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On the Nonexistence of Solutions for an Elliptic Problem in Unbounded Domains

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§ 1. Introduction.

In this paper we shall discuss Choquard's generalized equation, that is

$$-\Delta u-\alpha V(x)u-u(x)\int_{R^n} V(x-y)u^2(y)dy=\lambda u \tag{1.1}$$

where $x \in \boldsymbol{R}^n$, Δ denotes the Laplacian operator and V is a non-negative real-valued function in $\boldsymbol{R}^n$, which depends only on the radius, $r=|x|$. In (1.1) α and λ are real numbers and $\alpha \geq 0$. In our previous work [4] we studied equation (1.1) and we concluded that under suitable assumptions on V and for any $\lambda \geq 0$ there are no positive solutions u of (1.1), $u \not\equiv 0$ which belongs to $H^1(\boldsymbol{R}^n)$, that is the Sobolev space of order one. In this note we shall give some further results on the non-existence of such solutions of (1.1) in such a way that we will be allowed to consider V's in a much wider class.

Equation (1.1) in specific cases has been considered by a number of authors as an approximation to the Hartree-Fock theory for one component plasma. As P. L. Lions pointed out in [2], equation (1.1) provides solitary waves for the coupled Schrödinger-Klein Gordon system of equations.

In this paper we shall not discuss existence results which have been treated, for instance in [1], [2] and [3]. We shall use the standard notation: By $L^p(\boldsymbol{R}^n)$ $1 \leq p < \infty$ we denote the space of functions from $\boldsymbol{R}^n$ into $\boldsymbol{R}$ whose p-th powers are integrable. By $L^\infty(\boldsymbol{R}^n)$ we denote the space of measurable functions which are essentially bounded. By $H^m(\boldsymbol{R}^n)$ we shall denote the usual Sobolev space of order m.

§ 2. The results.

Lemma 2.1. *Assume that* V: $\boldsymbol{R}^n \to \boldsymbol{R}$ *is non-negative and let us suppose that there exists a solution* u *of* (1.1) *which is positive, radial, decreasing with* $r=|x|$ *and smooth*

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(say of class C^2), for some $\lambda>0$. Then, there exists a constant $C>0$ such that

$$u(|x|)\leq C\exp\left(-\frac{\lambda|x|^2}{4n}\right) \qquad \text{for } r=|x|\geq 1.$$

Proof. We shall write $u(r)$ instead of $u(x)=u(|x|)$. Let us write the Laplacian operator in spherical coordinates. Thus, from (1.1) we obtain

$$-u_{rr}-\frac{(n-1)}{r}u_r\geq\lambda u \tag{2.1}$$

where

$$u_r=\frac{\partial u}{\partial r} \quad \text{and} \quad u_{rr}=\frac{\partial^2 u}{\partial r^2}, \qquad r=|x|.$$

Clearly, from (2.1) if follows that

$$-(r^{n-1}u_r)_r\geq\lambda r^{n-1}u. \tag{2.2}$$

Let us integrate inequality (2.2) from $2^{-1/n}$ to r and use the fact that u is decreasing to obtain

$$u(r)\leq 2n\lambda^{-1}[2^{-(n-1)/n}u_r(2^{-1/n})-r^{n-1}u_r(r)]r^{-n} \tag{2.3}$$

provided that $r\geq 1$. Since u is decreasing in r it follows that

$$u(r)\leq -2n\lambda^{-1}r^{-1}u_r(r) \tag{2.4}$$

for $r\geq 1$. From (2.4) we obtain the inequality

$$\frac{d}{dr}\left(u(r)\exp\left(\frac{\lambda r^2}{4n}\right)\right)\leq 0 \tag{2.5}$$

for $r\geq 1$. Integration of (2.5) from 1 to r gives us

$$u(r)\leq C\exp\left(-\frac{\lambda r^2}{4n}\right)$$

where

$$C=u(1)\exp\left(\frac{\lambda}{4n}\right).$$

This proves the lemma

Theorem 2.1. *Suppose that $V\colon \boldsymbol{R}^n\to\boldsymbol{R}$ is a non-negative real-valued function which is radial and such that $V(x)\leq C(1+|x|)^{-n-\varepsilon}$ for some $\varepsilon>0$ and some constant*

$C>0$. *Then, for any* $\lambda>0$, *there is no solution* u *of* (1.1) *which is positive, radial, decreasing with* $r=|x|$, $u\not\equiv 0$ *and belonging to* $C^2(\boldsymbol{R}^n)\cap H^1(\boldsymbol{R}^n)$.

Proof. We shall use a device pointed out by W. Strauss in [5] based on Kato's theorem on the absence of positive discrete eigenvalues for the Schrödinger operator. Suppose that for some $\lambda>0$ there is a solution u of (1.1) with the above mentioned properties. Let us denote by $q_u(x)=-\alpha V(x)-(V*u^2)(x)$. Thus, we can write equation (1.1) as $-\Delta u+q_u(x)u=\lambda u$ in $\boldsymbol{R}^n$ with $\lambda>0$. We shall show that $q_u(x)$ satisfies the hypothesis of Kato's theorem. Clearly $V*u^2$ belongs to $L^\infty(\boldsymbol{R}^n)$. Thus, it remains to show that $V*u^2$ is of order $(1+|x|)^{-a}$ for some $a>1$. Let $x\in\boldsymbol{R}^n$ such that $|x|\geq 2$ and let us write

$$(V*u^2)(x)=\int_{\Omega_x}V(x-y)u^2(y)dy+\int_{\boldsymbol{R}^n-\Omega_x}V(x-y)u^2(y)dy \tag{2.6}$$

where $\Omega_x=\{y\in\boldsymbol{R}^n,\ y\cdot x\leq|x|^2/2\}$ and $y\cdot x$ denotes the inner product of y and x. It is clear that if $y\in\Omega_x$ then $|y-x|\geq|x|/2$. Since $V(x-y)\leq\theta(x-y)$ where $\theta(x)=C(1+|x|)^{-n-\varepsilon}$, then it follows that

$$\begin{aligned}\int_{\Omega_x}V(x-y)u^2(y)dy&\leq\int_{\Omega_x}\theta(x-y)u^2(y)dy\\&\leq\theta\left(\frac{|x|}{2}\right)\int_{\boldsymbol{R}^n}u^2(y)dy=\theta\left(\frac{|x|}{2}\right)\|u\|^2_{L^2}\end{aligned}\tag{2.7}$$

because θ is decreasing.

Similarly

$$\begin{aligned}\int_{\boldsymbol{R}^n-\Omega_x}V(x-y)u^2(y)dy&\leq\int_{\boldsymbol{R}^n-\Omega_x}V(x-y)w^2(y)dy\\&\leq w^2\left(\frac{|x|}{2}\right)\int V(x-y)dy=w^2\left(\frac{|x|}{2}\right)\|V\|_{L^1}\end{aligned}$$

where $w(x)=c\exp(-\lambda|x|^2/n)$ and we have used Lemma 2.1. Thus from (2.7) and (2.8) we obtain

$$(V*u^2)(x)\leq\theta\left(\frac{|x|}{2}\right)\|u\|^2_{L^2}+w^2\left(\frac{|x|}{2}\right)\|V\|_{L^1}$$

for $|x|\geq 2$. This proves that $V*u^2$ decays at the desired rate as $|x|\to+\infty$, which proves the theorem.

In what follows we shall consider the case in which $\lambda=0$.

Theorem 2.2. *Let* $V\colon\boldsymbol{R}^n\to\boldsymbol{R}$ *be a non-negative radial function such that* V *is continuously differentiable and* $V\in L^{n/2}(\boldsymbol{R}^n)+L^\infty(\boldsymbol{R}^n)$ $(n\geq 3)$. *Let us consider equation*

(1.1) *with* $\lambda=0$. *If* $2V(x)+\operatorname{grad} V(x)\cdot x>0$ *for all* $x\in \boldsymbol{R}^n$ *then any solution* u *of* (1.1) *with* $\lambda=0$ *such that* $u\in H^1(\boldsymbol{R}^n)$ *has to be zero almost everywhere.*

Proof. Suppose that u is a solution of (1.1) with $\lambda=0$. Thus, the functional $J\colon H^1(\boldsymbol{R}^n)\to \boldsymbol{R}$ given by

$$\begin{aligned}J(v)=&\frac{1}{2}\int_{R^n}|\operatorname{grad} v|^2\,dx-\frac{\alpha}{2}\int_{R^n}V(x)v^2(x)dx\\&-\frac{1}{4}\int_{R^n}\int_{R^n}V(x-y)v^2(x)v^2(y)dxdy\end{aligned}$$

is Frechet differentiable and u is a critical point of J, i.e., $J'(u)=0$ (see Theorem 5.1 in [3]). Let us consider the path $\varepsilon\to u(x/\varepsilon)$ ($\varepsilon>0$) which passes through the given solution u. By the "chain rule" we know that

$$0=\frac{d}{d\varepsilon}J\left(u\left(\frac{x}{\varepsilon}\right)\right)\Big|_{\varepsilon=1}.$$

Thus

$$\begin{aligned}0=&\frac{d}{d\varepsilon}\Big[\frac{1}{2}\int_{R^n}\left|\operatorname{grad} u\left(\frac{x}{\varepsilon}\right)\right|^2dx-\frac{\alpha}{2}\int_{R^n}V(x)u^2\left(\frac{x}{\varepsilon}\right)dx\\&-\frac{1}{4}\int_{R^n}\int_{R^n}V(x-y)u^2\left(\frac{x}{\varepsilon}\right)u^2\left(\frac{y}{\varepsilon}\right)dxdy\Big]\Big|_{\varepsilon=1}.\end{aligned}\tag{2.8}$$

Direct calculation on each term of (2.9) gives us

$$\begin{aligned}0=&\frac{n-2}{2}\int_{R^n}|\operatorname{grad} u|^2\,dx-\frac{n\alpha}{2}\int_{R^n}V(x)u^2(x)dx\\&-\frac{\alpha}{2}\int_{R^n}\operatorname{grad} V(x)\cdot xu^2(x)\,dx-\frac{n}{2}\int_{R^n}\int_{R^n}V(x-y)u^2(x)u^2(y)dxdy\\&-\frac{1}{4}\int_{R^n}\int_{R^n}\operatorname{grad} V(x-y)\cdot(x-y)u^2(x)u^2(y)dxdy.\end{aligned}$$

Let us multiply equation (1.1) by u and integrate by parts to obtain

$$0=\int_{R^n}|\operatorname{grad} u|^2\,dx-\alpha\int_{R^n}V(x)u^2(x)\,dx-\int_{R^n}\int_{R^n}V(x-y)u^2(x)u^2(y)dxdy$$

which together with the above identity gives us

$$\begin{aligned}0=&2\alpha\int_{R^n}[2V(x)+\operatorname{grad} V(x)\cdot x]u^2(x)dx\\&+\int_{R^n}\int_{R^n}[4V(x-y)+\operatorname{grad} V(x-y)\cdot(x-y)]u^2(x)u^2(y)dxdy.\end{aligned}$$

By hypothesis we know that $V \geq 0$ and $2V(x)+\text{grad } V(x) \cdot x$ is positive, thus we conclude that u has to be zero almost everywhere.

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References

[1] Lieb, E., Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Studies in Appl. Math., **57** (1977), 93–195.
[2] Lions, P. L., The Choquard equation and related questions, Nonlinear Analysis, **4–6** (1980), 1063–1073.
[3] Menzala, G. Perla, On regular solutions of a monlinear equation of Choquard's type, Proc. Roy. Soc. Edinburgh, Sect. A., **86** (1980), 293–301.
[4] ——, On an elliptic equation with a non-local nonlinearlity, Computational and Applied Mathematics, SBMA C (Brasil), **2**–1 (1983), 75–80.
[5] Strauss, W. A., Existence of solitary waves in higher dimensions, Comm. Math. Phys., **55** (1977), 149–162.

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