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Recurrent Solutions for Linear Almost Periodic Systems

By

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In this note, we shall show the following:

Theorem. *The sum of recurrent solutions of a linear almost periodic system is not necessarily recurrent.*

Define the set BU by

$BU=\{f(t): R\rightarrow R^n; f(t)$ is bounded and uniformly continuous on $R\}$,

where $R=(-\infty, \infty)$. It is known that the set BU is a real linear space if we define the sum of functions $f(t)$ and $g(t)$ in BU by $f+g=f(t)+g(t)$ for $t\in R$.

Define the set RE by

$RE=\{f\in BU; f(t)$ is recurrent$\}$,

where a function $f(t)$ is said to be recurrent, if for any $\varepsilon>0$ and compact set $S\subset R$, there exists a $T(\varepsilon, S)>0$ such that for any $t\in R$ and $a\in R$, there exists a $\tau\in[a, a+T(\varepsilon, S)]$ such that $|f(t+u)-f(\tau+u)|<\varepsilon$ for all $u\in S$, where $|\cdot|$ is the Euclidean norm of R^n. The concept that a function $f(t)$ is recurrent corresponds to the concept that the motion $\pi(t,f)=f_t=f(t+s)$, $s\in R$, in the dynamical system defined on $R\times$(metric space $C([R, R^n])$ introduced by the compact open topology) is recurrent (refer to [2]). It is known [1] that the set of recurrent functions is not a linear space. On the other hand, the sum of recurrent solutions for the linear periodic system is recurrent by Floquet Theory.

Proof of Theorem. Let $A(t)$ be an $n\times n$ matrix and a continuous, almost periodic function, $p(t)$ be an R^n-valued, continuous, almost periodic function and let $H(A)$ be the hull of $A(t)$, that is, $B(t)\in H(A)$ means that for a sequence $\{t_k\}$, $A(t+t_k)\rightarrow B(t)$ as $k\rightarrow\infty$ uniformly on R.

Consider the systems

$$\dot{x}=A(t)x \tag{1}$$

and

$$\dot{x}=A(t)x+p(t), \tag{2}$$

and we assume that System (2) has a bounded solution defined on $[0, \infty)$. It is known [3] that System (2) has a solution $u(t)$ with the minimal norm, that is, $u(t)$ is defined on R and

$$\sup_{t\in R}|u(t)|=\inf\{\sup_{t\in R}|v(t)|;\ v(t) \text{ is a bounded solution of (2) defined on } R\} =\lambda<\infty.$$

Let $H(u)$ and $H(u, A+p)$ be the hulls of $u(t)$ and $(u(t), A(t)x+p(t))$, respectively, that is, $(w(t), B(t)x+q(t)) \in H(u, A+p)$ means that for a sequence $\{t_k\}$, $u(t+t_k)\to w(t)$ as $k\to\infty$ uniformly on any compact interval in R and $A(t+t_k)x+p(t+t_k)\to B(t)x+q(t)$ as $k\to\infty$ uniformly on $R\times S$, $S=\{x \in R^n;\ |x|\leqq\lambda\}$. By [3], $w(t)$ is a solution of

$$(3)\qquad \dot{x}(t)=B(t)x+q(t)$$

defined on R and satisfies $\inf_{t\in R}|w(t)|=\lambda$. We can show that if $(w^1, B+q)$ and $(w^2, B+q)$ are in $H(u, A+p)$, then $\inf_{t\in R}|w^1(t)-w^2(t)|=0$. In fact, $x(t)=\{w^1(t)+w^2(t)\}/2$ is a solution of (3) and $y(t)=\{w^1(t)-w^2(t)\}/2$ is a solution of the homogeneous system

$$(4)\qquad \dot{x}=B(t)x.$$

Clearly, $\inf_{t\in R}|x(t)|\geqq\lambda$. We have

$$|x(t)|^2+|y(t)|^2=\{|w^1(t)|^2+|w^2(t)|^2\}/2\leqq\lambda^2$$

for all $t \in R$. If $\inf_{t\in R}|y(t)|\geqq\delta>0$, we have $\{\sup_{t\in R}|x(t)|\}^2\leqq\lambda^2-\delta^2<\lambda^2$, which is a contradiction.

Bender [2] has shown that System (3) has a recurrent solution $z(t)$ in $H(u)$. Let $(r^1, B+q)$ and $(r^2, B+q)$ be in $H(z, A+p)$. It is known that if $f(t) \in RE$, then every function $g(t)$ in $H(f)$ is in RE (Lemma 3.2 in [2]). Hence $r^1(t)$ and $r^2(t)$ are recurrent solutions of (3) that satisfy $\inf_{t\in R}|r^1(t)-r^2(t)|=0$. Therefore there exists an $(s, C) \in H(r^1-r^2, B)$ and a sequence $\{t_k\}$ such that $\lim_{k\to\infty}|r^1(t_k)-r^2(t_k)|=0$, $r^1(t+t_k)-r^2(t+t_k)\to s(t)$ as $k\to\infty$ uniformly on any compact interval in R and $B(t+t_k)\to C(t)$ as $k\to\infty$ uniformly on R. Since $s(t)$ is the solution of $\dot{x}=C(t)x$ through $(0, 0)$, $s(t)=0$ for all $t \in R$.

Assume that the solution $r^1(t)-r^2(t)$ of (4) is in RE. It is known that if $f(t) \in RE$, then $f \in H(g)$ for every $g \in H(f)$ (Lemma 3.3 in [2]). Hence we have $r^1(t)=r^2(t)$ for all $t \in R$, because $s(t) \in H(r^1-r^2)$, which implies that for every $B+g \in H(A+p)$, System (3) has only one solution in $H(z)$. By Theorem 5 in [4], System (2) has an almost periodic solution. However, Johnson's example gives a contradiction, that is, Johnson [5] has shown that there exists a scalar, almost periodic system $\dot{x}=a(t)x+b(t)$ which admits bounded solutions, but no almost periodic solutions.

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