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Perturbation Method for Linear Periodic Systems III

By

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§1. As we indicated in the paper [2], the problem to obtain Floquet representation of a fundamental matrix of linear periodic systems

$$\dot{x}=A(t)x, \quad x\in C^n, \quad t\in R, \quad \dot{}=d/dt$$

can be reduced to find a nondegenerate periodic solution of a matrix differential equation

$$(1) \qquad \dot{U}=A(t)U-UA(t).$$

(Here nondegeneracy of a matrix means that the eigenvalues belonging to different Jordan blocks of its Jordan's normal form are mutually different). However, since it is difficult to solve (1) in general, we condider only the case when $A(t)$ is, in a sense, sufficiently close to a constant matrix.

First we define two kinds of norms of a matrix to be used in this paper. For an arbitrary matrix $[h_{ij}(t)]$ with continuously differentiable entries $h_{ij}(t)$, we define

$$\|[h_{ij}(t)]\|=\sup_{t\in R}\max_{i}\sum_{j=1}^{r}|h_{ij}(t)|$$

and

$$|||[h_{ij}(t)]|||=\|[h_{ij}(t)]\|+\|[\dot{h}_{ij}(t)]\|.$$

Now let $t\in R$, $\varepsilon\in C$, and $A(t,\varepsilon)$ be an n-by-n matrix with complex-valued entires satisfying the following conditions:

(i) $A(t,\varepsilon)$ is continuously differentiable and periodic in t with period T,

(ii) $A(t,\varepsilon)=\sum_{k=0}^{\infty}A_k(t)\varepsilon^k$, and $\dot{A}(t,\varepsilon)=\sum_{k=0}^{\infty}\dot{A}_k(t)\varepsilon^k$ are convergent when $|\varepsilon|$ is small, and

$$|||A_k(t)|||<\alpha\rho^{-k}, \qquad k=0,1,2,\cdots,$$

where α and ρ are constants,

(iii) $A_0=A_0(t)=\begin{bmatrix}\lambda & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda\end{bmatrix}$

The aim of this paper is to get a recurrence formula for constructing a nondegenerate periodic solution of

$$\dot{U}=A(t, \varepsilon)U-UA(t, \varepsilon). \tag{2$_\varepsilon$}$$

The process follows the same outline as is indicated in [3] where $(2)_\varepsilon$ was considered under the assumption

$$A_0=\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

and

$$A_0=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

instead of (iii). First we show that the existence of a nondegenerate periodic solution $U(t, \varepsilon)$ of $(2)_\varepsilon$ with a condition

$$U(t, 0)=U_0(t)$$

where $U_0(t)$ is a nondegenerate periodic solution of $(2)_0$ (§ 2). Next we determine a nondegenerate periodic solution formally and finally show its convergence (§§ 3–5).

Throughout this paper, we assume $T=2\pi$ for simplicity. Obviously this assumption does not harm any generality. Also, regarding the indefinite symbol $\int$, we adopt a following convention:

$$\int e^{\mu t}dt=\begin{cases} \dfrac{1}{\mu}e^{\mu t} & if\ \mu\neq 0 \\ t & if\ \mu=0. \end{cases} \tag{3}$$

§ 2. In this section we prove the following lemma.

Lemma. *For an arbitrary nondegenerate periodic solution $U_0(t)$ of $(2)_0$, there exists a nondegenerate periodic solution of $(2)_\varepsilon$ such that*

$$U(t, 0)=U_0(t)$$

if $|\varepsilon|$ is sufficiently small.

Proof. The general solution of $(2)_\varepsilon$ has the form such as

$$U(t, \varepsilon)=X(t, \varepsilon)C(\varepsilon)X(t, \varepsilon)^{-1}$$

where $X(t, \varepsilon)$ is a fundamental matrix of the system

$$\dot{x}=A(t, \varepsilon)x$$

and $C(\varepsilon)$ is a matrix holomorphic in ε and independent of t. Therefore $U(t, \varepsilon)$ is periodic in t if and only if

$$M(\varepsilon)C(\varepsilon)=C(\varepsilon)M(\varepsilon).$$

Since $U(t, 0)=U_0(t)$, we should have

$$C(0)=C_0$$

if $U_0(t)=X(t, 0)C_0X(t, 0)^{-1}$. Therefore it is sufficient to show the existence of a holomorphic matrix $C(\varepsilon)$ such that

$$(4) \qquad \begin{cases} M(\varepsilon)C(\varepsilon)=C(\varepsilon)M(\varepsilon) \\ M_0C_0=C_0M_0 \end{cases} \qquad (M_0=M(0))$$

for the proof of this lemma.

For simplicity, we choose $X(t, 0)$ whose monodromy matrix has been already reduced to Jordan's normal form. Namely, if we put

$$H=\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix},$$

then

$$M_0=e^{2\pi\lambda}I+H.$$

The solution of (4) satisfies

$$(5) \qquad \begin{cases} (M(\varepsilon)-e^{2\pi\lambda}I)C(\varepsilon)=C(\varepsilon)(M(\varepsilon)-e^{2\pi\lambda}I) \\ HC_0=C_0H \end{cases}$$

and the solution of (5) also satisfies (4). So we may solve (5) instead of (4).

Now put

$$M_0=H, \quad M(\varepsilon)=\sum_{k=0}^{\infty} M_k\varepsilon^k, \quad C(\varepsilon)=\sum_{k=0}^{\infty} C_k\varepsilon^k$$

in (5), then

$$(6) \qquad M_0C_0=C_0M_0$$

$$(7) \qquad M_0C_k-C_kM_0=\sum_{q=0}^{k-1}(C_qM_{k-q}-M_{k-q}C_q)$$

From (6),

$$C_0=c_1I+\sum_{r=1}^{n-1}c_{r+1}H^r$$

where c_r are scalars. If we denote the solution of (5) by $C_r(\varepsilon)$ in case of $C_0=H^r$, $r=1, \cdots, n-1$, then

$$C(\varepsilon)=c_1I+\sum_{r=1}^{n-1}c_{r+1}C_r(\varepsilon)$$

satisfies (5) in case of $C_0=c_1I+\sum_{r=1}^{n-1}c_{r+1}H^r$. So it is sufficient to show the existence of the solution of (5) under the assumption $C_0=H^s$, $s=1, \cdots, n-1$.

Now we claim that

$$C_k=\sum_{p_1+\cdots+p_s=k}M_{p_1}\cdots M_{p_s}\qquad(k=0, 1, 2, \cdots)\tag{8}$$

satisfy (6) and (7). In fact if $k=0$, then (8) is compatible with the assumption $C_0=H^s$ $(s=1, \cdots, n-1)$. If $k=1$, then from (7) we obtain

$$M_0C_1-C_1M_0=M_0^sM_1-M_1M_0^s,\tag{9}$$

and direct calculation shows (9) has a solution

$$C_1=\sum_{p_1+\cdots+p_s=1}M_{p_1}\cdots M_{p_s}=\sum_{r=0}^{s-1}M_0^{s-1-r}M_1M_0^r.$$

This implies that (8) satisfies (7) if $k=1$. Next we assume that (8) satisfies (6) and (7) for $k=0, \cdots, j$. Then from (7) and the hypothesis of induction we have

$$\begin{aligned}
&M_0C_{j+1}-C_{j+1}M_0\\
&=\sum_{r=0}^{j}(C_rM_{j+1-r}-M_{j+1-r}C_r)\\
&=\sum_{r=0}^{j}\{(\sum_{p_1+\cdots+p_s=r}M_{p_1}\cdots M_{p_s})M_{j+1-r}-M_{j+1-r}(\sum_{p_1+\cdots+p_s=r}M_{p_1}\cdots M_{p_s})\}\\
&=\sum_{\left\{\substack{p_1+\cdots+p_{s+1}=j+1\\ p_{s+1}\neq 0}\right\}}M_{p_1}\cdots M_{p_{s+1}}-\sum_{\left\{\substack{p_1+\cdots+p_{s+1}=j+1\\ p_1\neq 0}\right\}}M_{p_1}\cdots M_{p_{s+1}}\\
&=\sum_{\left\{\substack{p_1+\cdots+p_{s+1}=j+1\\ p_1\neq 0;\ p_{s+1}\neq 0}\right\}}M_{p_1}\cdots M_{p_{s+1}}-\sum_{\left\{\substack{p_1+\cdots+p_{s+1}=j+1\\ p_1\neq 0;\ p_{s+1}\neq 0}\right\}}M_{p_1}\cdots M_{p_{s+1}}\\
&\quad+\sum_{q=1}^{s}(M_0^q\sum_{\left\{\substack{p_1+\cdots+p_{s-q+1}=j+1\\ p_1\neq 0;\ p_{s-q+1}\neq 0}\right\}}M_{p_1}\cdots M_{p_{s-q+1}}-\sum_{\left\{\substack{p_1+\cdots+p_{s-q+1}=j+1\\ p_1\neq 0;\ p_{s-q+1}\neq 0}\right\}}M_{p_1}\cdots M_{p_{s-q+1}}M_0^q)\\
&=\sum_{q=1}^{s}(M_0^q\sum_{\left\{\substack{p_1+\cdots+p_{s-q+1}=j+1\\ p_1\neq 0;\ p_{s-q+1}\neq 0}\right\}}M_{p_1}\cdots M_{p_{s-q+1}}-\sum_{\left\{\substack{p1+\cdots+p_{s-q+1}=j+1\\ p_1\neq 0;\ p_{s-q+1}\neq 0}\right\}}M_{p_1}\cdots M_{p_{s-q+1}}M_0^q).
\end{aligned}$$

Since we may regard M_1 in (9) as an arbitary matrix, we can replace M_1 by

$$\sum_{\left\{\substack{p_1+\cdots+p_{s-q+1}=j+1\\ p_1\neq 0;\ p_{s-q+1}\neq 0}\right\}} M_{p_1}\cdots M_{p_{s-q+1}}$$

and s by q in (9). Thus we obtain

$$\begin{aligned} C_{j+1} &= \sum_{q=1}^{s}\sum_{r=0}^{q-1} M_0^{q-1-r} \sum_{\left\{\substack{p_1+\cdots+p_{s-q+1}=j+1\\ p_1\neq 0;\ p_{s-q+1}\neq 0}\right\}} M_{p_1}\cdots M_{p_{s-q+1}} M_0^r \\ &= \sum_{p_1+\cdots+p_s=j+1} M_{p_1}\cdots M_{p_s}. \end{aligned}$$

Since $M(\varepsilon)=\sum_{k=0}^{\infty} M_k\varepsilon^k$ is convergent, there exist constants ξ and η such that

$$\|M_k\| \leq \xi\eta^{-k}.$$

Therefore

$$\|C_k\| \leq \sum_{p_1+\cdots+p_s=k} \xi\eta^{-p_1}\cdots\xi\eta^{-p_s} \leq \binom{s+k-1}{k}\xi^s\eta^{-k}$$

and

$$\lim_{k\to\infty}\frac{\|C_k\|}{\|C_{k+1}\|} \leq \eta.$$

Consequently $C(\varepsilon)=\sum_{k=0}^{\infty} C_k\varepsilon^k$ defined by (8) converges for $|\varepsilon|<\eta$ and the proof is completed.

§ 3. Here we put formally

$$U(t, \varepsilon)=\sum_{k=0}^{\infty} U_k(t)\varepsilon^k$$

and substituting this into $(2)_\varepsilon$ then we obtain

(10) $$\dot{U}_0 = A_0U_0 - U_0A_0$$

(11) $$\dot{U}_k = A_0U_k - U_kA_0 + F_k(t)$$

(12) $$F_k(t)=\sum_{q=1}^{k} (A_q(t)U_{k-q} - U_{k-q}A(t)).$$

Let $X(t, \varepsilon)$ be a fundamental matrix of a linear periodic system

$$\dot{x} = A(t, \varepsilon)x, \qquad x \in \boldsymbol{C}^n$$

with an initial condition

$$X(0, \varepsilon) = I.$$

Then, as was shown in [1], (11) is solved in a form:

$$U_k(t)=X_0(t)\left[\int X_0(t)^{-1}F_k(t)X_0(t)dt+C_k\right]X_0(t)^{-1} \tag{13}$$

where

$$X_0(t)=X(t, 0)$$

and C_k is a constant matrix. Since C_k is not uniquely determined, our main task in this paper is to determine these C_k's properly.

Now we represent (11) by its elements. Since

$$A_0=\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix},$$

we obtain

$$X_0(t)=e^{\lambda t}\begin{bmatrix} 1 & t & \cdots & \dfrac{t^{n-1}}{(n-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ 0 & & & 1 \end{bmatrix}$$

and

$$X_0(t)^{-1}=e^{-\lambda t}\begin{bmatrix} 1 & -t & \cdots & \dfrac{(-1)^{n-1}}{(n-1)!}t^{n-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & -t \\ 0 & & & 1 \end{bmatrix}$$

Suppose that, stating from some $U_0(t)$, we can obtain $U_r(t)$ $(r=1, \cdots, k-1)$ to be periodic in t. Then $F_k(t)$ are obviously periodic and can be represented by convergent Fourier series

$$F_k(t)=\sum_{m=-\infty}^{\infty}\hat{F}_k(m)\exp(m\sqrt{-1}t).$$

If we set

$$\hat{F}_k(m)=[\hat{f}_{ij}^{(k)}(m)]$$

then

$$X_0(t)^{-1}F_k(t)X_0(t)$$
$$=\left[\sum_{m=-\infty}^{\infty}\sum_{q_1=i}^{n}\sum_{q_2=1}^{j}\frac{(-1)^{q_1-i}}{(q_1-i)!\,(j-q_2)!}\hat{f}^{(k)}_{q_1q_2}(m)t^{q_1-1+j-q_2}\exp(m\sqrt{-1}t)\right].$$

However, if μ is a non-zero complex number and η is a nonnegative integer, then

$$\int t^{\eta}e^{\mu t}dt=\sum_{k=0}^{\eta}\frac{(-1)^{\eta-k}\eta!}{k!\,\mu^{\eta-k+1}}t^{k}e^{\mu t}$$

from our convention (3). Hence

$$\int X_0(t)^{-1}F_k(t)X_0(t)dt$$
$$=\left[\sum_{m=-\infty}^{\infty}\sum_{q_1=i}^{n}\sum_{q_2=1}^{j}\frac{(-1)^{q_1-i}}{(q_1-i)!\,(j-q_2)!}\hat{f}^{(k)}_{q_1q_2}(m)\int t^{q_1-i+j-q_2}\exp(m\sqrt{-1})\,dt\right]$$
$$=\left[\sum_{q_1=i}^{n}\sum_{q_2=1}^{j}\frac{(-1)^{q_1-i}\hat{f}^{(k)}_{q_1q_2}(0)}{(q_1-i)!\,(j-q_2)!\,(q_1-i+j-q_2+1)}t^{q_1-i+j-q_2+1}\right.$$
$$+\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{q_1=i}^{n}\sum_{q_2=0}^{j}\sum_{q_3=0}^{q_1-i+j-q_2}$$
$$\left.\frac{(-1)^{j-q_2-q_3}(q_1-i+j-q_2)!\,\hat{f}^{(k)}_{q_1q_2}(m)}{(q_1-i)!\,(j-q_2)!\,q_3!\,(m\sqrt{-1})^{q_1-i+j-q_2-q_3+1}}t^{q_3}\exp(m\sqrt{-1}\,t)\right],$$

and from (13)

$$(14)\quad U_k(t)=\left[\sum_{q_4=i}^{n}\sum_{q_5=1}^{j}\sum_{q_1=q_4}^{n}\sum_{q_2=1}^{q_5}\right.$$
$$\frac{(-1)^{q_1-q_4+j-q_5}\hat{f}^{(k)}_{q_1q_2}(0)}{(q_4-i)!\,(j-q_5)!\,(q_1-q_4)!\,(q_5-q_2)!\,(q_1-q_4+q_5-q_2+1)}t^{q_1-i+j-q_2+1}$$
$$+\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{q_4=i}^{n}\sum_{q_5=1}^{j}\sum_{q_1=q_4}^{n}\sum_{q_2=1}^{q_5}\sum_{q_3=0}^{q_1-q_4+q_5-q_2}$$
$$\frac{(-1)^{j-q_2-q_3}(q_1-q_4+q_5-q_2)!\,\hat{f}^{(k)}_{q_1q_2}(m)t^{q_4-i+j-q_5+q_3}}{(q_4-i)!\,(j-q_5)!\,(q_1-q_4)!\,(q_5-q_2)!\,q_3!\,(m\sqrt{-1})^{q_1-q_4+q_5-q_2-q_3+1}}\exp(\sqrt{-1}t)$$
$$\left.+\sum_{q_4=i}^{n}\sum_{q_5=1}^{j}\frac{(-1)^{j-q_5}}{(q_4-i)!\,(j-q_5)!}c^{(k)}_{q_4q_5}t^{q_4-i+j-q_5}\right].$$

If $U_k(t)$ is periodic, then all terms including $t^{q}(q\geq 1)$ must vanish and hence

$$(15)\qquad U_k(t)=[g_{ij}(t)+c^{(k)}_{ij}]$$

where

$$g^{(k)}_{ij}(t)=\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{q_1=i}^{n}\sum_{q_2=1}^{j}\frac{(-1)^{j-q_2}(q_1-i+j-q_2)!\,\hat{f}^{(k)}_{q_1q_2}(m)\exp(m\sqrt{-1}t)}{(q_1-i)!\,(j-q_2)!\,(m\sqrt{-1})^{q_1-i+j-q_2+1}}.$$

Indeed, in (14) the first term must vanish, the second term appears only if $q_4=i$, $q_5=j$, $q_3=0$, and the third term appear only if $q_4=i$, $q_5=j$. Since linear terms of t in (14) must vahish, we obtain

$$\hat{f}_{ij}^{(k)}(0)+c_{i+1j}^{(k)}-c_{ij-1}^{(k)}=0, \tag{16}$$

where $c_{i+1j}^{(k)}=0$ if $i=n$ and $c_{ij-1}^{(k)}=0$ if $j=1$. From (16) we obtain

$$c_{ij}^{(k)}=-\sum_{q=0}^{j-1}\hat{f}_{i-j+q\ q+1}^{(k)}(0), \qquad i=j+1, \cdots, n, \tag{17}$$

In fact if $j=1$, then from (16)

$$\hat{f}_{i1}^{(k)}(0)+c_{i+1\,1}^{(k)}=0,$$

namely

$$c_{i1}^{(k)}=-\hat{f}_{i-1\,1}^{(k)}(0) \qquad (i=2, \cdots, n).$$

Therefore (17) is valid for $j=1$. If (17) is valid for $j=1, \cdots, r$, then

$$\hat{f}_{ir+1}^{(k)}(0)+c_{i+1\ r+1}^{(k)}-c_{ir}^{(k)}=0$$

and therefore

$$\begin{aligned} c_{i+1\ r+1}^{(k)} &= -\hat{f}_{ir+1}^{(k)}(0)-\sum_{q=0}^{r-1}\hat{f}_{i-r+q\ q+1}^{(k)}(0) \\ &= -\sum_{q=0}^{r}\hat{f}_{i-q+r\ q+1}^{(k)}(0). \end{aligned}$$

Namely

$$c_{ir+1}^{(k)}=-\sum_{q=0}^{r}\hat{f}_{i-(r+1)+q\ q+1}^{(k)}(0) \qquad (i=r+2, \cdots, n). \tag{18}$$

Therefore by induction (17) is valid.

Next we consider the case when $i\leq j$. If $j-i=1+r$ $(r=0, 1, \cdots, n-2)$, then

$$\hat{f}_{i\,i+1+r}^{(k)}(0)+c_{i+1\,i+1+r}^{(k)}-c_{i\,i+r}^{(k)}=0, \qquad i=1, \cdots, n-r-1$$

from (16) and hence

$$c_{i+1\,i+1+r}^{(k)}=c_{1\,1+r}^{(k)}-\sum_{q=1}^{i}\hat{f}_{q\,q+1+r}^{(k)}(0), \qquad i=1, \cdots, n-r-1.$$

Here we remark that $c_{1\,1+r}^{(k)}$ cannot be determined. Changing the indices $i+1+r$ and $i+1$ for j and i respectively, we obtain

(19) $$c_{ij}^{(k)}=c_{1\,1+j-i}^{(k)}-\sum_{q=1}^{i-1}\hat{f}_{q\,q+1+j-i}^{(k)}(0), \qquad 2\leq i\leq j.$$

From (17) and (19)

$$[c_{ij}^{(k)}]=[\hat{g}_{ij}^{(k)}(0)]+\begin{bmatrix} c_{11}^{(k)} & c_{12}^{(k)} & \cdots & c_{1n}^{(k)} \\ & \ddots & \ddots & \vdots \\ & & \ddots & c_{12}^{(k)} \\ 0 & & & c_{11}^{(k)} \end{bmatrix}$$

where

(20) $$\hat{g}_{ij}^{(k)}(0)=\begin{cases} -\sum_{q=0}^{j-1}\hat{f}_{i-j+q\,q+1}^{(k)}(0) & \text{if } i>j, \\ -\sum_{q=1}^{i-1}\hat{f}_{q\,q+1+j-i}^{(k)}(0) & \text{if } i\leq j \text{ and } i=2,\cdots,n, \\ 0 \quad \text{if } i\leq j \quad \text{and} \quad i=1 \end{cases}$$

From the form of $X_0(t)$, the undetermined part

$$\begin{bmatrix} c_{11}^{(k)} & c_{12}^{(k)} & \cdots & c_{1n}^{(k)} \\ & \ddots & \ddots & \vdots \\ & & \ddots & c_{12}^{(k)} \\ 0 & & & c_{11}^{(k)} \end{bmatrix}$$

of C_k is found to be commutative with $X_0(t)$. In what follows, we will denote this undetermined part of C_k by D_k.

§ **4.** In this section we shall show that D_k $(k=1, 2, \cdots)$ can be taken to be zero. The verification of this follows the same outline as in § 5 of [3], and so we shall state only the outline of the proof.

First we define operators as follows. For every matrix-valued periodic function

$$H(t)=[h_{ij}(t)]=\left[\sum_{m=-\infty}^{\infty}\hat{h}_{ij}(m)\exp(\sqrt{-1}t)\right],$$

we put

$$\mathscr{A}_q H=A_q(t)H(t)-H(t)A_q(t), \qquad q=0, 1, \cdots,$$
$$\mathscr{X}H=X_0(t)\left[\int X_0(t)^{-1}H(t)X_0(t)dt+\Phi(H)\right]X_0(t)^{-1}$$

where

$$\Phi(H)=[\varphi_{ij}(H)]$$

$$\varphi_{ij}(H)=\begin{cases} -\sum_{q=0}^{j-1}\hat{h}_{i-j+q\ q+1}(0) & \text{if } i>j, \\ -\sum_{q=1}^{i-1}\hat{h}_{q\ q+1+j-i}(0) & \text{if } i\leq j \text{ and } i=2,\cdots,n, \\ 0 & \text{if } i\leq j \text{ and } i=1, \end{cases}$$

and

$$\mathscr{Y}_q=\mathscr{X}\circ\mathscr{A}_q.$$

Using these operators, $U_k(t)$ can be expressed as

$$U_k(t)=\sum_{q=0}^{k}\mathscr{Z}_{k-q}D_q \qquad (k=0, 1, 2, \cdots) \tag{21}$$

where

$$\mathscr{Z}_q=\begin{cases}\xi \text{ (identity operator)} & \text{if } q=0, \\ \sum_{\Sigma p_i=q}\prod_i \mathscr{Y}_{p_i} & \text{if } q=1, 2, \cdots. \end{cases}$$

In fact (21) can be shown by induction.

Next we shall show that secular terms of $U_k(t)$ vanish for every D_k belonging to Δ where Δ denotes the totality of the matrices of the form

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ & \ddots & \ddots & \vdots \\ & & \ddots & c_2 \\ 0 & & & c_1 \end{bmatrix}$$

Let $S_k(t, D_k)$ be secular terms of $U_k(t)=\sum_{q=0}^{k}\mathscr{Z}_{k-q}D_q$, and $\mathscr{S}$ be an operator defined by

$$S_k(t, D_s)=\mathscr{S}F_k(t).$$

Then we have

$$S_k(t, D)=\sum_{s=0}^{k-1}\mathscr{W}(s, k)D_s$$

where

$$\mathscr{W}(s, k)=\sum_{q=s}^{k-1}\mathscr{S}\mathscr{A}_{k-q}\mathscr{Z}_{q-s}.$$

However by the same procedure stated in §2 of [3], we can show that $\mathscr{W}(s, k)$ are null operators defined on Δ. Therefore for every $D_k \in \Delta$, the secular terms $S_k(t, D)$ vanish.

Finally, we shall show that the formal solution $\sum_{k=0}^{\infty} U_k(t)\varepsilon^k$ obtained by putting $D_k=0$ $(k=1, 2, \cdots)$ does converge. Since secular terms vanish, we have

$$\mathscr{X}F_k=[g_{ij}^{(k)}(t)+\hat{g}_{ij}^{(k)}(0)]$$

and hence

$$\begin{aligned}\|\mathscr{X}F_k\| &\leq \left\|\left[\sum_{\substack{m=-\infty\\m\neq 0}}^{\infty}\sum_{q_1=i}^{n}\sum_{q_2=1}^{j}\frac{(q_1-i+j-q_2)!}{(q_1-i)!\,(j-q_2)!\,|m|^{q_1-i+j-q_2+2}}\right]\right\|\cdot\|\dot{F}_k(t)\| \\ &\quad +(n-1)\|F_k(t)\| \\ &\leq \beta|||F_k(t)|||\end{aligned}$$

where

$$\beta=\max_i \sum_{j=1}^{\infty}\sum_{\substack{m=-\infty\\m\neq 0}}^{\infty}\sum_{q_1=i}^{n}\sum_{q_2=1}^{j}\frac{(q_1-i+j-q_2)!}{(q_1-i)!\,(j-q_2)!\,|m|^{q_1-i+j-q_2+2}}.$$

Moreover

$$\begin{aligned}\left\|\frac{d}{dt}(\mathscr{X}F_k)\right\| &\leq \left\|\frac{d}{dt}(X_0(t)\int X_0(t)^{-1}F_k(t)X_0(t)dt+\Phi(F_k))X_0(t)^{-1}\right\| \\ &\leq \|\mathscr{A}_0\mathscr{X}F_k(t)+F_k(t)\| \\ &\leq (2\alpha\beta+1)|||F_k(t)|||.\end{aligned}$$

Therefore

$$|||\mathscr{X}F_k|||\leq \gamma|||F_k(t)|||$$

where

$$\gamma=\beta+2\alpha\beta+1.$$

However

$$|||F_k|||\leq \sum_{q=1}^{k}|||\mathscr{A}_q|||\cdot|||U_{k-q}(t)|||\leq \sum_{q=1}^{k}2\alpha\rho^{-q}|||U_{k-q}(t)|||$$

and therefore

$$|||U_k(t)|||\leq|||\mathscr{X}F_k(t)|||\leq \gamma\sum_{q=1}^{k}2\alpha\rho^{q}|||U_{k-q}(t)|||.$$

If we choose

$$0<\eta<\frac{\rho}{2\alpha\gamma+1}, \qquad \mu=\|D_0\|$$

then

$$\text{(22)} \qquad |||U_k(t)||| \leq \mu\eta^{-k}.$$

In fact, if $k=0$, then (22) is evidently valid, and if (22) is valid for $k=1, \cdots, r-1$, then

$$|||U_r(t)||| \leq \gamma \sum_{q=1}^{r} 2\alpha\rho^{-q}\mu\eta^{-(r-q)}$$

$$\leq 2\alpha\gamma \frac{\eta/\rho}{1-\eta/\rho}\mu\eta^{-r}$$

$$\leq \mu\eta^{-r}.$$

Therefore $\sum_{k=0}^{\infty} U_k(t)\varepsilon^k$ converges for $|\varepsilon|<\rho/(2\alpha\gamma+1)$.

§5. It follows from the hypothesis (i) in §1 that (i,j)-elements $a_{ij}^{(k)}(t)$ of $A_k(t)$ can be expressed by uniformly convergent Fourier series such as

$$a_{ij}^{(k)}(t) = \sum_{m=-\infty}^{\infty} \hat{a}_{ij}^{(k)}(m) \exp(m\sqrt{-1}t).$$

By the discussion in the previosu sections we can put

$$C_k = [\hat{g}_{ij}^{(k)}(0)], \qquad k=1, 2, \cdots,$$

in (13) where $\hat{g}_{ij}^{(k)}(0)$ are defined by (20). If $\hat{g}_{ij}^{(k)}(m)$ denote Fourier coefficients of $\hat{g}_{ij}^{(k)}(t)$, then from (12) nad (15), we have

$$\text{(23)} \qquad \hat{g}_{ij}^{(k)}(m) = \sum_{q_1=i}^{n} \sum_{q_2=1}^{j} \frac{(-1)^{j-q_2}(q_1-i+j-q_2)!\hat{f}_{q_1q_2}^{(k)}(m)}{(q_1-i)!(j-q_2)!(m\sqrt{-1})^{q_1-i+j-q_2+1}} (m \neq 0),$$

$$\text{(24)} \qquad \hat{g}_{ij}^{(k)}(0) = \begin{cases} -\sum_{q=0}^{j-1} \hat{f}_{i-j+q\, q+1}^{(k)}(0) & (i>j) \\ -\sum_{q=1}^{i-1} \hat{f}_{q\, q+1+j-i}^{(k)}(0) & (i \leq j,\ i=2, \cdots, n) \\ 0 & (i \leq j,\ i=1) \end{cases}$$

$$\text{(25)} \qquad \hat{f}_{ij}^{(k)}(m) = \sum_{q=1}^{k} \sum_{s=1}^{n} \sum_{m_1=-\infty}^{\infty} \{\hat{a}_{is}^{(q)}(m_1)\hat{g}_{sj}^{(k-q)}(m-m_1) - \hat{g}_{is}^{(k-q)}(m-m_1)\hat{a}_{sj}^{(q)}(m_1)\}.$$

Substituting (25) into (23) and (24), we obtain the desired recurrence formula.

Theorem. *Let there be given a matrix differential equation* $(2)_\varepsilon$ *with coefficient* $A(t, \varepsilon)$ *satisfying* (i), (ii) *and* (iii) *in* §1. *Then we can construct a nondegenerate periodic solution of* $(2)_\varepsilon$

$$U(t,\varepsilon)=\sum_{k=0}^{\infty}U_k(t)\varepsilon^k,\qquad U_k(t)=[g_{ij}^{(k)}(t)],$$

by the following recurrence formula:

$$[g_{ij}^{(0)}(t)]=\begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ & \ddots & \ddots & \vdots \\ & & \ddots & c_2 \\ 0 & & & c_1 \end{bmatrix}$$

where $c_1,\cdots,c_n$ *are arbitrary constants which do not vanish at the same time.*

$$\hat{g}_{ij}^{(k)}(m)=\sum_{m_1=-\infty}^{\infty}\sum_{q_1=i}^{n}\sum_{q_2=1}^{j}\sum_{q=1}^{k}\sum_{s=1}^{n}\frac{(-1)^{j-q_2}(q_1-i+j-q_2)!}{(q_1-i)!\,(j-q_2)!\,(m\sqrt{-1})^{q_1-i+j-q_2+1}}$$
$$\times\{\hat{a}_{q_1 s}^{(q)}(m_1)\hat{g}_{sq_2}^{(k-q)}(m-m_1)-\hat{g}_{q_1 s}^{(k-q)}(m-m_1)\hat{a}_{sq_2}^{(q)}(m_1)\}$$

if $m\neq 0$ *and*

$$\hat{g}_{ij}^{(k)}(0)=\begin{cases} -\displaystyle\sum_{q=0}^{j-1}\sum_{q_1=i}^{k}\sum_{s=1}^{n}\sum_{m_1=-\infty}^{\infty}\{\hat{a}_{i-j+q\,s}^{(q_1)}(m_1)\hat{g}_{s\,q+1}^{(k-q_1)}(-m_1)-\hat{g}_{i-j+q\,s}^{(k-q_1)}(-m_1)\hat{a}_{s\,q+1}^{(q_1)}(m_1)\} & (\text{if } i>j) \\ -\displaystyle\sum_{q=1}^{i-1}\sum_{q_1=1}^{k}\sum_{s=1}^{n}\sum_{m_1=-\infty}^{\infty}\{\hat{a}_{qs}^{(q_1)}(m_1)\hat{g}_{s\,q+1+j-i}^{(k-q_1)}(-m_1)-\hat{g}_{qs}^{(k-q_1)}(-m_1)\hat{a}_{s\,q+1+j+1}^{(q_1)}(m_1)\} & (\text{if } i\leq j \text{ and } i=2,\cdots,n) \\ 0 & (\text{if } i\leq j \text{ and } i=1) \end{cases}$$

where $\hat{g}_{ij}^{(k)}(m)$ *are Fourier coefficients of* $g_{ij}^{(k)}(t)$.

Moreover the nondegenerate periodic solution $U(t,\varepsilon)=\sum_{k=0}^{\infty}U_k(t)\varepsilon^k$ *thus obtained is valid at least for*

$$|\varepsilon|<\frac{\rho}{2\alpha\gamma+1}$$

where

$$\beta=\max_i\sum_{j=1}^{n}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{q_1=i}^{n}\sum_{q_2=1}^{j}\frac{(q_1-i+j-q_2)!}{(q_1-i)!\,(j-q_2)!\,|m|^{q_1-i+j-q_2+2}}$$
$$\gamma=\beta+2\alpha\beta+1.$$

During the above discussion we obtain the following expression of a nondegenerate periodic solution of $(2)_\varepsilon$.

Corollary. *For every matrix-valued periodic function*

$$H(t)=\left[\sum_{m=-\infty}^{\infty}\hat{h}_{ij}(m)\exp(m\sqrt{-1}t)\right],$$

define

$$\mathscr{A}_q H=A_q(t)H(t)-H(t)A_q(t), \qquad q=1,2,\cdots,$$

$$\varphi_{ij}(H)=\begin{cases}-\sum_{q=1}^{j-1}\hat{h}_{i-j+q\ q+1}(0) & \text{if } i>j,\\ -\sum_{q=1}^{i-1}\hat{h}_{q\ q+1+j-i}(0) & \text{if } i\leq j \text{ and } i=2,\cdots,n,\\ 0 & \text{if } i\leq j \text{ and } i=1\end{cases}$$

$$\mathscr{X}H=\left[\sum_{m=-\infty}^{\infty}\sum_{q_1=i}^{n}\sum_{q_2=1}^{j}\frac{(-1)^{j-q_2}(q_1-i+j-q_2)!\,\hat{h}_{q_1q_2}(m)}{(q_1-i)!\,(j-q_2)!\,(m\sqrt{-1})^{q_1-i+j-q_2+1}}\exp(m\sqrt{-1}t)+\varphi_{ij}(H)\right]$$

$$\mathscr{Y}_q=\mathscr{X}\cdot\mathscr{A}_q$$

$$\mathscr{Z}_q=\begin{cases}E\ (\textit{identitity operator}) & \text{if } q=0\\ \sum_{\Sigma p_i=q}\prod_i \mathscr{Y}_{p_i} & \text{if } q=1,2,\cdots.\end{cases}$$

Then there exists a nondegenerate periodic solution expressed by

$$U(t,\varepsilon)=\sum_{k=0}^{\infty}\mathscr{Z}_k D_0\varepsilon^k$$

where $D_0\in\Delta$ *is a nondegerate constant matrix.*

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References

[1] Saito, T., On a singular point of a second order linear differential equation containing a prameter, Funkcial. Ekvac. **5** (1963), 1–29.

[2] Saito, T. and Tsukamoto, I., Perturbation method for linear periodic systems I, Funkcial. Ekvac. **27** (1984).

[3] Tsukamoto, I., Perturbation method for linear periodic systems II, Funkcial. Ekvac. **27** (1984).

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