On Systems of Variational Inequalities with Mixed Boundary Conditions

By

Kazuya Hayasida and Haruo Nagase

(Kanazawa University and Suzuka College of Technology, Japan)

§ 1. Introduction

In this paper we consider the solution $U$ of a system of variational inequalities with mixed boundary conditions. Our aim is to study the regularity behavior of $U$ at the $(n-2)$-dimensional submanifold $S$, across which a boundary condition is broken. Our main result is a $W^{m, p}$ estimate of $U$ near $S$ with $1 < p < 10/9$.

Let $\Omega$ be a smoothly bounded domain in $\mathbb{R}^n$. Let $\partial_1\Omega$ and $\partial_2\Omega$ be two disjoint open subsets of the boundary $\partial\Omega$ such that $\partial\Omega = \overline{\partial_1\Omega} \cup \partial_2\Omega$ and $S = \overline{\partial_1\Omega} \cap \overline{\partial_2\Omega}$. We first consider a closed convex set of $H^1(\Omega)$:

$$K = \{ u \in H^1(\Omega); u=0 \text{ on } \partial_1\Omega \text{ and } u \geq 0 \text{ on } \partial_2\Omega \}$$

and a solution $u \in K$ of the following single variational inequality

$$\langle Fu, V(v-u) \rangle \geq \langle f, v-u \rangle \text{ for all } v \in K$$

(1.1)

where $\langle , \rangle$ is the inner product of $L^2(\Omega)$ and $f$ is a given continuous function in $\overline{\Omega}$. It was proved by H. Beirão da Veiga [2] and by M.K.V. Murthy-G. Stampacchia [12] that such $u$ is Hölder continuous up to the boundary. Further the latter studied the regularity for $u$ in a Sobolev space $W^{m, p}(\Omega)$. Their method is to reduce the problem with obstacle on $\partial_2\Omega$ to a suitable one with obstacle throughout the whole of $\Omega$.

When $\partial_1\Omega = \emptyset$ in particular, the variational inequality (1.1) was previously treated by J.L. Lions [11] and H. Brézis [3]. In this case, M.G. Garroni [7] has recently given a regularity property in $W^{m, p}(\Omega)$ for nonlinear operators. When $\partial_1\Omega \neq \emptyset$, there is also a work of G.M. Troianiello [13], which gives an estimate of the form $-\Delta u \leq f+(-f)^+$.

Secondly we consider systems of variational inequalities in place of (1.1), where $K$ is replaced by a closed convex subset $K$ in $[H^1(\Omega)]^N$. If $V$ is a closed subspace of $[H^1(\Omega)]^N$ containing $[H^1_0(\Omega)]^N$ and, in particular, if $K$ equals

$$\{(u_1, \ldots, u_N) \in V; u_i \geq \psi_i \text{ in } \Omega, i=1, \ldots, N\},$$

1) Here we consider only the most simple form.
the regularity results were obtained by J. Frehse [6]. More precisely, in [6] an $L^2$ estimate with a local Morrey condition was obtained for the second order derivatives of solutions. Recently A.A. Arhipova [1] has treated diagonal systems for more extensive $K$, his result being an extension of J. Frehse's previous work [5] concerning single variational inequalities, where it is shown that their solutions are in a Sobolev space $W^{2,\infty}(\Omega)$.

On the other hand the study of variational inequalities for vector-valued functions is closely connected with differential geometric problems in parametric form. In this connection we note the work of S. Hildebrandt and K.O. Widman [8], where each $K$ is given in more general form, for instance,

$$K = \{(u_1, \ldots, u_N) \in [H^1(\Omega)]^N; u_i(x) \in Z \text{ a.e. in } \Omega \text{ for } i = 1, \ldots, N\}$$

and $Z$ is a closed convex set in $R^N$. For such $K$ they have obtained H"older continuity of solutions under certain assumptions. For other topics in this field we refer some results of [4], [15] for systems of variational inequalities.

The aim of this paper is the study of systems of variational inequalities with mixed boundary conditions. Form the above mentioned it is a natural course of events to consider $K$ as follows

$$K = \{(u_1, \ldots, u_N) \in [H^1(\Omega)]^N; u_i = 0 \text{ on } \partial_i \Omega \text{ and } u_i(x) \in Z \text{ a.e. in } \partial_j \Omega \text{ for } i = 1, \ldots, N\}.$$ 

Indeed, N.N. Ural'tseva [14] studied systems of variational inequalities with such $K$ to derive regularity of solutions except at $S$. Her technique is the penalty method and her result is that each solution is in $H^2(\Omega')$ for every compact subset $\Omega'$ of $\Omega - S$. However we are interested in the regularity behavior of solutions near $S$.

We will prove regularity results near $S$ for any vector-valued solution of variational inequalities (see Theorem 2 in Section 2). For this a weaker regularity theorem is prepared, which is valid for $\overline{\partial_i \Omega}$ irregdular at $S$ (see Theorem 1 in Section 2). In [10] this result has already been given for single linear elliptic equations under the assumption that $u = 0$ throughout $\partial \Omega$, that is, the Dirichlet data of $u$ is zero.

In the proof of Theorem 1, our variational method yields $H^1(\Omega)$ estimates with an adequate weight function. In order to prove Theorem 2, we shall use a parallel translation of solutions with a weight, where we use the same method as that in Theorem 8.6 of [11, Chapter 2] for the case of $\partial_i \Omega = \phi$. In this paper we consider only cases in the upper-half plane for the sake of brevity.

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\section*{§ 2. Preliminaries and Results}

Let $\beta$ be any fixed positive number less than $1/2$. In Section 5 we choose $\beta$ so that it is close to $1/2$. Let $r$ be a positive number such that
Hereafter we consider the case in the upper half space \( \{ x_n \geq 0 \} \) of \( \mathbb{R}^n \) with co-ordinates \((x_1, \ldots, x_n)\). Let \( r = (x_1^2 + \cdots + x_n^2)^{1/2} \), \( \rho = (x_{n-1}^2 + x_n^2)^{1/2} \) and \( \theta = \operatorname{Arg}(x_{n-1} + ix_n) \). We set
\[
\chi(x_{n-1}, x_n) = \rho^{-\beta} \sin(\beta \theta + \gamma).
\]
By (2.1) there is a positive constant \( c \leq 1 \) such that
\[
(2.2) \quad \chi \geq c \rho^{-\beta} \quad \text{for } x_n \geq 0.
\]

For any fixed \( R \) with \( 0 < R < 1 \) we define
\[
\Sigma = \{(x_1, \ldots, x_n); r < R, x_n > 0\}
\]
and
\[
\Gamma = \{x_n = 0\} \cap \{x_{n-1} < 0, x_1^2 + \cdots + x_{n-1}^2 < R^2\}.
\]
Let \( G \) be a subdomain of \( \Sigma \) such that the boundary \( \partial G \) contains \( \Gamma \) (see Figure 1). Thus \( \partial G \) may be non-smooth except for \( \Gamma \). We denote \( \partial G - \Gamma \) by \( \partial'G \). In what follows, we consider in \( G \) instead of \( \Omega \).

Now we set
\[
C_{(0)}^1(\overline{G}) = \{u \in C^1(\overline{G}); u = 0 \text{ in a neighborhood of } \partial'G\}.
\]
If \( G = \Sigma \) particularly, \( u \in C_{(0)}^1(\overline{G}) \) means that \( u \in C^1(\Sigma) \) and \( u = 0 \) near \( r = R \) and \( \{x_n = 0, x_{n-1} \geq 0\} \). The norm and the inner product in \( L^2(G) \) are simply denoted by \( \| \| \) and \((, )\) respectively. We define \( \|u\| = \|Fu\| + \|u\| \) for \( u \in C_{(0)}^1(\overline{G}) \), which is equivalent to \( \|Fu\| \) by Poincaré's inequality. The completion of \( C_{(0)}^1(\overline{G}) \) with respect to the norm \( \|u\| \) is denoted by \( H_{(0)}^1(G) \).

For any \( u, v \in C_{(0)}^1(\overline{G}) \) we define the inner product and the norm as follows:
\[
((u, v)) = (\chi^2 Fu, Fv) + (\chi^2 - \rho^{-2(\beta + 1)} u, v), \quad \|u\| = ((u, u))^{1/2}.
\]
where $\delta$ is any fixed positive number less than 1. In Section 5, $\delta$ may be chosen as close to 1 as desired. By virtue of (2.2), $||$ $||$ satisfies the condition of the norm. We denote by $\widetilde{H}_{(0)}^{1}(G)$ the completion of $C_{(0)}^{1}(\overline{G})$ with respect to the norm $||$ $||$.

When $X$ is a Sobolev space of real valued functions defined on $G$, $[X]^N$ denotes the space of $N$-vector valued functions with components in $X$. Let $U=(u_1, \ldots, u_N)$ and $V=(v_1, \ldots, v_N)$. Then we define

$$
\|U\|^2 = \sum_{i=1}^{N} \|u_i\|^2, \quad (U, V) = \sum_{i=1}^{N} (u_i, v_i),
$$

$$
||U||^2 = \sum_{i=1}^{N} ||u_i||^2, \quad ((U, V)) = \sum_{i=1}^{N} ((u_i, v_i)).
$$

For any set $\Omega \subset \mathbb{R}^n$ and for any $p>1$, $L^p(\Omega)$ is the class of all measurable functions $u$ defined in $\Omega$, for which $\int_{\Omega} |u(x)|^p \, dx < \infty$. We write

$$
\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u|^p \, dx \right)^{1/p} \quad \text{and} \quad \|U\|_{L^p(\Omega)} = \sum_{i=1}^{N} \|u_i\|_{L^p(\Omega)}.
$$

We consider the following bilinear form

$$
B[U, V] = \sum_{i=1}^{N} (\nabla u_i, \nabla v_i) + \sum_{i=1}^{N} (c_{ij}u_j, v_i),
$$

where $c_{ij} \in C^1(\overline{G})$ $(1 \leq i, j \leq N)$ and we assume

(2.3) \quad \max_{i,j,\delta} |c_{ij}| \leq 9c^\delta/(32NR^2),

where $c$ is the positive constant in (2.2). Obviously $B[U, V]$ is continuous in $[H_{(0)}^{1}(G)]^N \times [H_{(0)}^{1}(G)]^N$. Since $\max_{i,j,\delta} |c_{ij}| \leq (2NR)^{-1}$ from (2.3), we have

$$
|\sum_{i,j} (c_{ij}u_i, u_j)| \leq (\max_{i,j,\delta} |c_{ij}|) \sum_{i,j} \|u_i\| \cdot \|u_j\| \leq N(\max_{i,j,\delta} |c_{ij}|) \left( \sum_{i} \|u_i\|^2 \right).
$$

By Poincaré's inequality,

$$
\|u_t\|^2 \leq (2R)^2(2n)^{-1} \|\nabla u_t\|^2.
$$

More precisely, this follows from

$$
\|u_t\|^2 \leq \frac{1}{2} (2R)^2 \|\partial_x u_t\|^{2}. \tag{2.2}
$$

Hence we get

$$
|\sum_{i,j} (c_{ij}u_i, u_j)| \leq \frac{1}{2} \sum_{i} \|\nabla u_i\|^2.
$$

\[\text{See e.g., page 158 in the book of R. A. Adams "Sobolev Spaces."}\]
This implies that there is a positive constant \( c \) such that
\[
B[U, U] \geq c \|FU\|^2 \quad \text{for all } U \in [H^1_{00}(G)]^N.
\]
Accordingly \( B[U, V] \) satisfies Lax-Milgram's condition on \([H^1_{00}(G)]^N\).

Let \( Z \) be a closed convex set in \( R^N \) which contains the origin.

Then we set
\[
H = \{ V \in [C^1_{0}(\overline{G})]^N ; u(x) \in Z \text{ for all } x \in \Gamma \}.
\]

The completion of \( H \) with respect to the norm \( \| \| \) (resp. \( \|\|\| \)) is denoted by \( K \) (resp. \( \overline{K} \)). Clearly \( K \) (resp. \( \overline{K} \)) is a closed convex set in \([H^1_{00}(G)]^N \) (resp. \([\overline{H}^1_{00}(G)]^N \)) containing the zero vector of \( R^N \), respectively. It is well-known that for any given \( F \in [L^2(G)]^N \) there is a unique solution \( U \in K \) of
\[
(2.4) \quad B[U, V - U] \geq (F, V - U) \quad \text{for all } V \in K
\]
(see e.g., [9]).

For \( \varepsilon > 0 \) we denote by \( \Sigma_{\varepsilon} \) the set \( \{ r < R/2 \} \cap \{ x_{n-1} > \varepsilon |x_{n-1}| \} \). And we denote simply by \( \partial^k \) any \( k \)-th order derivative with respect to the variables \( x_j \) (1 \( \leq j \leq n \)).

Under the assumptions as above we shall prove the following theorem:

**Theorem 1.** Let \( U \) be the solution of (2.4). Then \( U \) is in \( \overline{K} \) and it satisfies
\[
|U| \leq C \rho^{1-\beta/2} F,
\]
where \( \beta, \delta \) are the numbers appearing in the definition of the norm \( \|\|\| \) and \( C \) is independent of \( F \) and \( U \).

**Theorem 2.** Suppose that \( G = \Sigma_{\varepsilon} \) in particular. Let \( U \) be the solution of (2.4). Then for any \( \varepsilon_1, \varepsilon_2 > 0 \) it holds that
\[
\partial^2 U \in [L^{10/9-\varepsilon_1}(\Sigma_{\varepsilon_2})]^N
\]
and
\[
|\partial^2 U|_{L^{10/9-\varepsilon_1}(\Sigma_{\varepsilon_2})} \leq C \rho^{5/4} F,
\]
where \( C \) is independent of \( F \) and \( U \), but depending on \( \varepsilon_1 \) and \( \varepsilon_2 \).

§ 3. **Lemmas**

In what follows, let \( 0 < \beta, \gamma < 1/2 \) and let us assume (2.1). Further let \( 0 < \delta < 1 \).

First we have

**Lemma 1.** Let \( \chi \) be as in the previous section. Then it holds that
\[ \Delta \chi^\delta = \beta^2 (\partial - 1) \chi^{\delta-2} \rho^{-2(\beta+1)}, \quad |F \chi^\delta| = \beta \partial \chi^{\delta-1} \rho^{-(\beta+1)} \]

and

\[ \partial_{x_n} \chi^\delta |_{x_n=0, x_{n-1}<0} < 0. \]

Proof. We can rewrite

\[ \chi = -\mathrm{Im} e^{-\gamma i} (x_{n-1} + ix_n)^{-\beta}, \]

which means that \( \chi \) is a harmonic function with respect to the variables \((x_{n-1}, x_n)\). On the other hand we see

\[ \partial_{x_{n-1}} \chi = -\beta \rho^{-(\beta+1)} \sin ((\beta+1)\theta + \gamma) \]

and

\[ \partial_{x_n} \chi = \beta \rho^{-(\beta+1)} \cos ((\beta+1)\theta + \gamma), \]

from which

\[ |F \chi| = \beta \rho^{-(\beta+1)}. \]

Hence the first and second equalities follow. The last inequality is valid from (2.1). Q.E.D.

Lemma 2. Let \( F \in [L^2(G)]^N \) and \( V \in [\tilde{H}_0^1(G)]^N \). Then

\[ |(\chi^\delta F, V)| \leq C \rho^{1-\beta(\delta/2)} |F| \cdot ||V||. \]

Proof. By Schwarz' inequality we have

\[ |(\chi^\delta F, V)| \leq C \rho^{1-\beta(\delta/2)} |F| \cdot \rho^{-1-\beta(\delta/2)} |V| \leq C \rho^{1-\beta(\delta/2)} |F| \cdot ||V||. \] Q.E.D.

In succession we consider cases in the upper half space \( \{x_n \geq 0\} \). Let \( \alpha \) be any fixed real number with \( 4/5 < \alpha < 1 \), which may be chosen so close to \( 4/5 \). Let \( \tilde{\rho} = (x_{n-1}^2 + x_n^{2\alpha})^{1/2} \). Since \( x_n \leq \tilde{\rho}^{1/\alpha} \), we have

\[ (3.1) \quad |\partial_{x_{n-1}} \tilde{\rho}| \leq C, \quad |\partial_{x_n} \tilde{\rho}| \leq C \tilde{\rho}^{1-1/\alpha}. \]

Hereafter let \( h \) be positive and sufficiently small. We define the following mapping from \( R^n \) into itself:

\[ \Phi_h: \begin{cases} y_j = x_j & (j \neq n-1) \\ y_{n-1} = x_{n-1} + h \tilde{\rho}. \end{cases} \]

Then we see that
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\[
\begin{pmatrix}
\partial_{x_1}y_1 & \cdots & \partial_{x_1}y_n \\
\cdots & \cdots & \cdots \\
\partial_{x_n}y_1 & \cdots & \partial_{x_n}y_n
\end{pmatrix}
= \begin{pmatrix}
1 & & 0 \\
0 & 1 & 0 \\
0 & 1 + h\partial_{x_1}\tilde{\rho} & 1
\end{pmatrix}
\]

(3.2)

where

\[
* = \frac{1}{1 + h\partial_{x_n-1}\tilde{\rho}} \begin{pmatrix}
1 & 0 \\
0 & 1 + h\partial_{x_{n-1}}\tilde{\rho}
\end{pmatrix}
\]

Let $J_h$ be the Jacobian of $\Phi_h$, which is equal to $1 + h\partial_{x_{n-1}}\tilde{\rho}$. From (3.1) there is a positive constant $c$ such that $c \leq J_h \leq c^{-1}$ in $R^n$. Hence $\Phi_h$ is one-to-one mapping from $\{x_n \geq 0\}$ onto itself. Combining (3.1) and (3.2) one finds

(3.1)

\[
|\partial_{y_{n-1}}\tilde{\rho} \circ \Phi_h^{-1}| \leq C \quad \text{and} \quad |\partial_{y_n}\tilde{\rho} \circ \Phi_h^{-1}| \leq C(\tilde{\rho} \circ \Phi_h^{-1})^{1-1/\alpha}.
\]

In what follows, the $y$-variable is always connected with the $x$-variable by $y = \Phi_h(x)$. For the notational convenience, we usually write $\tilde{\rho}$ in place of $\tilde{\rho} \circ \Phi_h^{-1}$, because the dependence of $\tilde{\rho}$ on $x$ or $y$ will be understood from the context.

We put $x'' = (x_1, \ldots, x_{n-1})$ and $y'' = (y_1, \ldots, y_{n-1})$ for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. The following operators are defined:

\[
(S_h u)(x) = u(x'', x_{n-1} + h\tilde{\rho}, x_n), \quad (T_h u)(y) = u(y'', y_{n-1} - h\tilde{\rho}, y_n),
\]

\[
(P_h u)(x) = h^{-1}((S_h u)(x) - u(x)), \quad (Q_h u)(y) = h^{-1}((T_h u)(y) - u(y)).
\]

If we set

\[
\nabla x(S_h u) = S_h \nabla u + hF_h(\nabla u)
\]

and

\[
\nabla y(T_h u) = T_h \nabla u + hG_h(\nabla u),
\]

it is easily seen that

(3.3)

\[
\begin{align*}
F_h(\nabla x u) &= \begin{pmatrix} 0, \ldots, 0, \partial_{x_{n-1}}\tilde{\rho} \cdot S_h \partial_{x_{n-1}} u, \partial_{x_n}\tilde{\rho} \cdot S_h \partial_{x_{n-1}} u \end{pmatrix}, \\
G_h(\nabla y u) &= \begin{pmatrix} 0, \ldots, 0, -\partial_{y_{n-1}}\tilde{\rho} \cdot T_h \partial_{y_{n-1}} u, -\partial_{y_n}\tilde{\rho} \cdot T_h \partial_{y_{n-1}} u \end{pmatrix},
\end{align*}
\]
The following lemma follows immediately from (3.1), (3.1') and (3.3).

**Lemma 3.** For all \( u \in C^1_{(0)}(\overline{G}) \)

\[
|F_h(F_x u)| \leq C \delta^{1-1/a} |S_h \partial_{x_{n-1}} u|
\]

and

\[
|G_h(F_y u)| \leq C \delta^{1-1/a} |T_h \partial_{y_{n-1}} u|
\]

where \( C \) is independent of \( h \) and \( u \).

We note that any function \( u \in H^1_{(0)}(G) \) can be extended over \( \{x_n \geq 0\} \) in such a way that \( u=0 \) in \( \{x_n \geq 0\} \cap \Gamma^c \).

**Lemma 4.** Suppose that \( G = \sum \) in particular. Let \( \beta \delta/2 > 1/\alpha - 1 \) and let \( \zeta \) be a function in \( C^\infty((r<R)) \) such that \( 0 \leq \zeta \leq 1 \). Then \( \zeta S_h u \) and \( \zeta T_h u \in K \) for any \( U \in K \).

**Proof.** Recalling the definition of \( K \), one takes a sequence \( \{U_j\} \in H \) such that

\[
\|\rho^{-\beta/2} F(U_j - U)\| + \|\rho^{-\beta/2} - 1(U_j - U)\| \to 0 \quad \text{as} \quad j \to \infty.
\]

Since \( \delta \geq \rho \) and \( \beta \delta/2 > 1/\alpha - 1 \), it follows that

(3.4) \[
\|U_j - U\| + \|\delta^{1-1/a} F(U_j - U)\| \to 0 \quad \text{as} \quad j \to \infty.
\]

Clearly \( \zeta S_h U_j \in H \) and, by (3.4),

\[
\lim_{j \to \infty} \|\zeta S_h (U_j - U)\| = \lim_{j \to \infty} \|\zeta S_h F(U_j - U)\| = 0;
\]

so that it suffices to show

\[
\lim_{j \to \infty} \|F_h(F_x(U_j - U))\| = 0
\]

to complete the proof. Note that Lemma 3 implies

\[
|F_h(F_x(U_j - U))| \leq C \delta^{1-1/a} |S_h \partial_{x_{n-1}} (U_j - U)|.
\]

Since \( \delta^{1-1/a} \) is bounded from above by \( C(\rho \circ \Phi_x)^{1-1/a} \), the proof will be completed with the aid of (3.4). \( \text{Q.E.D.} \)

We denote by \( (\cdot,\cdot)_x \) and \( \| \cdot \|_x \), \( (\cdot,\cdot)_y \) and \( \| \cdot \|_y \) the \( L^2 \)-inner product and the \( L^2 \)-norm of functions on \( \{x_n \geq 0\} \) \( \{y_n \geq 0\} \), respectively.

**Lemma 5.** For \( u \in H^1_{(0)}(\Sigma) \)

\( ^3 \) Namely, \( \zeta \) is in \( C^\infty \) with compact support in \( (r<R) \).
\[ ||\hat{\rho}^{-1}P_hu||_x \leq C \|\partial_{x_{n-1}}u\|_x \]

and

\[ ||\hat{\rho}^{-1}Q_hu||_y \leq C \|\partial_{y_{n-1}}u\|_y, \]

where \( C \) is independent of \( h \) and \( u \).

**Proof.** When \( u \in C^1_0(\Sigma) \) in particular, we see

\begin{equation}
(P_hu)(x) = \hat{\rho} \int_0^1 (\partial_{x_{n-1}}u)(x', x_{n-1} + \theta h \hat{\rho}, x_n) d\theta.
\end{equation}

By Schwarz' inequality,

\[ ||\hat{\rho}^{-1}P_hu||^2 \leq \int_0^1 \int_{x_{n-1}>0} \left[(\partial_{x_{n-1}}u)(x'', x_{n-1} + \theta h \hat{\rho}, x_n)\right]^2 dx d\theta \]

\[ \leq \int_0^1 \| (J^2_{\hat{\rho}})^{1/2} \partial_{y_{n-1}}u\|_o^2 d\theta \leq C \|\partial_{x_{n-1}}u\|_x^2. \]

Hence the first half has been proved for \( u \in C^1_0(\Sigma) \). When \( u \in H^1_0(\Sigma) \), it is sufficient to take an approximating sequence from \( C^1_0(\Sigma) \).

The proof of the later half is completely analogous. Q.E.D.

**Lemma 6.** Let \( u \in H^1_0(\Sigma) \). Then

a) \( \|P_hu - \hat{\rho}\partial_{x_{n-1}}u\|_x \to 0 \),

b) \( \|Q_hu + \hat{\rho}\partial_{y_{n-1}}u\|_y \to 0 \) as \( h \to 0 \).

**Proof.** If \( u \in C^1_0(\Sigma) \), we have from (3.5)

\[ \|P_hu - \hat{\rho}\partial_{x_{n-1}}u\|_x^2 \]

\[ \leq \int_0^1 \int_{x_{n-1}>0} \hat{\rho}^2 (\partial_{x_{n-1}}u)(x', x_{n-1} + \theta h \hat{\rho}, x_n) - \partial_{x_{n-1}}u(x)\|^2 dx d\theta, \]

which implies a) for \( u \in C^1_0(\Sigma) \).

If \( u \in H^1_0(\Sigma) \), there is a sequence \( \{u_j\} \subset C^1_0(\Sigma) \) such that \( u_j \to u \) in \( H^1_0(\Sigma) \).

Using Lemma 5, we see that

\[ \|P_hu - \hat{\rho}\partial_{x_{n-1}}u\|_x \leq \|P_h(u-u_j)\|_x + \|P_hu_j - \hat{\rho}\partial_{x_{n-1}}u_j\|_x + \|\partial_{x_{n-1}}u_j - \partial_{x_{n-1}}u\|_x \]

\[ \leq C(\|P(u-u_j)\|_x + \|P_hu_j - \hat{\rho}\partial_{x_{n-1}}u_j\|_x). \]

Thus we have proved a). Similarly b) can be proved. Q.E.D.

§4. Proof of Theorem 1

Let us define a new bilinear form:

\[ \langle \hat{P}_h v, w \rangle = \int_{\Sigma} v \partial_{x_{n-1}} w \hat{\rho}^2 \delta(x_{n-1} - x_n) \]
In view of Lemma 1, it holds that for any $U, V \in [\tilde{H}^{1}(G)]^{N}$
$$|\tilde{B}[U, V]| \leq C ||U|| \cdot ||V||.$$
\[(\ell^2 v_i, v_i) \leq 4 R^2 (2 - \beta \bar{\delta})^{-2} c^{-2} (\ell^2 \nabla v_i, \nabla v_i) \leq \frac{16}{9} R^2 c^{-2} (\ell^2 \nabla v_i, \nabla v_i).\]

Therefore it follows that

\[
\left| \sum_{i,j} (c_{ij} \ell^2 v_i, v_i) \right| \leq N(\max_{t, j} |c_{ij}|) \sum_{i} (\ell^2 v_i, v_i) \leq \frac{16}{9} c^{-2} N R^2 (\max_{t, j} |c_{ij}|) \sum_{i} (\ell^2 \nabla v_i, \nabla v_i).
\]

By virtue of (2.3), (4.1) becomes

\[\tilde{B}[V, V] \geq c \|V\|^2 \quad \text{for all } V \in \mathcal{H}_{(0)}^{1}(G)^{N},\]

where \(c\) is a positive constant. Accordingly \(\tilde{B}[U, V]\) satisfies Lax-Milgram's condition.

By Lemma 2 and a well-known fact (see, e.g., page 24 in [9]) there is a unique solution \(\bar{U} \in \mathcal{K}\) of

\[\tilde{B}[\bar{U}, V - \bar{U}] \geq (\ell^2 F, V - \bar{U}) \quad \text{for all } V \in \mathcal{K}.\]

Let us put \(V' = (1 - t\zeta^{-2}) \bar{U} + t\zeta^{-2} W\) for any given \(W \in H\) and for sufficiently small \(t > 0\). Evidently \(V' \in \mathcal{K}\). Setting \(V = V'\) in (4.2) and dividing by \(t\), we have

\[B[\bar{U}, W - \bar{U}] \geq (F, W - \bar{U}).\]

which is valid also for any \(W \in K\). The unique solvability of (2.4) implies that \(\bar{U}\) coincides with the solution \(U\) of (2.4). Putting \(V = (0, \ldots, 0)\) in (4.2) and using Lemma 2, we arrive at the conclusion of Theorem 1. Q.E.D.

§ 5. Proof of Theorem 2

Let us take a function \(\zeta(x) \in C_{0}^{\infty}(\{r < R\})\) such that \(0 \leq \zeta \leq 1\) and \(\zeta = 1\) in a neighborhood of the origin. Let \(V' = (1 - \zeta^2) U + \zeta^2 V\) for any fixed \(V \in K\). Since \(V' \in K\), we have from (2.4)

\[B[U, \zeta^2(V - U)] \geq (F, \zeta^2(V - U)).\]

Let us define

\[H[U, V] = B[U, \zeta^2(V - U)] - B[\zeta U, \zeta(V - U)].\]

Then

\[H[U, V] = \sum_{i} [\langle \nabla u_i, \zeta(v_i - u_i) \nabla \zeta \rangle - (u_i \nabla \zeta \cdot \nu_{i} + \langle \zeta(v_i - u_i) \nabla \zeta \rangle)],\]

and also,
We observe that the Jacobian of the inverse mapping $\Phi_h^{-1}$, where $\Phi_h$ is the mapping defined in Section 3. We write $J_h = 1 + hJ_1$ and $J_h^{-1} = 1 + hJ_2$. More precisely, $dy = J_h(x)dx$, $dx = J_h^{-1}(y)dy$, $J_h(x) = 1 + hJ_1(x)$, $J_h^{-1}(y) = 1 + hJ_2(y)$ and $J_h(x)J_h^{-1}(y) = 1$. It is obvious that $J_1$, $J_2$ are bounded in \( \{x_n \geq 0\} \) and

\[
J_1 + J_2 = O(h) \quad \text{with} \quad y = \Phi_h(x) \quad (h \to 0).
\]

From now on we put $w_i = \zeta u_i$ for the sake of brevity. We observe that

\[
(\nabla w_i, S_h w_i)_x = (\nabla w_i, S_h w_i)_x + h(\nabla w_i, F_h(\nabla w_i))_x
\]

and

\[
(\nabla w_i, S_h w_i)_y = (J_h^{-1}T_h \nabla w_i, \nabla w_i)_y.
\]

Hence

\[
(\nabla w_i, S_h w_i)_y = (J_h^{-1}T_h \nabla w_i, \nabla w_i)_y = (J_h^{-1}T_h \nabla w_i, \nabla w_i)_y - h(\nabla w_i, F_h(\nabla w_i))_x + h(\nabla w_i, F_h(\nabla w_i))_x.
\]

Moreover we see that

\[
(T_h \nabla w_i, T_h \nabla w_i)_y = (T_h \nabla w_i, T_h \nabla w_i)_y + 2h(T_h \nabla w_i, G_h(\nabla w_i))_y
\]

\[
+ h(\nabla T_h \nabla w_i, \nabla w_i)_x + h(T_h \nabla w_i, \nabla w_i)_x.
\]

Consequently the following equality holds:

\[
(\nabla w_i, F_h w_i)_y = h^{-1}[(T_h \nabla w_i, T_h \nabla w_i)_y - (T_h \nabla w_i, \nabla w_i)_y - (T_h \nabla w_i, \nabla w_i)_y + (T_h \nabla w_i, \nabla w_i)_x]
\]

\[
= h^{-1}[2(T_h \nabla w_i, G_h(\nabla w_i))_y + (J_h \nabla w_i, \nabla w_i)_x + (J_h \nabla w_i, \nabla w_i)_y] + h(\nabla T_h \nabla w_i, \nabla w_i)_x.
\]

This is rewritten in the form

\[
(\nabla w_i, F_h w_i)_y = h^{-1}[I_1 + I_2] + h^{-1}[2I_3 + I_4 + I_5 + I_6 - I_7] + I_8.
\]

Now we shall prove the following inequality

\[
(\nabla w_i, F_h w_i)_y \leq C[\|\dot{\rho}^{-1/\sigma}U\| + \|\dot{\rho}^{-1/\sigma}F\| + \|\dot{\rho}^{-1/\sigma}U\| + \|\nabla Q_h(\zeta U)_y\|]
\]

\[
+ \|\dot{\rho}F\| \cdot \|\nabla Q_h(\zeta U)_y\|_y].
\]
To do this, we shall first prove

\begin{equation}
(5.5) \quad h^{-2}[I_1 + I_2] \leq \text{the right-hand side of (5.4)}.
\end{equation}

Let $\xi \in C_r^\gamma([r < R])$ be such that $0 \leq \xi \leq 1$ and $\xi = 1$ on the support of $\zeta$. By Lemma 4, $\xi S_h U \in K$. (5.1) yields

$$
B[\zeta U, \zeta(S_h U - U)]_x \geq (F, \zeta(S_h U - U))_x - H[U, \zeta S_h U]_x^4.
$$

Hence

\begin{equation}
(5.6) \quad I_1 \leq (f_i, \zeta(w_i - \zeta S_h u_i))_x + H[U, \zeta S_h U]_x \\
- (f'_w, \nabla S_h(w_i - \zeta S_h u_i))_x + (c_{ij} w_j, \zeta S_h u_i - w_i)_x,
\end{equation}

where $F = (f_1, \cdots, f_N)$. Almost similarly we have

\begin{equation}
(5.7) \quad I_2 \leq (f_i, \zeta(w_i - \zeta T_h u_i))_y + H[U, \zeta T_h U]_y \\
- (f'_w, \nabla(T_h w_i - \zeta T_h u_i))_y + (c_{ij} w_j, \zeta T_h u_i - w_i)_y.
\end{equation}

We calculate the sum of each first term on the right-hand sides of (5.6) and (5.7). We see that

\begin{align*}
(f_i, \zeta(w_i - \zeta S_h u_i))_x + (f_i, \zeta(w_i - \zeta T_h u_i))_y \\
= (f_i, \zeta(w_i - S_h w_i))_x + (f_i, \zeta(S_h \zeta - \zeta) S_h u_i)_x \\
+ (f_i, \zeta(w_i - T_h w_i))_y + (f_i, \zeta(T_h \zeta - \zeta) T_h u_i)_y \\
= (f_i, \zeta(w_i - S_h w_i))_x + (f_i, \zeta(S_h \zeta - \zeta) S_h u_i)_x \\
+ (S_h f_i, S_h \zeta \cdot (S_h w_i - w_i))_x + (S_h f_i, S_h \zeta \cdot (\zeta - S_h \zeta) u_i)_x \\
+ h[(J_S f_i, S_h \zeta \cdot (S_h w_i - w_i))_x + (J_S f_i, S_h \zeta \cdot (\zeta - S_h \zeta) u_i)_x],
\end{align*}

and

\begin{align*}
f_i \zeta S_h u_i - S_h f_i \cdot S_h \zeta \cdot u_i \\
= (f_i \zeta - S_h f_i \cdot S_h \zeta) S_h u_i + S_h f_i \cdot S_h \zeta \cdot (S_h u_i - u_i).
\end{align*}

Thus

\begin{equation}
(5.8) \quad h^{-2}|(f_i, \zeta(w_i - \zeta S_h u_i))_x + (f_i, \zeta(w_i - \zeta T_h u_i))_y| \\
\leq |(f_i, P_h \zeta \cdot P_h w_i)_x| + |(P_h f_i, S_h \zeta \cdot P_h w_i)_x| + |(f_i, (P_h \zeta) S_h u_i)_x| \\
+ |(P_h f_i, S_h \zeta \cdot P_h \zeta \cdot S_h u_i)_x| + |(S_h f_i, S_h \zeta \cdot P_h \zeta \cdot P_h u_i)_x| \\
+ |(J_S f_i, S_h \zeta \cdot P_h w_i)_x| + |(J_S f_i, S_h \zeta \cdot P_h \zeta \cdot u_i)_x|.
\end{equation}

4) The notation $B[\cdot, 1_x, H[\cdot, 1_x]$ means the forms with respect to variables $x$. The replacement of $x$ with $y$ is similar.
We estimate the second term on the right-hand side of (5.8). In general the following equality holds:

\[(P_h f, g)_x = -(S_h f, P_h g)_x + (J_z f, g)_y.\]

Hence from (3.4) and Lemma 5

\[|\langle P_h f_i, S_h \zeta \cdot P_h w_i \rangle| \leq C \| \rho f_i \| (\| \rho^{-1} P_h (S_h \zeta \cdot P_h w_i) \|_x + \| \rho^{-1} P_h w_i \|_x) \leq C \| \rho f_i \| (\| \partial_{x_{n-1}} (S_h \zeta \cdot P_h w_i) \|_x + \| \mathcal{F} w_i \|).\]

On the other hand we have from (3.2)

\[|\partial_{x_{n-1}} (S_h \zeta \cdot P_h w_i) | \leq C | \mathcal{F} \zeta Q_h w_i | \]

and by Poincaré's inequality

\[\| Q_h w_i \| \leq C \| \mathcal{F} (Q_h w_i) \|_y.\]

Consequently we get

\[|\langle P_h f_i, S_h \zeta \cdot P_h w_i \rangle| \leq \text{the right-hand side of (5.4)}.\]

We choose the preceding \( \zeta \) in such a form that \( \zeta(x) = \zeta(x_{n-1}, x_n) \cdot \zeta(x') \). Then \( P_h \zeta = \zeta_x P_h \zeta \), which implies that \( P_h \zeta = 0 \) in a neighborhood of the origin. From this the other terms on the right-hand side of (5.8) can be estimated more easily. Therefore it follows that

\[h^{-1}|(f, \zeta (w_i - \zeta S_h u_i))_x + (f, \zeta (w_i - \zeta T_h u_i))_y| \leq \text{the right-hand side of (5.4)}.\]

Next we estimate the sum of each third term on the right-hand sides of (5.6) and (5.7). Since \( \mathcal{F} w_i = A \mathcal{F} x \) for \( A = (\partial_y x_i)_{i,j=1,...,n} \), we see

\[\begin{align*}
\mathcal{F} w_i, \mathcal{F} (T_h w_i - \zeta T_h u_i) \\
= (J_z S_h \mathcal{F} w_i, A \mathcal{F} (w_i - S_h \zeta \cdot u_i))_x \\
= (S_h \mathcal{F} w_i, \mathcal{F} (w_i - S_h \zeta \cdot u_i))_x \\
+ h(J_z S_h \mathcal{F} w_i, \mathcal{F} (w_i - S_h \zeta \cdot u_i))_x \\
+ (J_z S_h \mathcal{F} w_i, (A - I) \mathcal{F} (w_i - S_h \zeta \cdot u_i))_x,
\end{align*}\]

where \( I \) is the unit matrix. On the other hand

\[h^{-1}|(\mathcal{F} w_i, \mathcal{F} (S_h w_i - \zeta S_h u_i))_x + (S_h \mathcal{F} w_i, \mathcal{F} (w_i - S_h \zeta \cdot u_i))_x | \leq |(P_h \mathcal{F} w_i, \mathcal{F} (P_h \zeta \cdot S_h u_i))_x| + |(S_h \mathcal{F} w_i, \mathcal{F} (P_h \zeta \cdot P_h u_i))_x|.\]

Recalling \( P_h \zeta = 0 \) in a neighborhood of the origin and using Lemmas 3, 5 and (5.9),
we see that the right-hand side of this inequality is estimated by that of (5.4). By (3.2), \( \hat{\alpha}^{1/a-1} (A - I) = O(h) \) \((h \to 0)\). Hence the remaining terms of (5.11) are similarly estimated. The sum of the remaining terms on the right-hand sides of (5.6) and (5.7) can be estimated more easily. Therefore (5.5) has been proved.

Next we shall estimate \( h^{-1} [2I_{h} + \cdots - I_{h}] \) with frequent use of Lemmas 3 and 5. From (3.2), we get

\[
\partial_{x_{n-1}} \hat{\beta} = (1 + h \partial_{x_{n-1}} \hat{\beta})^{-1} \partial_{x_{n-1}} \hat{\beta}, \\
\partial_{y_{n}} \hat{\beta} = (1 + h \partial_{x_{n-1}} \hat{\beta})^{-1} \partial_{x_{n}} \hat{\beta}.
\]

Hence we have

\[
\partial_{x_{n-1}} \hat{\beta} \cdot S_{h} \partial_{x_{n-1}} w_{t} - \partial_{y_{n-1}} \hat{\beta} \cdot T_{h} \partial_{y_{n-1}} w_{t} \\
= h(1 + h \partial_{x_{n-1}} \hat{\beta})^{-1} (\partial_{x_{n-1}} \hat{\beta})^{2} S_{h} \partial_{x_{n-1}} w_{t} \\
+ \partial_{y_{n-1}} \hat{\beta} \cdot (S_{h} \partial_{x_{n-1}} w_{t} - T_{h} \partial_{y_{n-1}} w_{t}).
\]

and

\[
\partial_{x_{n}} \hat{\beta} \cdot S_{h} \partial_{x_{n-1}} w_{t} - \partial_{y_{n}} \hat{\beta} \cdot T_{h} \partial_{y_{n-1}} w_{t} \\
= h(1 + h \partial_{x_{n-1}} \hat{\beta})^{-1} \partial_{x_{n}} \hat{\beta} \cdot S_{h} \partial_{x_{n-1}} w_{t} \\
+ \partial_{y_{n}} \hat{\beta} \cdot (S_{h} \partial_{x_{n-1}} w_{t} - T_{h} \partial_{y_{n-1}} w_{t}).
\]

On the other hand from (3.3)

\[
F_{h}(w_{t}) + G_{h}(w_{t}) \\
= (0, \cdots, 0, \partial_{x_{n-1}} \hat{\beta} \cdot S_{h} \partial_{x_{n-1}} w_{t} - \partial_{y_{n-1}} \hat{\beta} \cdot T_{h} \partial_{y_{n-1}} w_{t}, \\
\partial_{x_{n}} \hat{\beta} \cdot S_{h} \partial_{x_{n-1}} w_{t} - \partial_{y_{n}} \hat{\beta} \cdot T_{h} \partial_{y_{n-1}} w_{t}).
\]

This yields

\[
\| (F_{x_{t}}, G_{h}(w_{t}))_{x} \| \\
\leq C \| \hat{\beta}^{1-1/a} F_{w_{t}} \| (h \| S_{h} F_{w_{t}} \| + \| S_{h} F_{w_{t}} - (T_{h} F_{w_{t}}(y)) \|).\]

Since

\[
(T_{h} F_{w_{t}}, G_{h}(w_{t}))_{y} \\
= (F_{w_{t}}, G_{h}(w_{t}))(y)_{x} + h(J_{h} F_{w_{t}}, (G_{h}(w_{t}))(y))_{x},
\]

we get

\[
h^{-1} \| I_{h} + I_{h} \| \leq C \| \hat{\beta}^{1-1/a} F_{w_{t}} \| (\| F_{w_{t}} \| + \| Q_{h} F_{w_{t}} \|).
\]

Evidently
\[
\|Q_h F w_i\|_y \leq \|P Q_h w_i\|_y + C \|\tilde{\rho}^{1/a} P U\|.
\]

Therefore we obtain

\[
h^{-1}|I_8 + I_6| \leq \text{the right-hand side of (5.4)}.
\]

More easily we see that

\[
h^{-1}|I_8 - I_4|, \quad h^{-1}|I_8|, \quad I_8
\]

\[
\leq \text{the right-hand side of (5.4)},
\]

where we have used (5.2). Combining these inequalities and (5.5), we conclude the proof of (5.4).

From (5.4) we immediately obtain

\[
\|P Q_h (\zeta U)\|_y \leq C(\|\tilde{\rho} F\| + \|\tilde{\rho}^{1/4} P U\|).
\]

We can take \(\epsilon > 0\) so that \(\epsilon - 1/4 < 1 - 1/\alpha\) (\(<0\)). For such \(\epsilon\) we choose the real numbers \(\beta\) and \(\delta\) in Section 2 in such a way that \((-1/2)\delta \leq \epsilon - 1/4\). Then

\[
\|\tilde{\rho}^{1-1/a} P U\| \leq C \|\tilde{\rho}^{1-1/a} P U\| \leq C \|\tilde{\rho}^{(-1/2)} P U\|
\]

\[
\leq C \|\tilde{U}\|.
\]

Applying Theorem 1, we have

\[
\|F Q_h (\zeta U)\|_y \leq C \|\rho^{3/4} F\|
\]

where we have used the relation \(\tilde{\rho} \leq C \rho^a \leq C \rho^{3/4}\). Hence there are a subsequence \(\{h_n\}\) and a vector function \(V \in [H^1_{00}(\Sigma)]^N\) such that \(h_n \rightarrow +0\) as \(\nu \rightarrow \infty\) and \(Q_{h_n}(\zeta U)\) converges weakly to \(V\) in \([H^1_{00}(\Sigma)]^N\) as \(\nu \rightarrow \infty\). At the same time, from Lemma 6, \(Q_{h_n}(\zeta U)\) converges strongly to \(-\tilde{\rho} \partial_{y_{n-1}} U\) in \([L^2(\Sigma)]^N\) as \(\nu \rightarrow \infty\). Thus \(V = -\tilde{\rho} \cdot \partial_{y_{n-1}} (\zeta U)\) and

\[
\|P \tilde{\rho} \partial_{y_{n-1}} (\zeta U)\|_y \leq C \|\rho^{3/4} F\|.
\]

On the other hand, we use the usual parallel translation for \(y''\)-direction. Since \(\tilde{\rho}\) is independent of \(y''\), we have

\[
\|\tilde{\rho} P (\partial_{y''} (\zeta U))\|_y \leq C \|\tilde{\rho} F\| \leq C \|\rho^{3/4} F\|.
\]

Accordingly we obtain

\[
(5.12) \quad \|\tilde{\rho} \partial_{x_i} (\zeta U)\|_y \leq C \|\rho^{3/4} F\|
\]

for \(i = 1, \ldots, n-1\). From (2.4) it follows that \(-\Delta u_i + \sum_j c_{ij} u_j = f_i\) in \(\Sigma\). Hence (5.12) is also valid for \(i = n\).
Let $\varepsilon > 0$ be given. In advance we take $\zeta = 1$ in $\Sigma_{\varepsilon}$. Let $t$ be any real number with $1 < t < 10/9$. Then there is a real number $\alpha$ with $4/5 < \alpha < 1$ such that $t < 2/(1 + \alpha)$. If we put $p = 2/t$ and $q = 2/(2 - t)$, then $p, q > 1$ and $p^{-1} + q^{-1} = 1$. It is evident that $\rho^s \leq C_{\varepsilon} \rho$ in $\Sigma_{\varepsilon}$. Hence by Hölder's inequality we have

\begin{equation}
\int_{\Sigma_{\varepsilon}} (\partial^2 u_i)^p dx \leq \left( \int_{\Sigma_{\varepsilon}} \rho^{2 - t \alpha} \rho^{\partial^2 u_i} dx \right)^{1/q} \left( \int_{\Sigma_{\varepsilon}} \rho^{2p} \rho^{\partial^2 u_i} dx \right)^{1/p} \leq C_{\varepsilon} \left( \int_{\rho} \rho^{1 - \alpha t \alpha} d\rho \right)^{1/q} \left( \int_{\Sigma_{\varepsilon}} \rho^{2 \partial^2 u_i} dx \right)^{1/p}.
\end{equation}

Here $\alpha t \alpha < 2$, because $t < 2/(1 + \alpha)$. Thus, the first integral on the right-hand side of (5.13) is finite, and, in view of (5.12), the proof of Theorem 2 is completed. Q.E.D.

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nuna adreso:
Kazuya HAYASIDA
Department of Mathematics
Faculty of Science
Kanazawa University
Kanazawa, 920
Japan
Haruo NAGASE
Suzuka College of Technology
Suzuka, 510-02
Japan

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(Revizitia la 21-an de julio, 1984)