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Absolute and Asymptotic Stability of Closed Sets

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Introduction.

Absolute stability was originally defined by Ura using a prolongation, [10]. Auslander and Seibert showed that a compact subset M of a locally compact metric space is absolutely stable if and only if there is a continuous Liapunov function for M, [1]. From this paper, if not earlier, until the present the appropriateness of a stability concept is judged by whether it can be characterized in terms of Liapunov functions.

Using a prolongation Hajek extended the concept of absolute stability to non-compact sets, [4]. His results that are of interest here are:

Theorem A. *Let M be a subset of a phase space X that is Hausdorff, para-compact, and locally compact. Then M is closed and absolutely stable if and only if $M = \bigcap v_i^{-1}(0)$ for suitable Liapunov functions v_i: $X \to [0, 1]$.*

Theorem B. *Let M be a subset of a phase space X that is Hausdorff, para-compact, and locally compact. Then a closed G_δ set M is absolutely stable if and only if $M = v^{-1}(0)$ for some Liapunov function v: $X \to [0, 1]$. In particular, a closed subset M of a locally compact metric space X is absolutely stable if and only if $M = v^{-1}(0)$ for some Liapunov function v: $X \to [0, 1]$.*

In [7] the author characterized the absolute stability (in Hajek's sense) of a closed subset M of a locally compact metric space in terms of the open positively invariant neighborhoods of M. Specifically, such a set M is absolutely stable if and only if M possesses a family F of neighborhoods satisfying:

(i) If $U \in F$, then U is open and positively invariant.

(ii) $\bigcap F = M$.

(iii) If $U \in F$, then there is a $V \in F$ such that $\overline{V} \subset U$.

(iv) If U, $V \in F$ are such that $\overline{U} \subset V$, then there is a $W \in F$ such that $\overline{U} \subset W \subset \overline{W} \subset V$.

Notice that in all of these results the phase space is assumed to be locally compact. This restriction prevents these results from being applied to dynamical or semidynamical systems that do not have locally compact phase spaces such as those determined by Volterra integral equations ([8]), by functional differential equations ([2], [5]), or by differential equations without uniqueness ([9]).

We will give a definition that is equivalent to Hajek's definition whenever the phase space is locally compact and metric. With this definition of absolute stability we will show that a closed proper subset M of a connected Hausdorff space is absolutely stable if and only if there is a Liapunov function for M. We then conclude the paper with a characterization of asymptotic stability for such closed sets M.

Notation and Terminology.

A dynamical system on a topological space X is a continuous mapping π of $X \times \boldsymbol{R}$ onto X satisfying the following axioms (where $x\pi t=\pi(x, t)$):

(1) $x\pi 0=x$ for each $x \in X$,

(2) $(x\pi t)\pi s=x\pi(t+s)$ for each $x \in X$ and $t, s \in R$.

If, in the definition of dynamical system, $\boldsymbol{R}$ is replaced by $\boldsymbol{R}^+$, the result is called a semidynamical system. If $M\subset X$ and $N\subset R^+$, then $M\pi N$ will denote the set $\{x\pi t: x \in M, t \in N\}$. A subset M of X is called positively invariant if $M\pi R^+=M$.

A Liapunov function for a closed subset M of X is a continuous mapping v of a neighborhood W of M into R^+ such that $v^{-1}(0)=M$ and $v(x\pi t)\leq v(x)$ for each $x \in W$ and $t \in R^+$.

Absolute Stability.

Definition 1. A closed nonempty subset M of a Hausdorff space X, on which a semidynamical system π is defined, is said to be absolutely stable with respect to π if there exists a set $\mathscr{F}$ of open, positively invariant neighborhoods of M such that

(i) if $U, V \in \mathscr{F}$ with $U\neq V$, then either $\overline{U}\subset V$ or $\overline{V}\subset U$,

(ii) $\bigcap \mathscr{F}=M$,

(iii) for each $U \in \mathscr{F}$, the set $\{V \in \mathscr{F}: V\subset U\}$ is uncountable.

If M is compact and X is connected, locally compact, and metric, then this defi nition is equivalent to the usual definition of absolute stability, [6].

Lemma 2. *Let X be a connected Hausdorff space that satisfies the first axiom of countability and let N be a closed, nonempty proper subset of X. Let G be a set of open subsets of X such that*

(i) $N=\bigcap G$,

(ii) *if $U, V \in G$ with $U\neq V$, then $\overline{U}\subset V$ or $\overline{V}\subset U$.*

Then there exists a countable subset $\{U_i\}$ of G such that $\overline{U}_{i+1}\subset U_i$, $N=\bigcap U_i$, and for each $V \in G$ there is an i such that $\overline{U}_i\subset V$.

Proof. Since X is connected and N is closed, N is not also open. Hence, $\partial N\neq$

ϕ. Let $x \in \partial N$ and let $\{V_i\}$ be a countable fundamental system of open neighborhoods of x. We will define inductively the desired subset $\{U_i\}$ of G in such a way that $V_i \not\subset U_i$ for each i. Since each V_i is a neighborhood of $x \in \partial N$ we must have $V_i - N \neq \phi$ for each i. Moreover, for each i there must be an element W_i of G such that $V_i \not\subset W_i$. Otherwise $N = \cap G \supset V_i$ which is impossible because $V_i - N \neq \phi$.

Set $U_1 = W_1$ and suppose that for some positive integer k we have determined elements $U_1, \cdots, U_{k-1}$ of G such that $\bar{U}_i \subset U_{i-1}$ for $i=2, 3, \cdots, k-1$ and $V_i \not\subset U_i$, for $i=1, 2, \cdots, k-1$. Since U_{k-1} is a neighborhood of the closed, nonopen set N, properties (i) and (ii) assure us that there is a $U \in G$ such that $\bar{U} \subset U_{k-1}$. By property (ii) we have $\bar{U} \subset W_k$, $W_k \subset \bar{U}$, or $W_k = U$. In the first case set $U_k = U$ and in the latter two cases set $U_k = W_k$. Evidently we have determined an element U_k of G such that $\bar{U}_k \subset U_{k-1}$ and $V_k \not\subset U_k$. Therefore, we can determine inductively a countable subset $\{U_i\}$ of G such that $\bar{U}_i \subset U_{i-1}$ and $V_i \not\subset U_i$ for every i.

Since each U_i is an element of G we have that $N \subset U_i$ for each i. Suppose there is a $y \in \cap U_i - N$. Since $y \in \cup G$ and $y \notin N = \cap G$ there is a $V \in G$ such that $y \notin V$. Since $y \in U_i \in G$ for every i we conclude from property (ii) that $V \subset U_i$ for every i. Since $\{V_i\}$ is a fundamental system of neighborhoods of $x \in N \subset V$, there is a j such that $V_j \subset V$. This is impossible because $V_j \not\subset U_j$. Hence, we must have $N = \cap U_i$.

Now let V denote any element of G. By property (ii) we have $\bar{U}_i \subset V$, $\bar{V} \subset U_i$, or $V = U_i$ for each i. If $\bar{V} \subset U_i$ or $V = U_i$ for each i then $N = \cap U_i \supset V$, which is impossible because V is a neighborhood of N. Therefore, there is an i such that $\bar{U}_i \subset V$. This completes the proof.

Lemma 3. *Let G be as in Lemma 2 and satisfy*

(iii) *for each $U \in G$ the set $\{V \in G\colon V \subset U\}$ is uncountable.*

Then there is a $W \in G$ such that both of the sets $\{V \in G\colon V \subset W\}$ and $\{V \in G\colon W \subset V\}$ are uncountable.

Proof. Let $\{U_i\}$ be a subset of G as determined in Lemma 1. Since each $U_i \in G$, it suffices to show that for some i the set $\{V \in G\colon U_i \subset V\}$ is uncountable. Suppose the contrary. Then the set $G_i = \{V \in \mathscr{F}\colon U_i \subset V\}$ is countable. From Lemma 2 we have that if $W \in G$, then $W \in \{V \in G\colon U_i \subset V\}$ for some i. It follows that $G = \cup G_i$. This is impossible because G is uncountable. The desired result follows.

Theorem 4. *Let M be a closed, nonempty proper subset of a connected Hausdorff space that satisfies the first axiom of countability. Then M is absolutely stable with respect to a semidynamical system π on X if and only if there is a Liapunov function for M.*

Proof. Let v be a Liapunov function for M. Set

$$\mathscr{F} = \{v^{-1}([0, r))\colon r \text{ is in the range of } v \text{ and } v^{-1}([0, r)) \neq X\}.$$

We will show that $\mathscr{F}$ satisfies the three properties in Definition 1. It is clear that $v^{-1}([0, r))$ is positively invariant and that

$$\bigcap v^{-1}([0, r))=M.$$

Moreover, since v is continuous, we also have that $v^{-1}([0, r))$ is an open neighborhood of M and that

$$v^{-1}([0, r))\subset\overline{v^{-1}([0\ r))}\subset\overline{v^{-1}([0, r])}=V^{-1}([0, r])\subset v^{-1}([0, s)) \tag{1}$$

whenever $r<s$. If $v^{-1}([0, r))\neq X$, then $v^{-1}([0, r))$ is not both open and closed since X is connected. Therefore by (1), $v^{-1}([0, r))\neq v^{-1}([0, s))$ whenever $r<s$. Collecting together these properties of the elements of $\mathscr{F}$ we conclude that $\mathscr{F}$ satisfies the three properties in Definition 1. Therefore, M is absolutely stable.

Now assume that M is absolutely stable. Let $\mathscr{F}$ be a set of open positively invariant neighborhoods of M that satisfies properties (i)–(iii) of Definition 1. For each dyadic rational r, we will construct a set $U(r)\in\mathscr{F}$ such that $U(r)\subset U(s)$ whenever $r<s$. We first obtain from $\mathscr{F}$ a set of neighborhoods $\{U(2^{-n})$: n a positive integer$\}$ such that $U(2^{-n-1})\subset U(2^{-n})$ and the set $\{A\in\mathscr{F}: U(2^{-n-1})\subset A\subset U(2^{-n})\}$ is uncountable. This can be done by induction in the following manner. Let $\{N_i\}$ be a countable subset of $\mathscr{F}$ such that $\bigcap N_i=M$ (Lemma 2). Set $U(2^{-1})=N_1$. Suppose $U(2^{-n})$ has been defined. Since $U(2^{-n})$, $N_{n+1}\in\mathscr{F}$, either $U(2^{-n})\cap N_{n+1}=N_{n+1}$ or $U(2^{-n})\cap N_{n+1}=U(2^{-n})$. Set $G=\{V\in\mathscr{F}: V\subset U(2^{-n})\cap N_{n+1}\}$. By Lemma 3 there is a $B\in G$ such that $\{W\in\mathscr{F}: W\subset B\}$ and $\{W\in\mathscr{F}: B\subset W\subset U(2^{-n})\}$ are uncountable. Set $U(2^{-n-1})=B$. Now extend this system to one with the desired properties. To illustrate this extension we will construct $U(3/8)$. Set $C=\{W\in\mathscr{F}: U(1/4)\subset W\subset U(1/2)\}$, $G=\{W\in C; \{U\in C, U\subset W\}$ is uncountable$\}$, and $N=U(1/4)$. By Lemma 3 there is a $D\in G$ such that $\{V\in G: V\subset D\}$ and $\{V\in G: D\subset V\}$ are uncountable. Set $U(3/8)=D$. Now define $v: U(1)\rightarrow R^+$ by $v(x)=\inf\{r: x\in U(r)\}$. Evidently $v(x)=0$ if and only if $x\in M$. If $x\in U(r)$ and $t\in R^+$, then $x\pi t\in U(r)$ since $U(r)$ is positively invariant. It easily follows that $v(x\pi t)\leq v(x)$. The continuity of v is proved as in the proof of Urysohm's lemma. Thus, we have constructed a Liapunov function for M. This completes the proof.

Combining the previous theorem and Theorem B we obtain the following result.

Corollary. *Let X be locally compact, connected and metric. Then a closed subset M of X is absolutely stable if and only if M is absolutely stable according to Hajek's definition* ([3, Definition 12]).

In the proof of Theorem 4 we have actually proved more than is stated in Theorem 4. We have proved:

Theorem 5. *Let M be a closed subset of a Hausdorff space that satisfies the first* *axiom of countability. Let M be absolutely stable with respect to a semidynamical system π on X and let $\mathscr{F}$ be a set of open positively invariatn neighborhoods of M satisfying properties* (i)–(iii) *of Definition* 1. *Then there is a Liapunov function v for M such that $v^{-1}([0, r)) \in \mathscr{F}$ for every dyadic rational r.*

Asymptotic Stability.

Definition 6. A closed subset M of a Hausdorff space X, on which a semidynamical system π is defined, is said to be asymptotically stable with respect to π if there exists a set $\mathscr{F}$ of open, positively invariant neighborhoods of M such that

(i) $\bigcap \mathscr{F} = M$,
(ii) if $U, V \in \mathscr{F}$ with $U \neq V$, then either $\bar{U} \subset V$ or $\bar{V} \subset U$,
(iii) for each $U \in \mathscr{F}$, the set $\{V \in \mathscr{F} : V \subset U\}$ is uncountable,
(iv) for each $x \in \bigcup \mathscr{F}$ and each $U \in \mathscr{F}$, $x\pi t$ is eventually in U.

Theorem 7. *Let M be a closed subset of a Hausdorff space that satisfies the first axiom of countability. If M is asymptotically stable with respect to a semidynamical system π on X, then there is a Liapunov function v for M such that*

(i) *$v(x\pi t) \to 0$ as $t \to \infty$ for every x in the domain of v,*
(ii) *$v(x\pi t) < v(x)$ for every x in the domain of v and every $t > 0$.*

Proof. Let $\mathscr{F}$ be a set of open positively invariant neighborhoods of M satisfying properties (i)–(iv) of Definition 6. Clearly M is absolutely stable. By Theorem 5 there is a Liapunov function w for M mapping a neighborhood U of M into the interval $[0, 1]$ such that $w^{-1}([0, r)) \in \mathscr{F}$ for each dyadic rational r. Define $v \colon U \to R^+$ by

$$v(x) = \int_0^\infty e^{-s} w(x\pi s) ds.$$

Evidently v is continuous since w is continuous. For each $x \in U$ and dyadic rational r there is a t' such that $x\pi t \in w^{-1}([0, r))$ for all $t \geq t'$. Then

$$\begin{aligned} v(x\pi t) &= \int_0^\infty e^{-s} w(x\pi(t+s)) ds \\ &\leq r \int_0^\infty e^{-s} ds = r. \end{aligned}$$

It follows that $v(x\pi t) \to 0$ as $t \to \infty$. This proves (i). A standard argument that can be found on page 145 of [3] establishes (ii).

Elementary examples show that the existence of a Liapunov function satisfying property (ii) in Theorem 7 is not sufficient to assure the asymptotic stability of M.

However, the existence of a Liapunov function satisfying the first property does assure that M is asymptotically stable.

Theorem 8. *Let M be a closed subset of a Hausdorff space. If there is a Liapunov function v for M such that $v(x\pi t)\to 0$ as $t\to\infty$ for every x in the domain of v, then M is asymptotically stable.*

The proof of this Theorem is nearly identical to the first half of the proof of Theorem 4 and will be omitted.

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