \mathscr{M}$ is also a solution of (1.2). Hence $\mathscr{M}$ is a (nonempty) closed subset of $C(I, X)$ consisting of solutions of (1.2), thus $\mathscr{M}$ is a complete metric space under the metric of the uniform convergence.

For each $u \in X$, (t, x) and $(t_1, x_1) \in D$ we set

$$\|u-f(t_1, x_1)\|_{H(t,x)}=\inf\{r>0 \mid u-f(t_1, x_1) \in rH(t, x)\}, \tag{2.2}$$

where

$$H(t, x)=U(t, x)-f(t, x).$$

This definition is meaningful for the origin is an interior point of $H(t, x)$. Furthermore, for each $(t, x) \in D$ and $0\le r\le 1$, we put

$$U_r(t, x)=f(t, x)+r[U(t, x)-f(t, x)].$$

Let $\rho(t, x)=\sup\{s>0 \mid \bar{S}(f(t, x), s)\subset G(t, x)\}$, $(t, x) \in D$. Since $f(t, x) \in \text{int}\, G(t, x)$, $\rho(t, x)$ is well defined and positive. Observe that $\rho(t, x)$ (resp. $U_r(t, x)$, $H(t, x)$) is continuous (resp. Hausdorff continuous) as function of $(t, x) \in D$.

Proposition 2.1. *Let $u, u_i, v \in X$ and $(t, x), (t_i, x_i) \in D$ $(i=1, 2, \cdots, k)$. We have*

(a_1) $\|u-f(t_1, x_1)\|_{H(t,x)}=0$ *if and only if* $u=f(t_1, x_1)$.

(a_2) $\|\mu[u-f(t_1, x_1)]\|_{H(t,x)}=\mu\|u-f(t_1, x_1)\|_{H(t,x)}$ $(\mu\ge 0)$.

(a_3) $\left\|\sum_{i=1}^{k} \mu_i[u_i-f(t_i, x_i)]\right\|_{H(t,x)} \le \sum_{i=1}^{k} \mu_i\|u_i-f(t_i, x_i)\|_{H(t,x)}$ $\left(\sum_{i=1}^{k} \mu_i=1,\ \mu_i\ge 0\right)$.

(a_4) $\|u-f(t, x)\|_{H(t,x)}=1$ *if and only if* $u \in \partial U(t, x)$.

(a_5) $$\|u-f(t_1, x_1)\|_{H(t,x)}-\frac{|v|}{\rho(t, x)}\le \|u+v-f(t_1, x_1)\|_{H(t,x)} \le \|u-f(t_1, x_1)\|_{H(t,x)}+\frac{|v|}{\rho(t, x)}.$$

(a_6) *Let* $H(t, x) \supset \theta\bar{S}$, $\theta > 0$. *Then we have*

$$\|u - f(t_1, x_1)\|_{H(t,x)} \leq \|u - f(t_1, x_1)\|_{H(t_1,x_1)}[1 + \theta^{-1}h(H(t, x), H(t_1, x_1))].$$

(a_7) *Let* $H(t, x) \supset \theta\bar{S}$, $\theta > 0$. *Then we have* $\|u - f(t_1, x_1)\|_{H(t,x)} \leq \theta^{-1}|u - f(t_1, x_1)|$.

Proposition 2.2. *If* $x: I \to X$ *is continuous and* $z: I \to X$ *(Bochner) measurable and bounded, then* $\|z(t) - f(t, x(t))\|_{H(t, x(t))}$ *is a measurable and bounded function of* $t \in I$.

For each $0 < \sigma < 1$ define

$$\mathscr{M}_\sigma = \left\{ x \in \mathscr{M} \,\middle|\, \frac{1}{|I|} \int_I \|\dot{x}(t) - f(t, x(t))\|_{H(t, x(t))} dt > \sigma \right\},$$

where $|I| = 2T$. Under our hypotheses the integral makes sense by virtue of Proposition 2.2.

Theorem 2.3. *Let* X *be a real reflexive Banach space. Let* D, $\mathscr{K}$ *be as above and suppose that* $F: D \to \mathscr{K}$ *satisfies* (i)–(iii). *Then, for every* $0 < \sigma < 1$, *the set* $\mathscr{M}_\sigma$ *is open and desne in* $\mathscr{M}$.

By means of Theorem 2.3 we can prove our main result, that is the Cauchy problem (1.1) has solutions if $F: D \to \mathscr{K}$ satisfies (i)–(iii). In fact, denote by $\mathscr{M}_F$ the set of all $x \in \mathscr{M}$ which are solutions of (1.1). Let $\{\sigma_k\}$ ($0 < \sigma_k < 1$, $k \in N$) be a strictly increasing sequence converging to 1. Since $\mathscr{M}_{\sigma_k}$ is open and dense in the nonempty complete metric space $\mathscr{M}$, the set

$$\mathscr{M}^* = \bigcap_{k=1}^{\infty} \mathscr{M}_{\sigma_k}$$

is residual in $\mathscr{M}$. Hence $\mathscr{M}^*$ is dense in $\mathscr{M}$ and, in particular, $\mathscr{M}^*$ is nonempty. We claim that $\mathscr{M}^* = \mathscr{M}_F$. Let $x \in \mathscr{M}^*$. Clearly, $\dot{x}(t) \in G(t, x(t))$ a.e. in I and so

$$\dot{x}(t) - f(t, x(t)) \in G(t, x(t)) - f(t, x(t)) \subset U(t, x(t)) - f(t, x(t)) = H(t, x(t)).$$

This implies that $b(t) \leq 1$ a.e. in I, where $b(t) = \|\dot{x}(t) - f(t, x(t))\|_{H(t, x(t))}$. On the other hand

$$\frac{1}{|I|} \int_I b(t)\, dt \geq 1,$$

since $x \in \mathscr{M}_{\sigma_k}$ for each $k \in N$. Consequently, $b(t) = 1$ a.e. in I, and so by Proposition 2.1 (a_4), $\dot{x}(t) \in \partial U(t, x(t))$. By virtue of hypothesis (ii) we can conclude that x is a solution to (1.1), that is $x \in \mathscr{M}_F$. Conversely, in view of (ii), each $x \in \mathscr{M}_F$ satisfies $\dot{x}(t) \in \partial U(t, x(t))$ and hence, by Proposition 2.1 (a_4) $b(t) = 1$ a.e. in I. Thus $x \in \mathscr{M}_{\sigma_k}$, for each $k \in N$, that is $x \in \mathscr{M}^*$. Therefore $\mathscr{M}_F = \mathscr{M}^*$. We have proved the following

Theorem 2.4. *Let X be a real reflexive Banach space. Let D, $\mathscr{K}$ and $\mathscr{M}$ be as above and suppose that F: $D \to \mathscr{K}$ satisfies* (i)–(iii). *Then the set of all $x \in \mathscr{M}$ which satisfy* (1.1) *is a residual subset of $\mathscr{M}$. In particular $\mathscr{M}_F$ is nonempty and the Cauchy problem* (1.1) *has solutions.*

Let f: $\boldsymbol{R} \times X \to X$ be continuous. By a *solution* of the Cauchy problem

$$\dot{x} = f(x, t), \qquad x(t_0) = u \tag{2.3}$$

($u \in X$) we mean a continuously differentiable function x: $I_c \to X$, $I_c = (t_0 - c, t_0 + c)$, $c > 0$, satisfying (2.3) for each $t \in I_c$.

We are going to show that there exist multifunctions F: $\boldsymbol{R} \times X \to \mathscr{K}$ satisfying the hypotheses of Theorem 2.4 (thus the Cauchy problem (1.1) has solutions) while, for each continuous selection f of F, the Cauchy problem (2.3) has no solution.

In the following theorem X stands for the real infinite dimensional Hilbert space 1_2 and, accordingly, the spaces $\mathscr{K}$ and $\mathscr{B}$ are supposed to consist of subsets of 1_2.

Theorem 2.5. *There exists a Hausdorff continuous multifunction F: $\boldsymbol{R} \times X \to \mathscr{K}$ ($X = 1_2$) satisfying the following properties*: (j) $\overline{\text{co}}\, F(t, x) \in \mathscr{B}$ *for each* $(t, x) \in \boldsymbol{R} \times X$; (jj) *there exists a Hausdorff continuous multifunction U: $\boldsymbol{R} \times X \to \mathscr{B}$ such that $\partial U(t, x) \cap \overline{\text{co}}\, F(t, x) = F(t, x)$ for each $(t, x) \in \boldsymbol{R} \times X$*; (jjj) *$h(F(t, x), 0) < 2$ for each $(t, x) \in \boldsymbol{R} \times X$*; (jv) *the set $\{f_i\}$ of all continuous selections f_i of F is denumerable*; (v) *for each $i \in \boldsymbol{N}$ and u in a neighborhood of the origin (depending on i) the Cauchy problem*

$$\dot{x} = f_i(t, x), \qquad x(0) = u,$$

has no solution; (vj) *the Cauchy problem*

$$\dot{x} \in F(t, x), \qquad x(0) = x_0,$$

has solutions for each $x_0 \in X$.

§ 3. Proof of Propositions 2.1 and 2.2

Proof of Proposition 2.1. The statements (a_1) and (a_2) follow easily from the definition (2.2).

(a_3) Let $\varepsilon > 0$. For each $i = 1, 2, \cdots, k$, put $b_i = \|u_i - f(t_i, x_i)\|_{H(t,x)}$ and let $b_i \leq r_i < b_i + \varepsilon$ be such that $u_i - f(t_i, x_i) \in r_i H(t, x)$. Hence

$$\sum_{i=1}^{k} \mu_i [u_i - f(t_i, x_i)] \in \left(\sum_{i=1}^{k} \mu_i r_i \right) H(t, x),$$

from which

$$\left\| \sum_{i=1}^{k} \mu_i [u_i - f(t_i, x_i)] \right\|_{H(t,x)} \leq \sum_{i=1}^{k} \mu_i r_i < \sum_{i=1}^{k} \mu_i b_i + \varepsilon$$

follows. Since $\varepsilon>0$ is arbitrary, (a_3) is true.

(a_4) Suppose $u \in \partial U(t, x)$. We have $u-f(t, x) \in U(t, x)-f(t, x)=H(t, x)$, thus $b\leq 1$ where $b=\|u-f(t, x)\|_{H(t,x)}$. Assume $b<1$. Then, for some $b\leq r<1$, we have $u-f(t, x) \in rH(t, x)$ and hence, $u-f(t, x)+(1-r)H(t, x)\subset H(t, x)$. Since the origin is in the interior of $H(t, x)$, there is a $\theta>0$ such that $\theta S\subset(1-r)H(t, x)$ and so $u-f(t, x)+\theta S\subset H(t, x)$. Therefore $u+\theta S\subset H(t, x)+f(t, x)=U(t, x)$, a contradiction. Hence $b=1$. Conversely, let $b=1$. Suppose that $u \in \operatorname{int} U(t, x)$, that is $u+\theta S\subset U(t, x)$ for some $\theta>0$. Let $1<r<2$ be such that $(r-1)H(t, x)\subset\theta S$. We have $u-f(t, x)+(r-1)H(t, x)\subset u+\theta S-f(t, x)\subset U(t, x)-f(t, x)=H(t, x)$, that is $u-f(t, x)+(r-1)H(t, x)\subset(2-r)H(t, x)+(r-1)H(t, x)$ and, by Rådström's cancellation rule [21], $u-f(t, x) \in (2-r)H(t, x)$. This implies $b\leq 2-r$, that is $r\leq 1$, a contradiction. Now suppose $u \in \operatorname{int}(X\backslash U(t, x))$. Since $b=1$ there is a sequence $\{s_k\}$ ($s_k>1$) converging to 1, such that $u-f(t, x) \in s_kH(t, x)$, $k \in N$. It follows $u-f(t, x) \in H(t, x)+(s_k-1)H(t, x)$ and so $u \in U(t, x)+(s_k-1)H(t, x)$. Letting $k\to+\infty$, a contradiction follows. Therefore $u \in \partial U(t, x)$.

(a_5) Let $\varepsilon>0$. Let $b=\|u-f(t_1, x_1)\|_{H(t,x)}$. There is $b\leq r<b+\varepsilon$ such that $u-f(t_1, x_1) \in rH(t, x)$. Since $\rho(t, x)\bar{S}\subset H(t, x)$, we have $v \in (|v|/\rho(t, x))\rho(t, x)\bar{S}\subset (|v|/\rho(t, x))H(t, x)$, thus

$$u-f(t_1, x_1)+v \in rH(t, x)+\frac{|v|}{\rho(t, x)}H(t, x)=\left(r+\frac{|v|}{\rho(t, x)}\right)H(t, x).$$

From this, the second inequality in (a_5) follows at once. Let us prove the first inequality. Set $b_1=\|u-v-f(t_1, x_1)\|_{H(t,x)}$. Take $b_1\leq r_1<b_1+\varepsilon$ such that $u-v-f(t_1, x_1) \in r_1H(t, x)$. Hence $u-f(t_1, x_1) \in v+r_1H(t, x)\subset(|v|/\rho(t, x)+r_1)H(t, x)$, from which the first inequality in (a_5) follows at once.

(a_6) Let $\varepsilon>0$. There is $b\leq r<b+\varepsilon$, where $b=\|u-f(t_1, x_1)\|_{H(t_1,x_1)}$, such that $u-f(t_1, x_1) \in rH(t_1, x_1)$. Clearly $H(t_1, x_1)\subset H(t, x)+(h_0+\varepsilon)S$, $h_0=h(H(t, x), H(t_1, x_1))$. Thus

$$u-f(t_1, x_1) \in r\left[H(t, x)+\left(\frac{h_0+\varepsilon}{\theta}\right)\theta S\right]\subset r\left(1+\frac{h_0+\varepsilon}{\theta}\right)H(t, x),$$

because $\theta S\subset H(t, x)$. Hence

$$\|u-f(t_1, x_1)\|_{H(t,x)}\leq r\left(1+\frac{h_0+\varepsilon}{\theta}\right)<(b+\varepsilon)\left(1+\frac{h_0+\varepsilon}{\theta}\right)$$

and, since $\varepsilon>0$ is arbitrary, (a_6) is proved.

(a_7) Since $u-f(t_1, x_1) \in (\theta^{-1}|u-f(t_1, x_1)|)\theta\bar{S}$, and $\theta\bar{S}\subset H(t, x)$, also ($a_7$) is true. This completes the proof.

Observe that for each $u \in X$ and $A \in \mathscr{K}$, $e(u, sA)$ is continuous as function of

$s \geq 0$. To prove Proposition 2.2 we use the following lemma the proof of which is routine and is omitted.

Lemma 3.1. *Let* $\hat{u}(t) = \sum_i u_i \chi_{I_i}(t)$, $\hat{K}(t) = \sum_j K_j \chi_{I'_j}(t)$, *where* $u_i \in X$, $K_j \in \mathscr{B}$, *and* $t \in I$. *Suppose that the sets* I_i, *with i in a countable set (resp.* I'_j, *with j in a countable set) are measurable, pairwise disjoint, and such that* $\bigcup_i I_i = I$ *(resp.* $\bigcup_j I'_j = I$). *Then* $e(\hat{u}(t), \hat{K}(t))$ *is a measurable function of* $t \in I$.

Proof of Proposition 2.2. Set $u(t) = z(t) - f(t, x(t))$, $K(t) = H(t, x(t))$, $t \in I$. We have

$$b(t) = \|z(t) - f(t, x(t))\|_{H(t, x(t))} = \inf\{s > 0 \mid u(t) \in sK(t)\}$$
$$= \inf\{s > 0 \mid e(u(t), sK(t)) = 0\}, \qquad t \in I.$$

Since u is measurable and K is Hausdorff continuous, there exist sequences $\{\hat{u}_n\}$ and $\{\hat{K}_n\}$, with $\hat{u}_n$ and $\hat{K}_n$ satisfying the hypotheses of Lemma 3.1, such that $\hat{u}_n \to u$ a.e. in I, and $\hat{K}_n \to K$ uniformly on I. Set $c(t, s) = e(u(t), sK(t))$, $(t, s) \in I \times \boldsymbol{R}^+$. For fixed $t \in I$, $c(t, s)$ is continuous as function of $s \in \boldsymbol{R}^+$. For fixed $s \in \boldsymbol{R}^+$, $c(t, s) = \lim_{n \to +\infty} e(\hat{u}_n(t), s\hat{K}_n(t))$ for $t \in I$ a.e., thus using Lemma 3.1 it follows that $c(t, s)$ is a measurable function of $t \in I$. By [16, Theorem 6.4], the multifunction $t \mapsto \{s > 0 \mid c(t, s) = 0\}$ is measurable. Hence by [16, Theorem 6.6], so is the multifunction $t \mapsto \inf\{s > 0 \mid c(t, s) = 0\}$, and the measurability of b is proved. Since, for each $t \in I$, we have $H(t, x(t)) \supset \rho_0 \bar{S}$, where $\rho_0 = \min\{\rho(t, x(t)) \mid t \in I\}$ is positive, by Proposition 2.1 (a_7) we have $b(t) \leq \rho_0^{-1} |z(t) - f(t, x(t))|$ a.e. in I and so b is bounded. This completes the proof.

§ 4. Proof of Theorem 2.3 ($\mathscr{M}_\sigma$ is open)

In this section we shall prove that for each $0 < \sigma < 1$ the set $\mathscr{M}_\sigma$ is open in $\mathscr{M}$. It will suffice to show that $\tilde{\mathscr{M}}_\sigma = \mathscr{M} \setminus \mathscr{M}_\sigma$ is closed in $\mathscr{M}$. To this end let us consider a sequence $\{x_n\} \subset \tilde{\mathscr{M}}_\sigma$ which converges uniformly to $x \in \mathscr{M}$. We want to prove that $x \in \tilde{\mathscr{M}}_\sigma$. Since $\{\dot{x}_n\}$ is a bounded sequence contained in the reflexive Banach space $L^2(I, X)$, there exists a subsequence, say $\{\dot{x}_n\}$, which converges weakly to some $\omega \in L^2(I, X)$. By a corollary to Mazur's theorem [15, p. 36], there exists a sequence $\{\sum_{i=0}^{k_n} \mu_i^n \dot{x}_{n+i}\}$ ($\mu_i^n \geq 0$, $\sum_{i=0}^{k_n} \mu_i^n = 1$) which converges strongly to ω in $L^2(I, X)$ and so also in $L^1(I, X)$. A simple computation gives $x(t) = x_0 + \int_{t_0}^t \omega(s) ds$, from which one obtains $\dot{x}(t) = \omega(t)$ a.e. in I. Let $0 < \varepsilon < \rho_0$, where $\rho_0 = \min\{\rho(t, x(t)) \mid t \in I\}$. Since the multifunction H is Hausdorff continuous and $\{x_n\}$ converges uniformly to x, by Lebesgue's covering lemma, one can find and integer $n_0 \in \boldsymbol{N}$ such that

$$h(H(t, x_n(t)), H(t, x(t))) < \varepsilon \qquad \text{for each } n \geq n_0, \ t \in I. \tag{4.1}$$

Now, set $b(t)=\|\dot{x}(t)-f(t,x(t))\|_{H(t,x(t))}$, $t\in I$ a.e.. We have

$$b(t)=\left\|\sum_{i=0}^{k_n}\mu_i^n[\dot{x}_{n+i}(t)-f(t,x_{n+i}(t))+p_n(t)+q_n(t)]\right\|_{H(t,x(t))}$$

where

$$p_n(t)=\dot{x}(t)-\sum_{i=0}^{k_n}\mu_i^n\dot{x}_{n+i}(t),\qquad q_n(t)=\sum_{i=0}^{k_n}\mu_i^n[f(t,x_{n+i}(t))-f(t,x(t))].$$

Hence, by Proposition 2.1 (a_3), (a_5),

$$\begin{aligned}b(t)&\le\sum_{i=0}^{k_n}\mu_i^n\|\dot{x}_{n+i}(t)-f(t,x_{n+i}(t))+p_n(t)+q_n(t)\|_{H(t,x(t))}\\&\le\sum_{i=0}^{k_n}\mu_i^n\|\dot{x}_{n+i}(t)-f(t,x_{n+i}(t))\|_{H(t,x(t))}+\frac{|p_n(t)|+|q_n(t)|}{\rho(t,x(t))},\quad t\in I\quad\text{a.e..}\end{aligned}\tag{4.2}$$

Since $H(t,x(t))\supset\rho_0\bar{S}$, $t\in I$, by Proposition 2.1 (a_6), we have

$$\begin{aligned}\|\dot{x}_{n+i}(t)-f(t,x_{n+i}(t))\|_{H(t,x(t))}&\le\|\dot{x}_{n+i}(t)-f(t,x_{n+i}(t))\|_{H(t,x_{n+i}(t))}\\&\quad\times[1+\rho_0^{-1}h(H(t,x(t)),H(t,x_{n+i}(t)))],\quad t\in I\quad\text{a.e..}\end{aligned}$$

Observe that by (4.1) the quantity in brackets is less than $1+\rho_0^{-1}\varepsilon$ whenever $n\ge n_0$. Now, fix $n\ge n_0$. Then from (4.2) we obtain

$$b(t)\le\left(1+\frac{\varepsilon}{\rho_0}\right)\sum_{i=0}^{k_n}\mu_i^n\|\dot{x}_{n+i}(t)-f(t,x_{n+i}(t))\|_{H(t,x_{n+i}(t))}+\frac{|p_n(t)|+|q_n(t)|}{\rho_0},\qquad t\in I\quad\text{a.e..}$$

Therefore

$$\begin{aligned}\int_I b(t)dt&\le\left(1+\frac{\varepsilon}{\rho_0}\right)\sum_{i=0}^{k_n}\mu_i^n\int_I\|\dot{x}_{n+i}(t)-f(t,x_{n+i}(t))\|_{H(t,x_{n+i}(t))}dt\\&\quad+\frac{1}{\rho_0}\int_I(|p_n(t)|+|q_n(t)|)dt.\end{aligned}$$

Hence, dividing by $|I|$ and taking into account that $x_{n+i}\in\mathscr{M}_\sigma$, it follows

$$\frac{1}{|I|}\int_I b(t)dt\le\left(1+\frac{\varepsilon}{\rho_0}\right)\sigma+\frac{1}{|I|\rho_0}\int_I(|p_n(t)|+|q_n(t)|)dt.$$

Since the integral on the right hand side vanishes as $n\to+\infty$, and $\varepsilon>0$ is arbitrary, we obtain

$$\frac{1}{|I|}\int_I b(t)dt\le\sigma.$$

Thus $x\in\mathscr{M}_\sigma$ and the proof is complete.

§ 5. Proof of Theorem 2.3 ($\mathcal{M}_\sigma$ is dense)

In this section we shall prove that for each $0<\sigma<1$ the set $\mathcal{M}_\sigma$ is dense in $\mathcal{M}$. Indeed, fix $\bar{x} \in \mathcal{M}$ and let $\varepsilon>0$. From the definition of $\mathcal{M}$, it follows that there exists an $\tilde{x} \in \mathcal{V}_{r_{k_0}}$ (for some $k_0 \in N$) such that $|\tilde{x}(t)-\bar{x}(t)|\leq\varepsilon/2$, $t \in I$. Choose $k>k_0$ such that $\sigma<r_k<1$ ($\{r_k\}$) has been defined in Section 2). The density of $\mathcal{M}_\sigma$ in $\mathcal{M}$ is certainly established if we find an $x \in \mathcal{V}_{r_{k+2}}$ satisfying both the inequalities:

(5.1) $$|x(t)-\tilde{x}(t)|\leq\varepsilon/2 \qquad t \in I,$$

(5.2) $$\frac{1}{|I|}\int_I \|\dot{x}(t)-f(t, x(t))\|_{H(t,x(t))}dt>r_k.$$

Since $\tilde{x}$ is a polygonal solution of (2.1) (with k_0 in the place of k), there exists a countable family $\{I_q\}$ of nonempty pairwise disjoint open intervals $I_q\subset I$, with $m(I\setminus\bigcup_q I_q)=0$, such that $\dot{\tilde{x}}$ is constant in each interval I_q. Without loss of generality we can and do assume that no I_q contains t_0. Consider an arbitrary I_q ane let $\tau \in I_q$. On each (sufficiently small) closed interval $J_{\tau,\delta}=[\tau-\delta, \tau+\delta]$ ($\delta>0$) we shall define a function $x_{\tau,\delta}$: $J_{\tau,\delta}\to X$ which enjoys the properties stated in the following Proposition 5.1. Our purpose is to construct the desired $x \in \mathcal{V}_{r_{k+2}}$, satisfying (5.1) and (5.2) by pasting conveniently a countable subfamily of the $x_{\tau,\delta}$'s.

Set $D_1=\{t \in \boldsymbol{R}\,|\,|t-t_0|<a\}\times\{x \in X\,|\,|x-x_0|<R\}$. Clearly $D_1\subset D_2\subset D$. We recall that a, R, D_2, D are defined in Section 2.

Proposition 5.1. *Let $\tilde{x} \in \mathcal{V}_{r_{k_0}}$ and let $\varepsilon>0$. Let τ be in an open interval I_q from the countable family $\{I_q\}$ (where no I_q contains t_0). Let $k \in N$ ($k>k_0$) be such that $\sigma<r_k<1$. Then there is a $\delta_0=\delta_0(\tau, \varepsilon)>0$ such that for each $0<\delta<\delta_0$ the multivalued differential equation*

(5.3) $$\dot{x} \in \operatorname{int} G_{r_{k+2}}(t, x)$$

admits a polygonal solution $x_{\tau,\delta}$: $J_{\tau,\delta}\to X$ ($J_{\tau,\delta}=[\tau-\delta, \tau+\delta]\subset I_q$, $\delta>0$) satisfying

(5.4) $$x_{\tau,\delta}(\tau\pm\delta)=\tilde{x}(\tau\pm\delta)$$

(5.5) $$\dot{x}_{\tau,\delta}(t)-f(t, x_{\tau,\delta}(t)) \notin r_k H(t, x_{\tau,\delta}(t)) \qquad t \in J_{\tau,\delta} \quad \text{a.e.}$$

(5.6) $$|x_{\tau,\delta}(t)-\tilde{x}(t)|\leq\varepsilon/2 \qquad t \in J_{\tau,\delta}.$$

We postpone the lengthy proof of Proposition 5.1. Using this proposition we are ready to complete the proof of the density of $\mathcal{M}_\sigma$ in $\mathcal{M}$. To this end observe that the family $\{J_{\tau,\delta}\,|\,\tau \in I_q, 0<\delta<\delta_0(\tau, \varepsilon)\}$ of nondegenerate closed intervals $J_{\tau,\delta}\subset I_q$ is a Vitali's covering of I_q. Evidently no $J_{\tau,\delta}$ contains t_0. By Vitali's theorem there exists a countable subfamily of pairwise disjoint closed intervals $J_i^q=J_{\tau_i,\delta_i}\subset I_q$ such that $m(I_q\setminus\bigcup_i J_i^q)=0$. Repeating this procedure for any other interval of the countable family $\{I_q\}$ and relabelling the family of all J_i^q, we obtain a countable family, $\mathfrak{J}=\{J_i\}$, of pairwise disjoint nondegenerate closed intervals $J_i \subset I$ such that $t_0 \notin J_i$ and $m(I\setminus\bigcup_i J_i)=0$.

Let x_{τ_i,δ_i} correspond to $J_i \in \mathfrak{J}$. Set

$$\omega(t)=\sum_i \dot{x}_{\tau_i,\delta_i}(t)\chi_{J_i}(t) \qquad t \in I \quad \text{a.e.}$$

$$x(t)=x_0+\int_{t_0}^{t} \omega(s)ds \qquad t \in I.$$

It is easy to see that $x(t)=x_{\tau_i,\delta_i}(t)$ on each interval $J_i \in \mathfrak{J}$. To this end, consider an arbitrary $J_i \in \mathfrak{J}$ and set $J_i=[a_i, b_i]$. We have $t_0 \notin J_i$. Now, suppose $a_i>t_0$ (whenever $a_i<t_0$ the proof is similar). Denote by $\{J'_j\}$ the subfamily of $\mathfrak{J}$ consisting of those intervals belonging to the family $\mathfrak{J}$ which are contained in $[t_0, a_i]$. Notice that $[t_0, a_i]\setminus\bigcup_j J'_j$ has measure zero, and at the end points of each J'_j we have $x_{\tau_j,\delta_j}(\tau_j \pm \delta_j)=\tilde{x}(\tau_j \pm \delta_j)$. Therefore we can write

$$x(a_i)=x_0+\sum_j \int_{J'_j} \dot{x}_{\tau_j,\delta_j}(s)ds=x_0+\sum_j \int_{J'_j} \dot{\tilde{x}}(s)ds=\tilde{x}(a).$$

It follows $x(a_i)=x_{\tau_i,\delta_i}(a_i)$, since $x_{\tau_i,\delta_i}(a_i)=\tilde{x}(a_i)$, thus

$$x(t)=x(a_i)+\int_{a_i}^{t} \dot{x}_{\tau_i,\delta_i}(s)ds=x_{\tau_i,\delta_i}(a_i)+\int_{a_i}^{t} \dot{x}_{\tau_i,\delta_i}(s)ds=x_{\tau_i,\delta_i}(t), \quad t \in J_i.$$

Taking into consideration the definition of x and the fact that $x(t)=x_{\tau_i,\delta_i}(t)$, on each closed interval J_i of the countable family $\mathfrak{J}$, one has that x is a polygonal solution of (2.1) (with r_{k+2} in the place of r_k), that is $x \in \mathscr{V}_{r_{k+2}}$; furthermore x satisfies (5.1), and $\dot{x}(t)-f(t, x(t)) \notin r_k H(t, x(t))$, $t \in I$ a.e.. This implies that $\|\dot{x}(t)-f(t, x(t))\|_{H(t,x(t))}>r_k$ a.e. in I and thus, integrating on I, (5.2) follows. This completes the proof.

§ 6. Proof of Proposition 5.1

Let A, B be nonempty subsets of X. We set $w(A, B)=\inf\{|a-b|\,|\,a \in A, b \in B\}$.

Proof of Proposition 5.1. Let $\tilde{x} \in \mathscr{V}_{r_{k_0}}$ and let $\varepsilon>0$. Let t be in an interval $I_q=(b_1, b_2)$ from the countable family $\{I_q\}$ defined in Section 4. Take $k \in N$ $(k>k_0)$ so that $\sigma<r_k<1$, where $\{r_k\}$ is the sequence introduced in Section 2. Let θ satisfy

$$0<\theta<\rho_0 \min\{r_{k+1}-r_k, r_{k+2}-r_{k+1}\},$$

where $\rho_0=\min\{\rho(t, \tilde{x}(t))\,|\,t \in I\}$. Evidently $\rho_0>0$. Since U_{r_k}, $G_{r_{k+2}}$, and G_{r_k} are Hausdorff continuous at $(\tau, \tilde{x}(\tau)) \in D_1$, there exists a d, $0<d<\min\{a, R\}$, such that

(6.1) $$h(U_{r_k}(t, u), U_{r_k}(\tau, \tilde{x}(\tau)))<\theta/4$$

(6.2)$_s$ $$h(G_s(t, u), G_s(\tau, \tilde{x}(\tau)))<\theta/4 \quad (s=r_{k+1}, r_{k+2}),$$

for all $(t, u) \in S((\tau, \tilde{x}(\tau)), d)\subset D_2$. We have $U_{r_{k+1}}(t, u)=U_{r_k}(t, u)+(r_{k+1}-r_k)H(t, u)$, $(t, u) \in D$. From this (for $(t, u)=(\tau, \tilde{x}(\tau))$, observing that $H(\tau, \tilde{x}(\tau))\supset\rho(\tau, \tilde{x}(\tau))S\supset \rho_0 S$ and $\rho_0(r_{k+1}-r_k)>\theta$, we obtain

(6.3) $$U_{r_{k+1}}(\tau, \tilde{x}(\tau))\supset U_{r_k}(\tau, \tilde{x}(\tau))+\theta S.$$

Similarly

(6.4) $$G_{r_{k+2}}(\tau, \tilde{x}(\tau))\supset G_{r_{k+1}}(\tau, \tilde{x}(\tau))+\theta S.$$

From (6.2)$_{r_{k+2}}$, (6.4) and (6.2)$_{r_{k+1}}$ it follows

$$G_{r_{k+2}}(t, u)+\frac{\theta}{4}S\supset G_{r_{k+2}}(\tau, \tilde{x}(\tau))\supset G_{r_{k+1}}(\tau, \tilde{x}(\tau))+\frac{\theta}{4}S+\frac{3}{4}\theta S$$
$$\supset G_{r_{k+1}}(t, u)+\frac{3}{4}\theta S=G_{r_{k+1}}(t, u)+\frac{\theta}{2}S+\frac{\theta}{4}S$$

and, by Rådström cancellation rule [21],

(6.5) $$G_{r_{k+2}}(t, u)\supset G_{r_{k+1}}(t, u)+\frac{\theta}{2}S \qquad \text{for each } (t, u) \in S((\tau, \tilde{x}(\tau)), d).$$

Now, set

(6.6) $$C(\tau, \tilde{x}(\tau))=F_{r_{k+1}}(\tau, \tilde{x}(\tau))+\frac{\theta}{4}S,$$

where $F_r(t, u)=f(t, u)+r[F(t, u)-f(t, u)]$, $(t, u) \in D$, $0\leq r\leq 1$. Observe that for each $\alpha>0$ we have

$$G_r(t, u)=\overline{\text{co}}\, F_r(t, u)\subset \text{co}\, F_r(t, u)+\alpha S.$$

From (6.6), by virtue of (6.2)$_{r_{k+1}}$ and (5.5), we obtain

$$C(\tau, \tilde{x}(\tau))\subset \text{co}\, F_{r_{k+1}}(\tau, \tilde{x}(\tau))+\frac{\theta}{4}S\subset G_{r_{k+1}}(\tau, \tilde{x}(\tau))+\frac{\theta}{4}S$$
$$\subset G_{r_{k+1}}(t, u)+\frac{\theta}{4}S+\frac{\theta}{4}S\subset G_{r_{k+2}}(t, u)$$

and hence

(6.7) $$C(\tau, \tilde{x}(\tau))\subset \text{int}\, G_{r_{k+2}}(t, u) \qquad \text{for each } (t, u) \in S((\tau, \tilde{x}(\tau)), d).$$

On the other hand (6.3) implies

$$w\left(U_{r_k}(\tau, \tilde{x}(\tau))+\frac{\theta}{4}S,\ \partial U_{r_{k+1}}(\tau, \tilde{x}(\tau))+\frac{\theta}{4}S\right)>\frac{\theta}{4}.$$

From (6.1) one has $U_{r_k}(t, u)\subset U_{r_k}(\tau, \tilde{x}(\tau))+(\theta/4)S$, $(t, u)\in S((\tau, \tilde{x}(\tau)), d)$, while from (6.6)

$$\begin{aligned} C(\tau, \tilde{x}(\tau)) &\subset f(\tau, \tilde{x}(\tau))+r_{k+1}[F(\tau, \tilde{x}(\tau))-f(\tau, \tilde{x}(\tau))]+\frac{\theta}{4}S \\ &\subset f(\tau, \tilde{x}(\tau))+r_{k+1}[\partial U(\tau, \tilde{x}(\tau))-f(\tau, \tilde{x}(\tau))]+\frac{\theta}{4}S \\ &=\partial U_{r_{k+1}}(\tau, \tilde{x}(\tau))+\frac{\theta}{4}S, \end{aligned}$$

thus

$$\text{(6.8)}\qquad w(U_{r_k}(t, u),\ C(\tau, \tilde{x}(\tau)))>\frac{\theta}{4}\qquad \text{for each } (t, u)\in S((\tau, \tilde{x}(\tau)), d).$$

Now we are ready to construct the functions $x_{\tau,\delta}$ enjoying the properties stated in Proposition 5.1. Let $\tilde{x}$, τ, I_q, r_k, θ be as at the beginning of the proof. Recall that on the interval $I_q=(b_1, b_2)$ the function $\tilde{x}\in\mathscr{V}_{r_{k_0}}$ has constant derivative, say ξ; in particular at the point $\tau\in I_q$ we have $\dot{\tilde{x}}(\tau)=\zeta$. Evidently $\mathscr{V}_{r_{k_0}}\subset\mathscr{V}_{r_k}$, being $k>k_0$, thus $\tilde{x}\in\mathscr{V}_{r_k}$. Therefore $\xi\in\operatorname{int} G_{r_k}(\tau, \tilde{x}(\tau))$. Since

$$\begin{aligned} G_{r_k}(\tau, \tilde{x}(\tau)) &\subset G_{r_{k+1}}(\tau, \tilde{x}(\tau))\subset \operatorname{co} F_{r_{k+1}}(\tau, \tilde{x}(\tau))+\frac{\theta}{4}S \\ &=\operatorname{co}\left[F_{r_{k+1}}(\tau, \tilde{x}(\tau))+\frac{\theta}{4}S\right]=\operatorname{co} C(\tau, \tilde{x}(\tau)), \end{aligned}$$

it follows that ξ can be expressed as a convex combination of points

$$\text{(6.9)}\qquad z_i\in C(\tau, \tilde{x}(\tau))\qquad i=1, 2, \cdots, n,$$

that is, $\xi=\sum_{i=1}^n \mu_i z_i$, $\mu_i\geq 0$, $\sum_{i=1}^n \mu_i=1$. Let

$$0<\delta_0<\min\left\{\frac{d}{4(M+1)},\ \frac{\varepsilon}{6M},\ \tau-b_1,\ b_2-\tau\right\}$$

where M is defined in Section 2 (hypothesis iii). For each $0<\delta<\delta_0$, consider the closed interval $J_{\tau,\delta}=[\tau-\delta, \tau+\delta]$, which is contained in I_q. Devide $J_{\tau,\delta}$ into n nonempty pairwise disjoint intervals $J_i^*\subset J_{\tau,\delta}$ such that $m(J_i^*)=\mu_i m(J_{\tau,\delta})$ $(i=1, 2, \cdots, n)$, and $J_1^*\cup J_2^*\cup\cdots\cup J_n^*=J_{\tau,\delta}$. Define

$$\text{(6.10)}\qquad x_{\tau,\delta}(t)=\tilde{x}(\tau-\delta)+\int_{\tau-\delta}^t \omega_{\tau,\delta}(s)ds,\quad \omega_{\tau,\delta}(t)=\sum_{i=1}^n z_i\chi_{J_i^*}(t)\quad t\in J_{\tau,\delta}.$$

To complete the proof it remains to be shown that $x_{\tau,\delta}$ satisfies (5.3)–(5.6).

Let us prove (5.4). We have

$$x_{\tau,\delta}(\tau+\delta)=\tilde{x}(\tau-\delta)+\int_{\tau-\delta}^{\tau+\delta}\omega_{\tau,\delta}(s)ds=\tilde{x}(\tau-\delta)+\sum_{i=1}^{n}z_i m(J_i^*)$$
$$=\tilde{x}(\tau-\delta)+\Big(\sum_{i=1}^{n}\mu_i z_i\Big)m(J_{\tau,\delta})=\tilde{x}(\tau-\delta)+\xi m(J_{\tau,\delta})$$
$$=\tilde{x}(\tau-\delta)+\int_{\tau-\delta}^{\tau+\delta}\dot{\tilde{x}}(s)ds=\tilde{x}(\tau+\delta).$$

Evidently $x_{\tau,\delta}(\tau-\delta)=\tilde{x}(\tau-\delta)$, thus (5.4) is true.

Let us prove that $x_{\tau,\delta}$ satisfies (5.3) a.e.. Observe that for each $t\in J_{\tau,\delta}$ we have $|t-\tau|\leq\delta$, and

(6.11) $$|x_{\tau,\delta}(t)-\tilde{x}(t)|\leq|x_{\tau,\delta}(t)-\tilde{x}(\tau-\delta)|+|\tilde{x}(\tau-\delta)-\tilde{x}(t)|<4\delta M.$$

Since $\delta<\delta_0<d$ and $4\delta M<d$, it follows

(6.12) $$(t, x_{\tau,\delta}(t))\in S((\tau, \tilde{x}(\tau)), d)\qquad\text{for each } t\in J_{\tau,\delta}.$$

Taking into consideration (6.10), (6.9), (6.12), and (6.7) we obtain

$$\dot{x}_{\tau,\delta}(t)\in C(\tau, \tilde{x}(\tau))\subset\operatorname{int} G_{r_{k+2}}(t, x_{\tau,\delta}(t))$$

for $t\in J_{\tau,\delta}$ a.e., thus also (5.3) is satisfied.

The inequality (5.6) follows from (6.11), because $\delta<\varepsilon/(6M)$.

Finally let us consider (5.5). Observe that (6.12) and (6.8) imply that, for each $t\in J_{\tau,\delta}$, the sets $U_{r_k}(t, x_{\tau,\delta}(t))$ and $C(\tau, \tilde{x}(\tau))$ have empty intersection. Thus, by virtue of (6.10) and (6.9) we obtain $\dot{x}_{\tau,\delta}(t)\notin U_{r_k}(t, x_{\tau,\delta}(t))$, that is

$$\dot{x}_{\tau,\delta}(t)-f(t, x_{\tau,\delta}(t))\notin r_k[U(t, x_{\tau,\delta}(t))-f(t, x_{\tau,\delta}(t))]=r_k H(t, x_{\tau,\delta}(t)),$$

for $t\in J_{\tau,\delta}$ a.e.. Hence also (5.5) is satisfied. This completes the proof of Proposition 5.1.

§ 7. A singular example of a multivalued differential equation

This section is devoted to the proof of Theorem 2.5. From now on, X will denote the real (infinite dimensional) Hilbert space l_2. S, $\bar{S}$ stand respectively for the unit balls $S(0, 1)$, $\bar{S}(0, 1)$ in X. The space $\boldsymbol{R}\times X$ is supposed to be endowed with norm $|(t, x)|=\max\{|t|, |x|\}$, $(t, x)\in\boldsymbol{R}\times X$.

To prove Theorem 2.5, we establish some lemmas.

Lemma 7.1. *Let* $0<d_0<1$. *Then there is a denumerable set* $E=\{e_i\}\subset\partial S$ *satisfying the properties*: (a$_1$) $|e_i-e_j|>d_0$, *if* $i\neq j$; (a$_2$) $\partial S\subset E+dS$ *for each* $d>d_0$;

(a_3) *for each d such that* $d_0 < d < 1$, *one has* $(1-d)S \subset \overline{\text{co}}\, E$, *that is* $\overline{\text{co}}\, E \in \mathscr{B}$.

Proof. Let $0 < d_0 < 1$. Let $\tilde{E} = \{\tilde{e}_1, \tilde{e}_2, \cdots\}$ be a denumerable dense subset of ∂S. We associate to $\tilde{E}$ the family $\{\bar{S}(\tilde{e}_i, d_0)\}$, which is a denumerable closed covering of ∂S. Set $e_1 = \tilde{e}_1$. Hence define $e_2 = \tilde{e}_{i_2}$, where i_2 is the smallest integer $i > 1$ such that $\tilde{e}_i \notin \bar{S}(e_1, d_0)$. Similarly $e_3 = \tilde{e}_{i_3}$, where i_3 is the smallest integer $i > i_2$ such that $\tilde{e}_i \notin \bar{S}(e_1, d_0) \cup \bar{S}(e_2, d_0)$. Continuing in this manner one obtains a countable set $E = \{e_1, e_2, \cdots\} \subset \partial S$. (The proof that E is actually denumerable is postponed). It is evident that E satisfies (a_1). To prove (a_2), let $d > d_0$ and consider any $x \in \partial S$. Choose $\tilde{e}_k \in \tilde{E}$ such that $|x - \tilde{e}_k| < d - d_0$. From the construction of E, there exists on $e_i \in E$ such that $|\tilde{e}_k - e_i| < d_0$. Then $|x - e_i| \leq |x - \tilde{e}_k| + |\tilde{e}_k - e_i| < (d - d_0) + d_0 = d$, and so $x \in S(e_i, d)$. Since $x \in \partial S$ is arbitrary, one has $\partial S \subset E + dS$, and ($a_2$) is proved. Consider ($a_3$). From $\partial S \subset E + dS$ $(d_0 < d < 1)$, it follows $S \subset \overline{\text{co}}\, E + dS$, thus $(1-d)S + dS \subset \overline{\text{co}}\, E + dS$. Hence, by Rådström cancellation rule [21], $\overline{\text{co}}\, E \supset (1-d)S$, thus $\overline{\text{co}}\, E \in \mathscr{B}$. Since $\overline{\text{co}}\, E$ has nonempty interior and X is infinite dimensional, it follows that X is denumerable. This completes the proofs.

Lemma 7.2. *Let* $g: \boldsymbol{R} \times X \to X$ *be continuous. Let* $\varepsilon > 0$. *Then there is a* $\delta_0 > 0$ *such that, for each* $0 < \delta \leq \delta_0$, *there exists a function* $f_\delta: \boldsymbol{R} \times X \to X$ *satisfying the following properties*:

(b_1) $f_\delta(0, 0) = g(0, 0)$, *and* $f_\delta(t, x) = g(t, x)$ *when* $(t, x) \notin S((0, 0), \delta)$;

(b_2) $|f_\delta(t, x) - g(t, x)| < \varepsilon$ *for each* $(t, x) \in \boldsymbol{R} \times X$;

(b_3) *for each* $u \in (\delta/3)S$, *the Cauchy problem* $\dot{x} = f_\delta(t, x)$, $x(0) = u$ *has no solution.*

Proof. By Godunov's theorem [13] there exists a continuous function $g_0: \boldsymbol{R} \times X \to X$, with $g_0(0, 0) = 0$ $(x \in X)$, such that the Cauchy problem $\dot{x} = g_0(t, x)$, $x(0) = u$ has no solution whatever may be $u \in X$. As in [20], define $g_b: \boldsymbol{R} \times X \to X$ by $g_b(t, x) = g_0(t, x - bt) + b$ $(b = g(0, 0))$ for each $(t, x) \in \boldsymbol{R} \times X$, and observe that the Cauchy problem $\dot{x} = g_b(t, x)$, $x(0) = u$ has no solution whatever may be $u \in X$. In fact, if there were a solution $x: I_c \to X$, $I_c = (-c, c)$, $c > 0$, then x would satisfy $\dot{x}(t) = g_0(t, x(t) - bt) + b$ for every $t \in I_c$, and $x(0) = u$. Thus the function $z: I_c \to X$ given by $z(t) = x(t) - bt$, $t \in I_c$, would be a solution of the Cauchy problem $\dot{x} = g_0(t, x)$, $x(0) = u$, a contradiction. Now, take $\varepsilon > 0$. Since g and g_b are continuous and assume at $(0, 0)$ the same value b, there exists a $\delta_0 > 0$ such that

$$|g(t, x) - b| < \varepsilon/2, \quad |g_b(t, x) - b| < \varepsilon/2 \quad \text{for each } (t, x) \in \bar{S}((0, 0), \delta_0).$$

For each fixed $0 < \delta \leq \delta_0$, consider the function φ_δ defined by

$$\varphi_\delta(t, x) = \begin{cases} g_b(t, x) & \text{if } (t, x) \in \bar{S}((0, 0), \delta/3) \\ g(t, x) & \text{if } (t, x) \in \bar{S}((0, 0), \delta) \backslash S((0, 0), (2/3)\delta). \end{cases}$$

Clearly φ_δ is defined on a closed set $\Delta\subset\bar{S}((0, 0), \delta)$, is continuous, and satisfies $|\varphi_\delta(t, x)-b|<\varepsilon/2$ for each $(t, x)\in\Delta$. By Dugundji's theorem [9, p. 188], φ_δ admits a continuous extension f_δ defined on $\bar{S}((0, 0), \delta)$, and satisfying $|f_\delta(t, x)-b|<\varepsilon/2$, for each $(t, x)\in\bar{S}((0, 0), \delta)$. Putting $f_\delta(t, x)=g(t, x)$ outside $\bar{S}((0, 0), \delta)$, one has that f_δ is continuous all over $\boldsymbol{R}\times X$. It is straightforward to verify that f_δ satisfies (b_1)–(b_3).

Lemma 7.3. *Let $E=\{e_i\}\subset\partial S$ be a denumerable set satisfying the properties* (a_1) *and* (a_2) *of Lemma* 7.1 *(with $0<d_0<1$). Let $\{\varepsilon_i\}$ be a strictly decreasing sequence of positive numbers ε_i converging to zero, such that $0<\varepsilon_1<\min\{(1-d_0)/2, d_0/4\}$. Then there exists a strictly decreasing sequence $\{\delta_i\}$ of positive numbers δ_i, converging to zero, and there is a sequence $\{f_i\}$ $(f_i=f_{\delta_i})$ of functions $f_i\colon \boldsymbol{R}\times X\to X$ such that:*

(c_1) *$|f_i(t, x)-e_i|<\varepsilon_i$ for each $(t, x)\in\boldsymbol{R}\times X$, $f_i(0, 0)=e_i$;*

(c_2) *for every $i, j\in\boldsymbol{N}$ $(i\neq j)$, we have $|f_i(t, x)-f_j(t, x)|>d_0/2$ for all $(t, x)\in\boldsymbol{R}\times X$;*

(c_3) *the family $\{f_i\}$ is equicontinuous at each point $(t, x)\in\boldsymbol{R}\times X$;*

(c_4) *for each $u\in(\delta_i/3)S$, the Cauchy problem*

$$\dot{x}=f_i(t, x), \qquad x(0)=u \tag{7,1}$$

has no solution.

Proof. Let $\{g_i\}$ be a sequence of functions $g_i\colon \boldsymbol{R}\times X\to X$ which are equicontinuous at each point $(t, x)\in\boldsymbol{R}\times X$ and satisfy $|g_i(t, x)-e_i|<\varepsilon_i/2$, $(t, x)\in\boldsymbol{R}\times X$, $g_i(0, 0)=e_i$. Evidently there do exist such sequences. By Lemma 7.2 (taking $g=g_i$ and $\varepsilon=\varepsilon_i/2$) one can find a strictly decreasing sequence $\{\delta_i\}$ of positive numbers δ_i converging to zero, and a sequence $\{f_i\}$ of continuous functions $f_i\colon \boldsymbol{R}\times X\to X$ such that:

(1) $f_i(0, 0)=e_i$, and $f_i(t, x)=g_i(t, x)$ when $(t, x)\notin S((0, 0), \delta_i)$;

(2) $|f_i(t, x)-g_i(t, x)|<\varepsilon_i/2$ for each $(t, x)\in\boldsymbol{R}\times X$;

(3) for each $u\in(\delta_i/3)S$, the Cauchy problem (7.1) has no solution.

We shall verify that the sequence $\{f_i\}$ satisfies (c_1)–(c_4). In fact, by construction $f_i(0, 0)=g_i(0, 0)=e_i$; moreover for each $(t, x)\in\boldsymbol{R}\times X$ we have

$$|f_i(t, x)-e_i|\le|f_i(t, x)-g_i(t, x)|+|g_i(t, x)-e_i|<\varepsilon_i/2+\varepsilon_i/2=\varepsilon_i,$$

and so (c_1) is true. Also (c_2) is satisfied because, for any $i\neq j$ $(i, j\in\boldsymbol{N})$ and $(t, x)\in\boldsymbol{R}\times X$, we have

$$|f_i(t, x)-f_j(t, x)|\ge|e_i-e_j|-\varepsilon_i-\varepsilon_j>d_0-2(d_0/4)=d_0/2.$$

To prove (c_3), fix a point $(\bar{t}, \bar{x})\neq(0, 0)$. Take $i_0\in\boldsymbol{N}$ such that $(\bar{t}, \bar{x})\notin\delta_i\bar{S}$ for each $i>i_0$. This is possible for $\delta_i\to 0$ as $i\to+\infty$. Hence there is a $\theta_0>0$ such that for all $(t, x)\in S((\bar{t}, \bar{x}), \theta_0)$ we have $f_i(t, x)=g_i(t, x)$ $(i>i_0)$ and thus, since $\{g_i\}$ is equicontinuous at $(\bar{t}, \bar{x})$ also $\{f_i\}$ is so. Now, suppose $(\bar{t}, \bar{x})=(0, 0)$. Let $\varepsilon>0$. Since $\varepsilon_i\to 0$ as $i\to+\infty$, by virtue of (c_1) there an $i_0\in\boldsymbol{N}$ such that, whenever $i>i_0$, we have

$|f_i(t, x)-e_i|<\varepsilon$ for every $(t, x) \in \boldsymbol{R}\times X$. From this and the continuity of the functions f_i $(1\leq i\leq i_0)$, it follows that $\{f_i\}$ is equicontinuous also at (0, 0). Hence (c_3) is true. Since (c_4) is trivially fulfilled, the proof is complete.

Remark 7.1. Let the hypotheses of Lemma 7.3 be satisfied. Let $\{f_i\}$ be a sequence of functions f_i: $\boldsymbol{R}\times X\to X$ satisfying properties (c_1)–(c_4), whose existence has been established in Lemma 7.3. Define F: $\boldsymbol{R}\times X\to\mathscr{K}$ by

$$F(t, x)=\bigcup_{i=1}^{\infty} f_i(t, x), \qquad (t, x) \in \boldsymbol{R}\times X. \tag{7.2}$$

Observe that from (c_2) and (c_3) it follows that F is a Hausdorff continuous multifunction with values $F(t, x) \in \mathscr{K}$. In addition, as consequence of (c_1), one has that $h(F(t, x), 0)<2$ for each $(t, x) \in \boldsymbol{R}\times X$.

The following lemma has been proved in [8].

Lemma 7.4. *In addition to the hypotheses of Lemma* 7.3, *suppose that* ε_1 *is such that* $0<\varepsilon_1<d_0^2/384$. *Then there is a Hausdorff continuous multifunction* U: $\boldsymbol{R}\times X\to\mathscr{B}$ *satisfying* $\partial U(t, x)\cap\overline{\mathrm{co}}\,F(t, x)=F(t, x)$, *for each* $(t, x) \in \boldsymbol{R}\times X$.

Now we are ready to prove Theorem 2.5.

Proof of Theorem 2.5. Let $E=\{e_i\}\subset\partial S$ and $\{\varepsilon_i\}$ be as in Lemma 7.3. In addition, suppose that ε_1 is such that $0<\varepsilon_1<d_0^2/384$. Consider the multifunction F: $\boldsymbol{R}\times X\to\mathscr{K}$ defined by (7.2). Clearly F is Hausdorff continuous and satisfies (jjj). Properties (jv) and (v) follow from Lemma 7.3 (c_2) (c_4), while (jj) follows from Lemma 7.4. Now, let us prove (j). Let $(t, x) \in \boldsymbol{R}\times X$. By Lemma 7.3 ($c_1$), we obtain $h(F(t, x), F(0, 0))<\varepsilon_1$ and so $E=F(0, 0)\subset F(t, x)+\varepsilon_1 S$. Set $\tilde{d}=(1+d_0)/2$. Since $\tilde{d}>d_0$, by Lemma 7.1 (a_2), we have $\partial S\subset E+\tilde{d}S\subset F(t, x)+(\varepsilon_1+\tilde{d})\bar{S}$, which implies

$$\bar{S}\subset\overline{\mathrm{co}}\,F(t, x)+(\varepsilon_1+\tilde{d})\bar{S}.$$

Note that $\varepsilon_1+\tilde{d}<(1-d_0)/2+(1+d_0)/2=1$. Replacing $\bar{S}$ by $(\varepsilon_1+\tilde{d})\bar{S}+(1-\varepsilon_1-\tilde{d})\bar{S}$ on the left hand side of the above inclusion, and using the Rådström's cancellation rule [21], we obtain $\overline{\mathrm{co}}\,F(t, x)\supset(1-\varepsilon_1-\tilde{d})\bar{S}$. Hence $\overline{\mathrm{co}}\,F(t, x) \in \mathscr{B}$ and, since $(t, x) \in \boldsymbol{R}\times X$ is arbitrary, also (j) is satisfied. Finally (vj) follows by Theorem 2.4, since F satisfies (j)–(jjj). This completes the proof.

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