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Analogues of Cartan's Decomposition Theorem in Asymptotic Analysis

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In the present paper, the author introduces a new concept of asymptotic expansions of functions of several variables and proves its fundamental properties. These new asymptotic expansions are powerful tools for studying the pfaffian systems with singularities. For instance, we have a decomposition theorem (see Theorem 3 in § 3) analogous to Cartan's theorem, which plays an important role to solve the Riemann-Hilbert-Birkhoff problem of several variables case. Studies on pfaffian systems will be published elsewhere.

It was Y. Sibuya [6], [7] who obtained a decomposition theorem in asymptotic analysis of one variable. His theorem is related with Borel-Ritt theorem and reformulated in terms of cohomology of a sheaf by B. Malgrange [4]. To extend their results to several variables case, we need a new concept of asymptotic expansions, that is, "strong asymptotic developability" (see Definitions 1 and 2 in § 1) instead of asymptotic expansions of functions of several variables used by M. Hukuhara [2], R. Gérard-Y. Sibuya [1], K. Takano [8] etc..

In the first section, the definitions of strong asymptotic developability are given for a function holomorphic in an open polysector, and their elementary properties are listed. A basic idea for the definitions is to compare a function with certain kinds of formal series with respect to

$$|x|^N=\prod_{i=1}^n |x_i|^{N_i} \qquad (N=(N_1, \cdots, N_n)\in \boldsymbol{N}^n)$$

instead of

$$\sum_{i=1}^n |x_i|^p \qquad (p\in \boldsymbol{N}).$$

In the second section, we prove a theorem which asserts existence of a function holomorphic and strongly asymptotically developable to a given formal series. We call it the theorem of Borel-Ritt type.

The main theorems are given in the third section. The proof of Theorem 2 is done on an idea of Y. Sibuya used in one variable case. The idea is slightly different from that in [5] and he indicated it to the author during their stay at Strasbourg

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in 1980. Notice that Theorem 2 and Theorem 3 are considered as sharpened results of Cartan's decomposition theorem and that Theorem 4 is considered as that of Cousin's. We call them the theorems of Sibuya type.

We introduce a sheaf of germs of strongly asymptotically developable functions over real-n-dimensional torus T^n in the forth section. Combining the theorem of Borel-Ritt type with the theorems of Sibuya type, we obtain theorems in terms of cohomology of sheaves (see Theorems 5 and 6). These theorems in asymptotic analysis are compared with Theorems A and B in complex analysis.

In the last section, we treat the uniform strong asymptotic expansions of functions with parameters. All things in preceding sections are extended to this case.

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§ 0. Notation.

Throughout this paper, we use the following notation.

1) $\boldsymbol{N}=\{0, 1, \cdots\}$: set of non-negative integers.

2) For two integers n and n' such that $n<n'$, we put

$$[n, n']=\{i \in \boldsymbol{N};\ n\leqslant i\leqslant n'\}.$$

3) $\boldsymbol{R}^+$: set of all positive real numbers.

4) For $r=(r_1, \cdots, r_n)$, $r'=(r'_1, \cdots, r'_n)$ in $(\boldsymbol{R}^+)^n$, by $r\leqslant r'$ we mean that $r_i\leqslant r'_i$ for all $i=1, \cdots, n$.

5) $\boldsymbol{C}$: set of all complex numbers.

6) $D(r)=\{x \in \boldsymbol{C};\ |x|<r\}$

7) $S(\tau_-, \tau_+, r)=\{x \in \boldsymbol{C};\ 0<|x|<r,\ \tau_-<\arg x<\tau_+\}$: open sector.

8) $S[\tau_-, \tau_+, r]=\{x \in \boldsymbol{C};\ 0<|x|\leqslant r,\ \tau_-\leqslant\arg x\leqslant\tau_+\}$: closed sector.

9) For discs $D(r_i)$'s, $i=i, \cdots, n$, we put

$$\begin{aligned} &D(r_1)\times\cdots\times\check{D}(r_i)\times\cdots\times D(r_n) \\ &\quad =D(r_1)\times\cdots\times D(r_{i-1})\times D(r_{i+1})\times\cdots\times D(r_n) \end{aligned}$$

for a fixed i in $[1, n]$.

10) For a domain D in $\boldsymbol{C}^n$, we denote by $\mathcal{O}(D)$ the set of all holomorphic functions in D.

11) $\mathcal{O}(D(r_1)\times\cdots\times\check{D}(r_i)\times\cdots\times D(r_n))[[x_i]]$: $\boldsymbol{C}$-algebra of formal power series of one variable x_i with coefficients in $\mathcal{O}(D(r_1)\times\cdots\times\check{D}(r_i)\times\cdots\times D(r_n))$.

12) For $r=(r_1, \cdots, r_n) \in (\boldsymbol{R}^+)^n$, we put

$$\hat{\mathcal{O}}'_n(r)=\bigcap_{i=1}^n \mathcal{O}(D(r_1)\times\cdots\times\check{D}(r_i)\times\cdots\times D(r_n))[[x_i]].$$

13) $\hat{\mathcal{O}}'_n = \text{dir. lim.}_{r\to 0}\hat{\mathcal{O}}'_n(r)$.

14) $\hat{\mathcal{O}}_n$: $\boldsymbol{C}$-algebra of formal power series with coefficients in $\boldsymbol{C}$.

15) $\mathcal{O}_n$: $\boldsymbol{C}$-algebra of convergent power series with coefficients in $\boldsymbol{C}$.

16) For $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n)$ in $\boldsymbol{N}^n$, by $\alpha \geqslant \beta$ we mean that $\alpha_i \geqslant \beta_i$ for all $i=1, \cdots, n$, and so $\alpha \not\geqq \beta$ implies that $\alpha_i < \beta_i$ for some $i=1, \cdots, n$.

17) For a subset J of $[1, n]$, we denote by J^c the complement of J in $[1, n]$, and by $\sharp J$ the number of elements in J.

18) For a subset J of $[1, n]$, the set $\boldsymbol{N}^J=\{\alpha: J\to \boldsymbol{N}\}$ is regarded as the set of all multi-indices $\alpha_J=(\alpha_j)_{j\in J}$, $\alpha_j \in \boldsymbol{N}$ for all $j\in J$; x_J denotes the variables $(x_j)_{j\in J}$, and we put $x_J^{\alpha_J}=\prod_{j\in J} x_j^{\alpha_j}$.

19) For two subset J and J' of $[1, n]$ without intersection, we denote by $\alpha_{J\cup J'}$ or by $(\alpha_J, \alpha_{J'})$ the multi-index $(\alpha_j)_{j\in J\cup J'}$.

20) For a formal power series $\sum_{\alpha\in \boldsymbol{N}^n} f_\alpha x^\alpha$, for a subset J of $[1, n]$ and for a multi-index $\alpha_J=(\alpha_j)_{j\in J'}$ by

$$\sum\nolimits_{\beta\in \boldsymbol{N}^n} f_\beta x^\beta \qquad (\beta_i=\alpha_j,\ j\in J)$$

we mean the sum of all the terms $f_\beta x^\beta$ with the multi-indices $\beta=(\beta_1, \cdots, \beta_n)$ of which the j-th elements are equal to α_j for all $j\in J$. For the sake of simplicity, the sum is frequently written as follows,

$$x_J^{\alpha_J} \sum\nolimits_{\alpha_I\in \boldsymbol{N}^I} f_{\alpha_{J\cup I}} x_I^{\alpha_I}$$

with $I=J^c$.

21) I_m denotes the m by m unit matrix, and 0_m (resp. $0_{m,m'}$) denotes the m by m (resp. m') zero matrix.

22) For an m by m' matrix $A=(a_{ij})$, we put

$$|A|=\max\{|a_{ij}|;\ i=1, \cdots, m, j=1, \cdots, m'\}.$$

§ 1. Definitions and elementary properties.

In this section, we define the strong asymptotic developability of a function $f(x)$ in $x(x_1, \cdots, x_n)$ which is holomorphic in an open polysector $S=S_1\times\cdots\times S_n$ at the origin in $\boldsymbol{C}^n$. First, we give a definition that $f(x)$ is strongly asymptotically developable to a formal series $\hat{f}(x)$ which belongs to $\hat{\mathcal{O}}'_n$, and secondly we generalize the concept using formal series of other kinds.

Definition 1. We say that a holomorphic function $f(x)$ in an open polysector $S(r)=\prod_{i=1}^n S(\tau_{-i}, \tau_{+i}, r_i)$ is *strongly asymptotically developable to a formal series* $\hat{f}(x)=\sum_{\alpha\in \boldsymbol{N}^n} f_\alpha x^\alpha \in \hat{\mathcal{O}}'_n(r)$ *as* x *tends to* 0 *in* $S(r)$, if, for any $N\in \boldsymbol{N}^n$ and any closed subpolysector S' of $S(r)$, there exists a positive constant $K_{N,S'}$ such that

$$|f(x)-\sum\nolimits_{\alpha\in \boldsymbol{N}^n,\ \alpha\not\geqq N} f_\alpha x^\alpha| \leqslant K_{N,S'}|x|^N \tag{1.1}$$

for any $x \in S'$.

Given a formal series $f(x)=\sum_{\alpha \in N^n} f_\alpha x^\alpha \in \hat{\mathcal{O}}'_n$, for any non-empty subset J of $[1, n]$ and any $\alpha_J \in N^J$, we put $I=J^c$ and

$$(1.2) \qquad f(x_I;\alpha_J)=\sum_{\alpha_I \in N^I} f_{\alpha_{I\cup J}} x_I^{\alpha_I}.$$

Then, the *convergent* series $g_N(x)=\sum_{\alpha \in N^n,\, \alpha \ngeq N} f_\alpha x^\alpha$ for any $N \in N^n$ is written in the form

$$(1.3) \qquad g_N(x)=\sum_{\phi \neq J \subset [1,n]} (-1)^{\#J+1} \sum_{j \in J} \sum_{\alpha_j=0}^{N_j-1} f(x_{J^c};\alpha_J) x_J^{\alpha_J}.$$

Furthermore, $f(x_{J^c}\colon \alpha_J)$ can be expanded in the form

$$(1.4) \qquad f(x_{J^c};\alpha_J)=\sum_{\alpha_{I'} \in N^{I'}} f(x_{J^c \cap I'^c};\alpha_{J\cup I'}) x_{I'}^{\alpha_{I'}}$$

for any subset I' of J^c, so that

$$(1.5) \qquad f(x_I;\alpha_J)-g(x_I;\alpha_J;N_I)\equiv 0 \mod x_I^{N_I}$$

for any $N_I \in N^I$, where $I=J^c$ and $g(x_I;\alpha_J;N_I)$ is defined by

$$(1.6) \qquad g(x_I;\alpha_J;N_I)=\sum_{I' \subsetneqq J^c} (-1)^{\#I'+1} \sum_{i \in I'} \sum_{\alpha_i=0}^{N_i-1} f(x_{J^c \cap I'^c};\alpha_{J\cup I'}) x_{I'}^{\alpha_{I'}}.$$

The above equalities suggest a more general definition of "asymptotic expansion", that is

Definition 2. A holomorphic function $f(x)$ in S is said to be *strongly asymptotically developable as x tends to 0 in S*, if there exists a family of functions

$$(1.7) \qquad \{(f(x_{J^c};\alpha_J))\}_{J\subset[1,n],\, \alpha_J \in N^J}$$

with the following properties (1.8) and (1.9)

(1.8) $f(x_{J^c};\alpha_J)$ is holomorphic in $S_{J^c}=\prod_{i \in J^c} S_i$ for $J\neq[1,n]$ and $f(x_{[1,n]^c};\alpha_{[1,n]})$ is reduced to a constant f_α for $\alpha \in N^n$,

(1.9) for any $N_I \in N^I$ and for any closed subpolysector S'_I of S_I with $I=J^c$, there exists a positive constant K_{N_I,S'_I} such that

$$|f(x_I;\alpha_J)-g(x_I;\alpha_J;N_I)|\leq K_{N_I,S'_I}|x_I|^{N_I}$$

for any $x_I \in S'_I$, where $g(x_I;\alpha_J;N_I)$ is defined by (1.6) for $J\neq\phi$ and (1.3) for $J=\phi$ with the functions $f(x_{J^c\cap I'^c};\alpha_{J\cup I'})(I'\subsetneqq J^c)$ of the family (1.7), and $f(x_{\phi^c};\alpha_\phi)=f(x)$.

It is easy to prove the following

Proposition 1. *If $f(x)$ is strongly asymptotically developable, then the family of functions* (1.7) *satisfying the propoerties* (1.8) *and* (1.9) *is uniquely determined.*

Therefore, we call (1.7) the *total family of coefficients of strong asymptotic expansion* of $f(x)$ and denote it by $TA(f)$. For any non-empty subset J of $[1, n]$, we denote $f(x_{J^c}; \alpha_J)$ by $TA(f)_{\alpha_J}$ and define the formal series

$$(1.10) \qquad FA_J(f)=\sum_{\alpha_J \in N^J} TA(f)_{\alpha_J} x_J^{\alpha_J}$$

which is called the *formal series of strong asymptotic expansion of* $f(x)$ *for* $J \subseteq [1, n]$. In particular, for $J=[1, n]$, we use $FA(f)$ instead of $FA_{[1,n]}(f)$, and call it the *formal series of strong asymptotic expansion of* $f(x)$.

Definition 2 implies that, for any non-empty subset J of $[1, n]$ and for any α_J, the function $TA(f)_{\alpha_J}$ is also strongly asymptotically developable as a function of $(n-\# J)$ variables, and that, for any subset I' of J^c and for any $\alpha_{I'} \in N^{I'}$

$$(1.11) \qquad TA(TA(f)_{\alpha_J})_{\alpha_{I'}}=TA(f)_{\alpha_{J \cup I'}}.$$

From this point of view, we can define the strong asymptotic developability in an inductive way on the number of variables as follows.

For a function of a variable, the definition of strong asymptotic expansion is reduced to the usual definition of asymptotic expansion. Consider a function $f(x, y)$ of two variables holomorphic in an open polysector $S_1 \times S_2$ at the origin in C^2. The function $f(x, y)$ is strongly asymptotically developable as (x, y) tends to 0 in $S_1 \times S_2$, if there exist a formal series $\hat{f}(x, y)=\sum_{\alpha=0} \sum_{\beta=0} f_{\alpha\beta} x^\alpha y^\beta$, holomorphic functions $f_\beta(x)$ $(\beta \in N)$ asymptotic to $\sum_{\alpha=0}^{\infty} f_{\alpha\beta} x^\alpha$ *as* x tends to 0 in S_1 and holomorphic functions $f_\alpha(y)$ $(\alpha \in N)$ asymptotic to $\sum_{\beta=0} f_{\alpha\beta} y^\beta$ as y tends to 0 in S_2, and if for any $(M, N) \in N^2$ and for any closed subpolysector S_i' of S_i $(i=1, 2)$, there exists a positive constant $K_{M,N,S_1',S_2'}$ such that

$$|f(x, y)-\sum_{\alpha=0}^{M-1} f_\alpha(y) x^\alpha-\sum_{\beta=0}^{N-1} f_\beta(x) y^\beta+\sum_{\alpha=0}^{M-1} \sum_{\beta=0}^{N-1} f_{\alpha\beta} x^\alpha y^\beta| \leqslant K_{M,N,S_1',S_2'} |x|^M |y|^N$$

for any $(x, y) \in S_1' \times S_2'$. The formal power series $\hat{f}(x, y)$, the functions $f_\beta(x)$ $(\beta \in N)$ and $f_\alpha(y)$ $(\alpha \in N)$ are uniquely determined for the given function $f(x, y)$, and we denote $\hat{f}(x, y)$ by $FA(f)$, $f_{\alpha\beta}$ by $TA(f)_{\alpha\beta}$ $f_\beta(x)$ by $TA(f)_\beta$ and $f_\alpha(y)$ by $TA(f)_\alpha$. We assume that the strong asymptotic developability and the notation $FA(f)$, $TA(f)_*$ $(*=\alpha, \beta$ or $\alpha\beta)$ are defined for functions of n' variables in an open polysector $S_1 \times \cdots \times S_{n'}$, at the origin in $C^{n'}$ for $n'<n$, and then we proceed to a holomorphic function $f(x)$ of n variables in an open polysector $S_1 \times \cdots \times S_n$ at the origin in C^n. We say that $f(x)$ is strongly asymptotically developable as x tends to 0 in $S_1 \times \cdots \times S_{n'}$ if there exists a family (1.7) of holomorphic and strongly asymptotically developable functions with the properties (1.8), (1.11) and

(1.12) for any $N \in N^n$ and for any closed subpolysector S' of $S_1 \times \cdots \times S_n$, there exists a positive constant $K_{N,S'}$ such that

$$|f(x)-g_N(x)| \leqslant K_{N,S'} |x|^N$$

for any x in S', where $g_N(x)$ is defined by (1.3) with functions $f(x_{J^c}; \alpha_J)$ of the family (1.7).

We call $g_N(x)$ the *approximate function of degree N of strong asymptotic expansion for* $f(x)$, and the difference

$$e_N(x)x^N = f(x) - g_N(x) \tag{1.13}$$

the *error function of degree N for* $f(x)$ ($N \in \mathbf{N}^n$). As it is seen easily, we have the formulas

$$TA(f)_{N_J} x^{N_J} = \sum_{\phi \subseteq J' \subseteq J} (-1)^{\#J - \#J' - 1} - g_{(N_j+1)_{j \in J-J'}, N_{J'}, (0)_{i \in J^c}} \tag{1.14}$$

$$TA(f)_{N_J} = \sum_{\phi \subseteq J' \subseteq J} (-1)^{\#J - \#J'} e_{(N_j+1)_{j \in J-J'}, N_{J'}, (0)_{i \in J^c}} \prod_{j \in J - J'} x_j \tag{1.15}$$

and

$$\begin{aligned} g_N(x) = &\sum_{\phi \subsetneq J \subsetneq [1,n]} \sum_{j \in J} \sum_{\alpha_j=0}^{N_j-1} (f(x_{J^c}; \alpha_J) - g(x_{J^c}; \alpha_J; N_{J^c})) x_J^{\alpha_J} \\ &+ \sum_{j=1}^{n} \sum_{\alpha_j=0}^{N_j-1} f_\alpha x^\alpha \end{aligned} \tag{1.16}$$

for any non-empty subset J of $[1, n]$, for any $N_J \in \mathbf{N}^J$, and for any $N \in \mathbf{N}^n$.

Remark 1. Let D_i (resp. S_j) be a disc (resp. sector) at the origin of x_i-plane (resp. x_j-plane) for $i=1, \cdots, n'$ (resp. $j=n'+1, \cdots, n$). The Riemann's removable singularity theorem implies that if $f(x)$ is holomorphic and strongly asymptotically developable in $\prod_{i=1}^{n'} (D_i - \{0\}) \times \prod_{j=n'+1}^{n} S_j$, then $f(x)$ can be extended uniquely to a function holomorphic and strongly asymptotically developable in $\prod_{i=1}^{n'} D_i \times \prod_{j=n'+1}^{n} S_j$.

For an open polysector $S = S_1 \times \cdots \times S_n$, we denote by $A(S)$ the *set of all functions holomorphic and strongly asymptotically developable in S*. The set $A(S)$ is *closed* with respect to the fundamental operations: addition, multiplication, differentiation and integration. Moreover, each fundamental operations is *commutable* with the operation of taking strong asymptotic expansion.

Proposition 2. *For $f, g \in A(S)$, $a, b \in \mathbf{C}$ and any non-empty subset J of* $[1, n]$, *the following formulas are valid.*

(1) $af + bg \in A(S)$, $FA_J(af+bg) = aFA_J(f) + bFA_J(g)$,

(2) $fg \in A(S)$, $FA_J(fg) = FA_J(f)FA_J(g)$,

(3) *If $f(x) \neq 0$ in S and if $FA(f)(0) \neq 0$, then* $f^{-1} \in A(S)$, $FA_J(f)FA_J(f^{-1}) = 1$.

(4) *For any* $i \in [1, n]$, $(\partial/\partial x_i) f \in A(S)$,

$$TA((\partial/\partial x_i)f)_{(\alpha_j)_{j \in J}} = \begin{cases} (\partial/\partial x_i) TA(f)_{(\alpha_j)_{j \in J}} & \text{if } i \in J^c, \\ (\alpha_i + 1) TA_{(\alpha_j, j \neq i, \alpha_i + 1)} & \text{if } i \in J. \end{cases}$$

(5) *For any* $i \in [1, n]$, $\int_{\gamma_i} f(x)dx_i \in A(S)$,

$$TA\Big(\int_{\gamma_i} f(x)dx_i\Big)_{(\alpha_j)_{j \in J}} = \begin{cases} \alpha_i^{-1} TA(f)_{(\alpha_j, j \neq i,\ \alpha_i - 1)} & \text{if } i \in J, \\ \int_{\gamma_i} TA(f)_{(\alpha_j)_{j \in J}} dx_i & \text{if } i \in J^c, \end{cases}$$

where γ_i *is a curve* $(x_i(t))_{t \in (0,1]}$ *in* S_i *such that* $\lim_{t \to 0} x_i(t) = 0$.

Concerning infinite series and infinite products, we have

Ptoposition 3. *Let* $f_k(x)$ *be an element of* $A(S)$ *for any* $k \in N$, *and let* $e_{k,N}(x)x^N$ *be the error function of degree* N *for* $f_k(x)$ *for any* $N \in N^n$ *and any* $k \in N$.

(1) *Suppose that, for any* $N \in N^n$, *the infinite series* $\sum_{k=0}^{\infty} e_{k,N}(x)$ *is uniformly and absolutely convergent in any closed subpolysector* S' *of* S. *Then,* $\sum_{k=0}^{\infty} f_k(x)$ *is convergent, holomorphic and strongly asymptotically developable in* S, *and*

$$TA(\textstyle\sum_{k=0}^{\infty} f_k)_{\alpha_J} = \sum_{k=0}^{\infty} TA(f_k)_{\alpha_J}$$

for any non-empty subset J *of* $[1, n]$ *and any* $\alpha_J \in N^J$.

(2) *Suppose moreover that* $|f_k(x)| \leqslant M$ $(k \in N)$ *in* S *for some positive number* $M < 1$. *Then, the infinite product* $\prod_{k=0}^{\infty}(1 + f_k)$ *is uniformly and absolutely convergent, and is holomorphic, strongly asymptotically developable in* S, *and*

$$FA_J(\textstyle\prod_{k=0}^{\infty}(1 + f_k)) = \prod_{k=0}^{\infty}(1 + FA_J(f_k))$$

for any non-empty subset J *of* $[1, n]$.

The proofs of Propositions 2 and 3 are made in a way similar to one variable case (cf. W. Wasow [9], Y. Sibuya [5]).

If, instead of $A(S)$, we consider the set $A'(S)$ of *all functions* f *holomorphic and strongly asymptotically developable in* S, *such that* $FA(f)$ *belongs to* $\hat{\mathcal{O}}'_n$, we have the propositions 2′ and 3′ obtained from Propositions 2 and 3 by replacing $A(S)$ with $A'(S)$.

§ 2. Theorem of Borel-Ritt Type.

We shall prove the following theorem.

Theorem 1. (1) *For any formal power series* $\hat{f}(x) = \sum_{\alpha \in N^n} f_\alpha x^\alpha$ *and for any open polysector* $S(r) = \prod_{i=1}^{n} S(\tau_{-i}, \tau_{+i}, r_i)$ *at the origin in* C^n *there exists a holomorphic and strongly asymptotically developable function* $f(x)$ *in the given polysector such that* $FA(f)$ *coincides with the given formal series.*

(2) *Suppose in* (1) *that the formal power series* $\hat{f}(x)$ *belongs to* $\hat{\mathcal{O}}'_n(r)$. *Then, there exists a holomorphic function* $f(x)$ *in* $S(r)$ *such that* $f(x)$ *is strongly asymptotically developable to* $\hat{f}(x)$ *in* $S(r)$.

Proof of Theorem 1. (1) For the given series, we define $d_{\alpha,i}$ by

$$d_{\alpha,i}=\begin{cases}\min\{r_i^{-\alpha_i}|f_\alpha|^{-\alpha_i/|\alpha|},\ r_i^{-\alpha_i}\}, & \text{if } f_\alpha\neq 0\\ 0, & \text{if } f_\alpha=0.\end{cases}$$

We take p_i's and τ_i's such that

$$0<p_i<1,$$

and

$$\cos(\tau_i-p_i\arg x_i)\leqslant -1/2 \qquad \text{for } x_i\in S[\tau_{-i},\tau_{+i},r_i].$$

For this end, it is sufficient to choose p_i's and τ_i's so that

$$0<p_i<\min\{1,\ \pi/3(\tau_{+i}-\tau_{-i})\}$$

and

$$p_i\tau_{+i}+2k\pi+5\pi/6\leqslant\tau_i\leqslant p_i\tau_{-i}+2k\pi+7\pi/6$$

for some integer k. Putting

$$a_{\alpha,j}(x_j)=1-\exp(d_{\alpha,j}x_j^{-p_j}r_j^{p_j}\exp(\sqrt{-1}\,\tau_j))$$

for all $\alpha\in N^n$, we define a family of infinite series with the index set

$$\{(J,\alpha_J);\ J \text{ is a subset of } [1,n],\ \alpha_J=(\alpha_j)_{j\in J},\ \alpha_j\in N\}$$

as follows:

$$f(J,(\alpha_j)_{j\in J})=\sum_{\beta\in N^n}f_\beta\prod_{i\in J^c,\,\beta_i\neq 0}a_{\beta,i}x_i^{\beta_i}\ (\beta_j=\alpha_j,\ j\in J).$$

Then, the series $f(J,\alpha_J)$ is uniformly and absolutely convergent in $\prod_{i\in J^c}S[\tau_{-i},\tau_{+i},r_i]$, because

$$\begin{aligned}&\sum_{\beta\in N^n}|f_\beta\prod_{i\in J^c,\,\beta_i\neq 0}a_{\beta,i}(x)x_i^{\beta_i}|\ (\beta_j=\alpha_j,\ j\in J)\\ &\quad\leqslant\sum_{\beta\in N^n}\prod_{i\in J^c,\,\beta_i\neq 0}|(x_i/r_i)|^{\beta_i-p_i}\ (\beta_j=\alpha_j,\ j\in J)\\ &\quad\leqslant\prod_{i\in J^c}(1-|(x_i/r_i)|)^{-1}+\#J^c-1\end{aligned}$$

in $\prod_{i\in J^c}S[\tau_{-i},\tau_{+i},r_i]$ for any subset J of $[1,n]$ and any $\alpha_J\in N^J$. Moreover, if we put

$$g(\alpha_J;N_{J^c})=\sum_{J'\supsetneqq J}(-1)^{n+1-\#J'}\sum_{i\in J'^c}\sum_{\alpha_i=0}^{N_i-1}f(J\cup J'^c,(\alpha_h),\ h\in J\cup J'^c)\prod_{i\in J'^c}x_i^{\alpha_i}$$

then we have

$$\begin{aligned}&f(J,\alpha_J)-g(\alpha_J;N_{J^c})\\ &\quad=\sum_{J'\supset J''\supsetneqq J}\sum_{k\in J''\cap J^c}\sum_{\beta_k=N_k}^{\infty}\sum_{h\in J'\cap J''^c}\sum_{\beta_h=0}^{N_h-1}\sum_{i\in J'^c}\sum_{\beta_i=0}^{N_i-1}f_\beta\end{aligned}$$

$$\times \prod_{k \in J'' \cap J^c} a_{\beta,k} x_k^{\beta_k} \prod_{i \in J''^c} x_i^{\beta_i} \left(\prod_{i \in J'^c} a_{\beta,i} (-1)^{n+2-\# J'}\right)$$
$$(\beta_j = \alpha_j,\ j \in J)$$
$$= \sum_{J'' \supsetneqq J} \sum_{k \in J'' \cap J^c} \sum_{\beta_k = N_k}^{\infty} \sum_{i \in J''^c} \sum_{\beta_i = 0}^{N_i - 1} f_\beta$$
$$\times \prod_{k \in J'' \cap J^c} a_{\beta,k} x_k^{\beta_k} \prod_{i \in J''^c} x_i^{\beta_i} \prod_{i \in J''^c} (a_{\beta,i} - 1) \qquad (\beta_j = \alpha_j,\ j \in J).$$

Since $a_{\alpha,i}(x_i) - 1$ is asymptotic to zero as x_i tends to 0 in $S[\tau_{-i}, \tau_{+i}, r_i]$, there exists a constant $K_{J,\alpha_J,N_{J^c}}$ such that, for any subset J of $[1, n]$, any $\alpha_J \in N^J$ and any $N_{J^c} \in N^{J^c}$,

$$|f(J, \alpha_J) - g(\alpha_J; N_{J^c})| \leqslant K_{J,\alpha_J,N_{J^c}} |x_{J^c}|^{N_{J^c}}$$

in $\prod_{i \in J^c} S[\tau_{-i}, \tau_{+i}, r_i]$. This implies that the function $f(\phi; \alpha_\phi)$ is strongly asymptotically developable in $\prod_{i=1}^n S(\tau_{-i}, \tau_{+i}, r_i)$ and $FA(f(\phi; \alpha_\phi)) = \hat{f}(x)$.

(2) For the given series, we define d_α by

$$d_\alpha = \begin{cases} r^{-\alpha} |f_\alpha|^{-1}, & \text{if } f_\alpha \neq 0, \\ 0, & \text{if } f_\alpha = 0. \end{cases}$$

We take p_i's $(i = 1, \cdots, n)$ and τ such that

$$\cos(\tau - \textstyle\sum_{i \in I} p_i \arg x_i) \leqslant -1/2 \qquad \text{in } \prod_{i \in I} S[\tau_{-i}, \tau_{+i}, r_i]$$

for any non-empty subset I of $[1, n]$, and

$$0 < p_i < 1, \qquad i = 1, \cdots, n.$$

For example, we can choose p_i's and τ in such a way that

$$p = p_1 = \cdots = p_n,$$
$$p^{-1}\pi/3 > \max \{\textstyle\sum_{i \in J} \tau_{+i} - \sum_{j \in J'} \tau_{-j};\ J, J' \subset [1, n]\}$$

and

$$p \textstyle\sum_{i \in I} \arg x_i + 5\pi/6 \leqslant \tau \leqslant p \sum_{i \in I} \arg x_i + 7\pi/6$$

for any non-empty subset I of $[1, n]$ and any x_I in $\prod_{i \in I} S[\tau_{-i}, \tau_{+i}, r_i]$. Using the functions

$$a_\alpha(x) = \begin{cases} 1 - \exp(d_\alpha \prod_{i=1, \alpha_i \neq 0}^n x_i^{-p_i} r_i^{p_i} \exp(\sqrt{-1}\,\tau)) & \text{for } \alpha \neq 0, \\ 1 \quad \text{for } \alpha = 0, \end{cases}$$

we define an infinite series $f(x)$ by

$$f(x) = \sum_{\alpha \in N^n} f_\alpha x^\alpha a_\alpha(x).$$

Then, $f(x)$ is uniformly and absolutely convergent in $\prod_{i=1}^n S[\tau_{-i}, \tau_{+i}, r_i]$, because

$$\begin{aligned}\textstyle\sum_{\alpha\in N^n}|f_\alpha||x^\alpha||a_\alpha(x)| &\leqslant \textstyle\sum_{\alpha\in N^n}\prod_{i=1,\,\alpha_i\neq 0}^n|(x_i/r_i)|^{\alpha_i-p_i}\\ &\leqslant \textstyle\sum_{\beta\in N^n}|(x_1/r_1)|^{\beta_1}\times\cdots\times|(x_n/r_n)|^{\beta_n}+n-1\\ &\leqslant \textstyle\prod_{i=1}^n(1-|(x_i/r_i)|)^{-1}+n-1.\end{aligned}$$

By the assumption, for any non-empty subset J of $[1, n]$ and $\alpha_J\in N^J$, the infinite series

$$M(\alpha_J)=\textstyle\sum_{\alpha_I\in N^I}|f_{\alpha_{J\cup I}}|r_I^{\alpha_I}\qquad (I=J^c)$$

is convergent. For an element $N\in N^n$, we define M_N by

$$M_N=\max\{1,\ M(\alpha_J),\ J\subset[1, n],\ \alpha_J\in N^J,\ \alpha_j<N_j(j\in J)\}.$$

Then, for all $\beta\in N^n$, $\beta\ngeq N$, we have

$$|f_\beta|\leqslant M_N r^{-\beta},$$

from which

$$M_N^{-1}\leqslant d_\beta \qquad \text{if } f_\beta\neq 0,$$

and so

$$\begin{aligned}&\exp\,(d_\beta\textstyle\prod_{i\in I,\,\beta_i\neq 0}|x_i|^{-p_i}\cos\,(\sum_{i\in I,\,\beta_i\neq 0}p_i\arg x_i-\tau))\\ &\quad\leqslant\exp\,(-1/2M_N\textstyle\prod_{i\in I,\,\beta_i\neq 0}|x_i|^{p_i})\end{aligned}$$

for x_I in $\prod_{i\in I}S[\tau_{-i}, \tau_{+i}, r_i]$, for any non-empty subset I of $[1, n]$ and for any $\beta\in N^n$ with $f_\beta\neq 0$. Hence, we have

$$\begin{aligned}&|f(x)-\textstyle\sum_{\beta\in N^n,\,\beta\ngeq N}f_\beta x^\beta|\\ &\quad\leqslant\textstyle\sum_{\beta\in N^n,\,\beta\geqslant N}|f_\beta a_\beta(x)x^\beta|+\sum_{\beta\in N^n,\,0\neq\beta\ngeq N}|f_\beta x^\beta(1-a_\beta(x))|\\ &\quad\leqslant\textstyle\sum_{\beta\in N^n,\,\beta\geqslant N}|f_\beta a_\beta(x)x^\beta|+\sum_{\beta\in N^n,\,0\neq\beta\ngeq N}|f_\beta x^\beta|\exp\,(-1/2M_N\prod_{\beta_i\neq 0}x_i^{p_i})\end{aligned}$$

in $\prod_{i=1}^n S[\tau_{-i}, \tau_{+i}, r_i]$. This implies that there exists a positive constant K_N such that

$$|f(x)-\textstyle\sum_{\beta\in N^n,\,\beta\ngeq N}f_\beta x^\beta|\leqslant K_N|x^N|$$

in $\prod_{i=1}^n S[\tau_{-i}, \tau_{+i}, r_i]$. Q.E.D.

From Proposition 2 (resp. 2′) and the assertion (1) (resp. (2)) of Theorem 1, we see that $FA(*)$ is *surjective* homomorphism of the differential and integral algebra $A(S)$ (resp. $A'(S(r))$) onto the algebra $\hat{\mathcal{O}}_n$ (resp. $\hat{\mathcal{O}}'_n(r)$) over $\boldsymbol{C}$.

§ 3. Theorems of Sibuya type.

We now pass to Theorems of Sibuya type. Let $D(r_i)$ be a disc with radius r_i

and with the center at the origin in complex x_i-plane, and let $\{S(\tau_{-i,h_i}, \tau_{+i,h_i}, r_i)\}_{h_i=1,\cdots,l_i}$ be an open sectorial covering of $D(r_i)-\{0\}$ for all $i=1, \cdots, n$. For the sake of simplicity, we use the following notation:

$$D(r)^{n'-1}=D(r_1)\times\cdots\times D(r_{n'-1})$$
$$[n', n'']=\{n', n'+1, \cdots, n''\}$$
$$h[n', n'']=(h_{n'}, \cdots, h_{n''}) \qquad h_{n'}=1, \cdots, l_{n'}, \cdots, h_{n''}=1, \cdots, l_{n''}$$
$$\mathscr{L}[n', n'']=\{h[n', n'']; h_{n'}=1, \cdots, l_{n'}, \cdots, h_{n''}=1, \cdots, l_{n''}\}$$
$$S_{h[n',n'']}^{[n',n'']}(r)=\prod_{j=n'}^{n''} S(\tau_{-j,h_j}, \tau_{+j,h_j}, r_j)$$
$$S_{h[n',n''],h'[n',n'']}^{[n',n'']}(r)=S_{h[n',n'']}^{[n',n'']}(r)\cap S_{h'[n',n'']}^{[n',n'']}(r)$$
$$S_{h[n',n''],h'[n',n''],h''[n',n'']}^{[n',n'']}(r)=S_{h[n',n''],h'[n',n'']}^{[n',n'']}(r)\cap S_{h''[n',n'']}^{[n',n'']}(r)$$

for $1\leqslant n'\leqslant n''\leqslant n$. If $n'=n''$, we use the notation replacing $[n', n'']$ by n', and if $n'=1$ and $n''=n$, we use the notation without $[n', n'']$.

Let

$$\{D(r)^{n'-1}\times S_{n',h_{n'}}(r_{n'})\times S_{h[n'+1,n]}^{[n'+1,n]}(r)\}_{h_{n'}=1,\cdots,l_{n'}}$$

be an open coverging of

$$D(r)^{n'-1}\times(D(r_{n'})-\{0\})\times S_{h[n'+1,n]}^{[n'+1,n]}(r),$$

and let $(P_{h_{n'},h'_{n'}})_{h_{n'},h'_{n'}=1,\cdots,l_{n'}}$ be a family of m by m matrices whose elements are holomorphic and strongly asymptotically developable functions such that

(0, n') $\quad P_{h_{n'},h'_{n'}}$ is defined in $D(r)^{n'-1}\times S_{h_{n'},h'_{n'}}^{n'}(r_{n'})\times S_{h[n'+1,n]}^{[n'+1,n]}(r)$,

(1, n') $\quad TA(P_{h_{n'},h'_{n'}}-I_m)_{\alpha_J}=0_m$ for any non-empty subset J of $[n', n]$ and any $\alpha_J\in N^J$,

(2, n') $\quad P_{h_{n'},h'_{n'}}P_{h'_{n'},h''_{n'}}=P_{h_{n'},h'_{n'}}$ in $D(r)^{n'-1}\times S_{h_{n'}h'_{n'},h''_{n'}}^{n'}(r_{n'})\times S_{h[n'+1,n]}^{[n'+1,n]}(r)$.

Then, we have

Theorem 2 (*Theorem of Sibuya type*, cf. Y. Sibuya(6), (7)). *There exists a family $(Q_{h_{n'}})_{h_{n'}=1,\cdots,l_{n'}}$ of m by m matrices such that*

(3, n') $\quad$ *$Q_{h_{n'}}$ is holomorphic, invertible and strongly asymptotically developable in $D(r)^{n'-1}\times S_{h[n',n]}^{[n',n]}(r')$,*

(4, n') $\quad$ *$TA(Q_{h_{n'}}-I_m)_{\alpha_J}=0_m$ for any non-empty subset of $[n'+1, n]$ and any $\alpha_J\in N^J$,*

(5, n') $\quad$ *$Q_{h_{n'}}P_{h_{n'}h'_{n'}}=Q_{h'_{n'}}$ in $D(r')^{n'-1}\times S_{h_{n'}h'_{n'}}^{n'}(r'_{n'})\times S_{h[n'+1,n]}^{[n'+1,n]}(r')$,*

where r'_i is a positive constant less than r_i for $i=1, \cdots, n$.

Proof of Theorem 2. We can assume without loss of generality that

$$\tau_{-n',h_{n'}} < \tau_{+n',h_{n'}-1} < \tau_{-n',h_{n'}+1} < \tau_{+n',h_{n'}}$$

for any $h_{n'}=1, \cdots, l_{n'}$, where $\tau_{+n',0}=\tau_{+n',l_{n'}}$ and $\tau_{-n',l_{n'}+1}=\tau_{-n',1}$. Throughout the proof, we use the following notation:

$$\varepsilon' := (1/4) \min \{\tau_{+i,h_i} - \tau_{-i,h_i+1}, i=1, \cdots n, h_i=1, \cdots, l_i\}$$
$$\varepsilon'' := (1/2) \min \{r_i, i=1, \cdots, n\}$$
$$\varepsilon := \min \{\varepsilon', \varepsilon''\}$$
$$\delta_k := 1-2^{-k} \quad \text{for any non-negative integer } k$$
$$S_i[h_i, k] := S[\tau_{-i,h_i}+\varepsilon\delta_k, \tau_{+i,h_i}-\varepsilon\delta_k, r_i(1+\varepsilon\delta_k)^{-1}]$$
$$L_i(h_i, k, -) := (tr_i\,(1+\varepsilon\delta_k)^{-1} \exp(\sqrt{-1}\,(\tau_{-i,h_i}+\varepsilon\delta_k)))_{t\in(0,1]}$$
$$L_i(h_i, k, +) := (tr_i\,(1+\varepsilon\delta_k)^{-1} \exp(\sqrt{-1}\,(\tau_{+1,h_i}-\varepsilon\delta_k)))_{t\in(0,1]}$$
$$\tau_{i,h_i} := 2^{-1}(\tau_{+i,h_i-1}+\tau_{-i,h_i})$$
$$c_i(h_i, k, -) := (r_i(1+\varepsilon\delta_k)^{-1} \exp(\sqrt{-1}((1-t)(\tau_{-i,h_i}+\varepsilon\delta_k)+t\tau_{i,h_i}))) \text{ in } t\in[0, 1]$$
$$c_i(h_i, k, +) := (r_i(1+\varepsilon\delta_k)^{-1} \exp(\sqrt{-1}((1-t)(\tau_{+i,h_i}-\varepsilon\delta_k)+t\tau_{i,h_i+1}))) \text{ in } t\in[0, 1]$$
$$\gamma_i(h_i, k) := (tr_i\,(1+\varepsilon\delta_k)^{-1} \exp(\sqrt{-1}\,\tau_{i,h_i}))_{t\in(0,1]}$$
$$\gamma_i(h_i, k, *) := L_i(h_i, k, *) \cup c_i(h_i, k, *) \quad (*=+, -)$$
$$S_i[h_i, h_i+1, k] := S_i[h_i, k] \cap S_i[h_i+1, k]$$

for $i=n', \cdots, n$,

$$S^{[n'+1,n]}_{h[n'+1,n]}[1] := S_{n'+1}[h_{n'+1}, 1] \times \cdots \times S_n[h_n, 1],$$
$$D_i[k] := \text{closure of } D(r_i-\varepsilon\delta_k)$$
$$\gamma_i(k) := ((r_i-\varepsilon\delta_k) \exp(2\pi\sqrt{-1}\,t))_{t\in[0,1]}$$

for $i=1, \cdots, n'-1$,

$$(D[k])^{n'-1} := D_1[k] \times \cdots \times D_{n'-1}[k]$$

and

$$y=(x_1, \cdots, x_{n'-1})$$
$$z=(x_{n'+1}, \cdots, x_n).$$

For a family of functions $(b_{h_{n'}})_{h_{n'}=1,\cdots,l_{n'}}$ such that $b_{h_{n'}}$ is defined in $(D[k])^{n'-1}\times S_{n'}[h_{n'}-1, h_{n'}, k]\times S^{[n'+1,n]}_{h[n'+1,n]}[1]$ or $S_{n'}[h_{n'}-1, h_{n'}, k]\times S^{[n'+1,n]}_{h[n'+1,n]}[1]$, we define

$$\text{Integ}\,(h_{n'}, k; b_j, j\in[1, l_{n'}]) := (2\pi\sqrt{-1})^{-1} \sum_{j=1,\neq h_{n'},h_{n'}+1}^{l_{n'}} \int_{\gamma_{n'}(j,k)} b_j d\zeta_{n'}$$

$$+(2\pi\sqrt{-1})^{-1}\int_{\gamma_{n'}(h_{n'},k,-)} b_{h_{n'}}d\zeta_{n'}+(2\pi\sqrt{-1})^{-1}\int_{\gamma_{n'}(h_{n'},k,+)} b_{h_{n'}+1}d\zeta_{n'}$$

for $h_{n'}=1, \cdots, l_{n'}$. For abbreviation, we use Integ $(h_{n'}, k; b_j)$ instead of Integ $(h_{n'}, k; b_j, j\in[1, l_{n'}])$.

It is easy to see that

$$S_i[h_i, k+1]\subset S_i[h_i, k], \qquad S_i[h_i, h_i+1, k]\supset \gamma_i(h_i+1, k),$$
$$\partial S_i[h_i, h_i+1, k]=\gamma_i(h_i+1, k, -)\cup -\gamma_i(h_i, k, +),$$

and for $x_{n'}\in S_{n'}[h_{n'}, h_{n'}+1, k+1]$, and for $\zeta_{n'}\in \partial S_{n'}[h_{n'}, h_{n'}+1, k]$

$$|x_{n'}^{-1}-\zeta_{n'}^{-1}|\geqslant r_{n'}^{-1}(1+\varepsilon\delta_k)\tan(\varepsilon(\delta_{k+1}-\delta_k))\geqslant r_{n'}^{-1}\varepsilon(\delta_{k+1}-\delta_k).$$

The following estimation is used frequently in the proof.

Lemma 1. *For any* $h_{n'}=1, \cdots, l_{n'}$, *we have*

$$|\text{Integ}(h_{n'}, k; |\zeta_{n'}|^2|\zeta_{n'}-x_{n'}|^{-1})|\leqslant 2^k\sigma_1$$

for $x_{n'}\in S_{n'}[h_{n'}, k+1]$, *where* $\sigma_1=2r_{n'}^2(1+\varepsilon^{-1})(1+\pi^{-1}l_{n'})$.

We shall prove Lemma 1. Since

$$|\zeta_{n'}|^2|\zeta_{n'}-x_{n'}|^{-1}\leqslant|\zeta_{n'}^{-1}-x_{n'}^{-1}|^{-1}+|\zeta_{n'}|,$$

we have

$$|\text{Integ}(h_{n'}, k; |\zeta_{n'}|^2|\zeta_{n'}-x_{n'}|^{-1})|\leqslant(2\pi)^{-1}(2r_{n'}l_{n'}+2\pi r_{n'})(2^{k+1}r_{n'}\varepsilon^{-1}+r_{n'})$$

for $x_{n'}\in S_{n'}[h_{n'}, k+1]$.

We construct the matrices $Q_{h_{n'}}$'s in the form of infinite products.

First, we define $G_{h_{n'},1}$ by

$$(6, 1)\qquad G_{h_{n'},1}=P_{h_{n'},h_{n'}+1}-I_m, \qquad h_{n'}=1, \cdots, l_{n'}.$$

It follows from the assumption $(1, n')$ that, for any $N\in \boldsymbol{N}$ and for any $L\in \boldsymbol{N}^{n-n'}$, there exists a constant $K_{N,L}$ such that

$$(7, 1)\qquad |G_{h_{n'},1}(x)|\leqslant 2^{-n'-1}K_{N,L}|x_{n'}|^N|z|^L$$

for any $x\in D[0]^{n'-1}\times S_{n'}[h_{n'}, h_{n'}+1, 1]\times S^{[n'+1,n]}_{h[n'+1,n]}[1]$. Suppose that, for a positive integer k, the matricial functions $G_{h_{n'},k}(x)$ are defined in such a way that

$$(7, k)\qquad |G_{h_{n'},k}(x)|\leqslant 2^{-k(n'+1)}K_{N,L}|x_{n'}|^N|z|^L$$

for any $N \in \boldsymbol{N}$, any $L \in \boldsymbol{N}^{n-n'}$ and any $x \in D[k-1]^{n'-1} \times S_{n'}[h_{n'}, h_{n'}+1, k] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$. Because of holomorphy of $G_{h_{n'},k}(y, x_{n'}, z)$ with respect to y, $G_{h_{n'},k}(y, x_{n'}, z)$ has a power series expansion in y,

$$(8, k) \qquad G_{h_{n'},k}(y, x_{n'}, z) \sum_{\beta \in \boldsymbol{N}^{n'-1}} g_{k_{n'},k,\beta}(x_{n'}, z) y^{\beta}.$$

We see by (7, k) that the coefficients are majorized by

$$(9, k) \qquad |g_{h_{n'},k,\beta}(x_{n'}, z)| \leqslant (r-\varepsilon)^{-\beta} 2^{-k(n'+1)} K_{N,L} |x_{n'}|^{N} |z|^{L}$$

for any $N \in \boldsymbol{N}$, any $L \in \boldsymbol{N}^{n-n'}$, and any $(x_{n'}, z) \in S_{n'}[h_{n'}, h_{n'}+1, k] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$. If, for any $M \in \boldsymbol{N}^{n'-1}$, we put

$$(10, k) \qquad \begin{aligned} E_{h_{n'},k,M}(x) &= G_{h_{n'},k}(x) - \sum_{\beta \ngeq M} g_{h_{n'},k,\beta}(x_{n'}, z) y^{\beta} \\ &= \sum_{\beta \geqslant M} g_{h_{n'},k,\beta}(x_{n'}, z) y^{\beta}, \end{aligned}$$

then we have the integral representation

$$(11, k) \qquad \begin{aligned} E_{h_{n'},k,M}(x) = (2\pi\sqrt{-1})^{1-n'} \int_{\gamma_1(k-1)} \cdots \int_{\gamma_{n'-1}(k-1)} G_{h_{n'},k}(\zeta, x_{n'}, z) \\ \times \prod_{i=1}^{n'-1} x_i^{M_i} \zeta_i^{-M_i} (\zeta_i - x_i)^{-1} d\zeta_1 \cdots d\zeta_{n'-1}, \end{aligned}$$

from which

$$(12, k) \qquad |E_{h_{n'},k,M}(x)| \leqslant 2^{-2k} (r-\varepsilon)^{-M} \sigma_2 K_{N,L} |y|^{M} |x_{n'}|^{N} |z|^{L}$$

for any $M \in \boldsymbol{N}^{n'-1}$, any $N \in \boldsymbol{N}$, any $L \in \boldsymbol{N}^{n-n'}$ and any $(y, x_{n'}, z) \in D[k]^{n'-1} \times S_{n'}[h_{n'}, h_{n'}+1, k] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$, where $\sigma_2 = (2\varepsilon^{-1})^{n'-1}$.

Now, we define a holomorphic matricial function $F_{h_{n'},k+1}$ by

$$(13, k) \qquad F_{h_{n'},k+1}(x) = \mathrm{Integ}\,(h_{n'}, k; (\zeta_{n'} - x_{n'})^{-1} G_{j,k}(y, \zeta_{n'}, z))$$

for $(y, x_{n'}, z) \in D[k]^{n'-1} \times S_{n'}[h_{n'}, k+1] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$. Then, by using Lemma 1 and (7, k) with $N=2$, we can estimate it as follows,

$$(14, k) \qquad |F_{h_{n'},k+1}(x)| \leqslant 2^{-kn'} \sigma_1 K_{2,L} |z|^{L}$$

for any $L \in \boldsymbol{N}^{n-n'}$. We shall show that $F_{h_{n'},k+1}(x)$ is strongly asymptotically developable. To do this, we define $f_{h_{n'},k+1,\beta}(x_{n'}, z)$, $f_{h_{n'},k+1,\gamma,\beta}(z)$, $e_{h_{n'},k+1,N,\beta}(x_{n'}, z)$ and $e_{h_{n'},k+1,M,N}(y, x_{n'}, z)$ as follow,

$$(15, k) \qquad f_{h_{n'},k+1,\beta}(x_n, z) = \mathrm{Integ}\,(h_{n'}, k; (\zeta_{n'} - x_{n'})^{-1} g_{j,k,\beta}(\zeta_{n'}, z))$$

for $(x_{n'}, z) \in S_{n'}[h_{n'}, k+1] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$ and for $\beta \in \boldsymbol{N}^{n'-1}$,

$$(16, k) \qquad f_{h_{n'},k+1,\gamma,\beta}(z) = \mathrm{Integ}\,(h_{n'}, k; \zeta_{n'}^{-\gamma-1} g_{j,k,\beta}(\zeta_{n'}, z))$$

for $z \in S^{[n'+1,n]}_{h[n'+1,n]}[1]$, for $\beta \in \boldsymbol{N}^{n'-1}$ and for $\gamma \in \boldsymbol{N}$, (17, k) $$e_{h_{n'},k+1,N,\beta}(x_{n'}, z)f_{h_{n'},k+1,\beta}(x_{n'}, z)-\sum_{\gamma=0}^{N-1} f_{h_{n'},k+1,\gamma,\beta}(z)x_{n'}^{\gamma}$$

for any $N \in \boldsymbol{N}$, and

(18, k) $$\begin{aligned} e_{h_{n'},k+1,N,M}(x) &= F_{h_{n'},k+1}(x)-\sum_{\beta\not\geq M} f_{h_{n'},k+1,\beta}(x_{n'}, z)y^{\beta} \\ &\quad -\sum_{\gamma=0}^{N-1} \sum_{\beta\geq M} f_{h_{n'},k+1,\gamma,\beta}(z)y^{\beta}x_{n'}^{\gamma} \\ &= \sum_{\beta\geq M} e_{h_{n'},k+1,N,\beta}(x_{n'}, z)y^{\beta} \end{aligned}$$

for any $N \in \boldsymbol{N}$ and for $M \in \boldsymbol{N}^{n'-1}$. The last two functions have the integral representations

(19, k) $$e_{h_{n'},k+1,N,\beta}(x_{n'}, z)=\mathrm{Integ}\,(h_{n'}, k;\, x_{n'}^{N}\zeta_{n'}^{-N}(\zeta_{n'}-x_{n'})^{-1}g_{j,k,\beta}(\zeta_{n'}, z))$$

and

(20, k) $$e_{h_{n'},k+1,N,M}(x)=\mathrm{Integ}\,(h_{n'}, k;\, x_{n'}^{N}\zeta_{n'}^{-N}(\zeta_{n'}-x_{n'})^{-1}E_{j,k,M}(y, \zeta_{n'}, z))$$

from the definitions. By using the estimations (9, k) and (12, k), we have

(21, k) $$|f_{h_{n'},k+1,\gamma,\beta}(z)|\leqslant 2^{-k(n'+1)}(1+\pi^{-1}l_{n'})r_{n'}(r-\varepsilon)^{-\beta}K_{\gamma+1,L}|z|^{L}$$

for any $L \in \boldsymbol{N}^{n-n'}$ and for any $z \in S_{h[n'+1,n]}^{[n'+1,n]}[1]$,

(22, k) $$|e_{h_{n'},k+1,N,\beta}(x_{n'}, z)|\leqslant 2^{-kn'}(r-\varepsilon)^{-\beta}\sigma_1 K_{N+2,L}|x_{n'}|^{N}|z|^{L}$$

for any $N \in \boldsymbol{N}$, any $L \in \boldsymbol{N}^{n-n'}$ and any $(x_{n'}, z) \in S_{n'}[h_{n'}, k+1]\times S_{h[n'+1,n]}^{[n'+1,n]}[1]$, and

(23, k) $$|e_{h_{n'},k+1,N,M}(x)|\leqslant 2^{-k}(r-\varepsilon)^{-M}\sigma_1\sigma_2 K_{N+2,L}|y|^{M}|x_{n'}|^{N}|z|^{L}$$

for any $(M, N, L) \in \boldsymbol{N}^{n}$ and any $x \in D[k+1]^{n'-1}\times S_{n'}[h_{n'}, k+1]\times S_{h[n'+1,n]}^{[n'+1,n]}[1]$.

We define the $(k+1)$-th matricial function $G_{h_{n'},k+1}(x)$ by

(24, k) $$G_{h_{n'},k+1}(x)=F_{h_{n'},k+1}(x)G_{h_{n'},k}(x)(I_m+F_{h_{n'}+1,k+1}(x))^{-1}$$

in $D[k+1]^{n'+1}\times S_{n'}[h_{n'}, h_{n'}+1, k+1]\times S_{h[n'+1,n]}^{[n'+1,n]}[1]$. By the definition (13, k) and Cauchy's theorem, we have

(25, k) $$G_{h_{n'},k}(x)=F_{h_{n'}+1,k+1}(x)-F_{h_{n'},k+1}(x).$$

The equalities (24, k) and (25, k) yield

(26, k) $$(I_m+F_{h_{n'},k+1})(I_m+G_{h_{n'},k})=(I_m+G_{h_{n'},k+1})(I_m+F_{h_{n'}+1,k+1})$$

in the domain of definition of $G_{h_{n'},k+1}$. From the inequality

$$|G_{h_{n'},k+1}|\leqslant|F_{h_{n'},k+1}||G_{h_{n'},k}||(I_m+F_{h_{n'}+1,k+1})|^{-1},$$

we have the estimation

$$(7, k+1) \qquad |G_{h_{n'},k+1}(x)| \leqslant 2^{-(k+1)(n'+1)} K_{N,L} |x_{n'}|^N |z|^L$$

for any $N \in \boldsymbol{N}$, any $L \in \boldsymbol{N}^{n-n'}$ and any $x \in D[k+1]^{n'-1} \times S_{n'}[h_{n'}, h_{n'}+1, k+1] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$, if the radii r_i's are chosen so small that

$$r_{n'+1}, \cdots, r_n \leqslant 1$$
$$2^{-n'}\sigma_1 K_{2,0} = 2^{-n'+1} r_{n'}^2 (1+\pi^{-1} l_{n'})(1+\varepsilon^{-1}) K_{2,0} < 1$$

and

$$2^{-n'}\sigma_1 K_{2,0}(1-2^{-n'}\sigma_1 K_{2,0})^{-1} \leqslant 2^{-n'-1}.$$

Hence, this procedure enables us to define a sequence of matricial, holomorphic and strongly asymptotically developable functions $F_{h_{n'},k+1}$, (x) $(k \in \boldsymbol{N})$ for $h_{n'} = 1, \cdots, l_{n'}$, in such a way that the infinite series

$$\sum_{k=1}^{\infty} f_{h_{n'},k+1,\gamma,\beta}(z) z^{-L}$$
$$(\text{resp. } \sum_{k=1} e_{h_{n'},k+1,N,\beta}(x_{n'}, z) x_{n'}^{-N} z^{-L}$$
$$\text{and resp. } \sum_{k=1} e_{h_{n'},k+1,N,M}(x) y^{-M} x_{n'}^{-N} z^{-L})$$

is uniformly and absolutely convergent in $S^{[n'+1,n]}_{h[n'+1,n]}[1]$ for any $\gamma \in \boldsymbol{N}$, $\beta \in \boldsymbol{N}^{n'-1}$, and $L \in \boldsymbol{N}^{n-n'}$ (resp. $S_{n'}[h_{n'}, \infty] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$ for any $N \in \boldsymbol{N}$, $\beta \in \boldsymbol{N}^{n'-1}$ and $L \in \boldsymbol{N}^{n-n'}$ and resp. $D[\infty]^{n'-1} \times S_{n'}[h_{n'}, \infty] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$ for any $N \in \boldsymbol{N}$, $M \in \boldsymbol{N}^{n'-1}$ and $L \in \boldsymbol{N}^{n-n'}$). By the assertion (2) of Proposition 3, the infinite product

$$\tilde{Q}_{h_{n'}}(x) = \lim_{k\to\infty} (I_m + F_{h_{n'},k+1}(x)) \times \cdots \times (I_m + F_{h_{n'},2}(x))$$

is uniformly and absolutely convergent, and represents a matricial holomorphic and strongly asymptoticaly developable function, with the properties $(3, n')$ and $(4, n')$ in $D[\infty]^{n'-1} \times S_{n'}[h_{n'}, \infty] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$. Moreover, by using the definition of $\tilde{Q}_{h_{n'}}$, $(36, k)$ and $(7, k)$, we can verify easily that

$$\tilde{Q}_{h_{n'}}(x) P_{h_{n'},h_{n'}+1}(x) = \tilde{Q}_{h_{n'}+1}(x)$$

for any x in $D[\infty]^{n'-1} \times S_{n'}[h_{n'}, h_{n'}+1, \infty] \times S^{[n'+1,n]}_{h[n'+1,n]}[1]$ for all $h_{n'} = 1, \cdots, l_{n'}$. Therefore, if we put

$$Q_{h_{n'}}(x) = \begin{cases} \tilde{Q}_{h_{n'}-1}(x) P_{h_{n'}-1,h_{n'}}(x) \\ \qquad \text{for } x \text{ in } D[\infty]^{n'-1} \times S^{n'}_{h_{n'}-1,h_{n'}}(r_{n'}-\varepsilon) \times S^{[n'+1,n]}_{h[n'+1,n]}[1], \\ \tilde{Q}_{h_{n'}}(x) \qquad \text{for } x \text{ in } D[\infty]^{n'-1} \times S_{n'}[h_{n'}, \infty] \times S^{[n'+1,n]}_{h[n'+1,n]}[1], \\ \tilde{Q}_{h_{n'}+1}(x) P_{h_{n'},h_{n'}+1}(x)^{-1} \\ \qquad \text{for } x \text{ in } D[\infty]^{n'-1} \times S^{n'}_{h_{n'},h_{n'}+1}(r_{n'}-\varepsilon) \times S^{[n'+1,n]}_{h[n'+1,n]}[1], \end{cases}$$

we find that the family $\{Q_{h_{n'}}(x)\}_{h_{n'}=1,\ldots,l_{n'}}$ satisfies the conditions $(3, n')$, $(4, n')$ and $(5, n')$ with $r'_i = r_i - \varepsilon$. Q.E.D

Consider an open covering $\{S_h(r)\}_{h\in\mathscr{L}}$ of the product of n punctured discs $(D(r_1)-\{0\})\times\cdots\times(D(r_n)-\{0\})$, and let $\{P_{hh'}\}_{h,h'\in\mathscr{L}}$ be a family of m by m matrices such that

(0, 1) *$P_{hh'}$ is holomorphic and invertible in $S_{hh'}(r)$,*

(1, 1) *$P_{hh'}$ is strongly asymptotically developable to I_m in $S_{hh'}(r)$,*

(2, 1) *$P_{hh'}P_{h'h''}=P_{hh''}$ in $S_{hh'h''}(r)$ for h, h' and $h''\in\mathscr{L}$.* Then, we have

Theorem 3 (*Theorem of Sibuya type*). *There exists a family $\{P_h\}_{h\in\mathscr{L}}$ of m by m matricial functions such that*

$(3, n+1)$ *P_h is holomorphic and invertible in $S_h(r'')$,*

$(4, n+1)$ *there exists a formal series $\hat{P}$ in $GL(m, \hat{\mathcal{O}}'_n(r''))$ such that P_h is strongly asymptotically developable to $\hat{P}$ in $S_h(r'')$ for all $h\in\mathscr{L}$,*

$(5, n+1)$ *$P_hP_{hh'}=P_{h'}$ in $S_{hh'}(r'')$ for any h, $h'\in\mathscr{L}$, where $r''\in(\boldsymbol{R}^+)^n$, $r''\leqslant r$.*

Proof of Theorem 3. We construct the matrices P_h's in the form of products of n matrices $Q(h_n)\times\cdots\times Q(h[n', n])\times\cdots\times Q(h)$.

First, we denote by $P(1, h, h')$ the given matricial function $P_{hh'}$ for $h, h'\in\mathscr{L}$. Suppose for a positive integer $n'\leqslant n$ that, associated to the covering

$$\{D(r)^{n'-1}\times S^{[n',n]}_{h[n',n]}(r)\}h[n', n]\in\mathscr{L}[n', n]$$

of $D(r)^{n'-1}\times(D(r_{n'})-\{0\})\times\cdots\times(D(r_n)-\{0\})$, we define a family

$$\{P(n', h[n', n], h'[n', n])\}_{h[n',n],h'[n',n]\in\mathscr{L}[n',n]}$$

of matrices such that

$(0, n')^\#$ $P(n', h[n', n], h'[n', n])$ is holomorphic, invertible and strongly asymptotically developable in $D(r)^{n'-1}\times S^{[n',n]}_{h[n',n]h'[n',n]}(r)$,

$(1, n')^\#$ $TA(P(n', h[n', n], h'[n', n])-I_m)_{\alpha_J}=0_m$ for any nonempty subset J of $[n', n]$ and any $\alpha_J\in\boldsymbol{N}^J$,

$(2, n')^\#$ $P(n', h[n', n], h'[n', n])P(n', h'[n', n], h''[n', n])=P(n', h[n', n], h''[n', n])$ in $D(r)^{n'-1}\times S^{[n',n]}_{h[n',n]h'[n',n]h''[n',n]}(r)$.

Then, for a fixed index $h[n'+1, n]\in\mathscr{L}[n'+1, n]$, we can apply Theorem 2 to the family

$$\{P(n', (h_{n'}, h[n'+1, n]), (h'_{n'}, h[n'+1, n]))\}_{h_{n'},h'_{n'}\in[1,l_{n'}]}$$

associated to the covering

$$D(r)^{n'-1}\times S_{n',h_{n'}}(r_{n'})\times S^{[n'+1,n]}_{h[n'+1,n]}(r)$$

of

$$D(r)^{n'-1}\times(D(r_{n'})-\{0\})\times S^{[n'+1,n]}_{h[n'+1,n]}(r).$$

We denote by

$$\{Q(n',(h_{n'},h[n'+1,n]))\}_{h_{n'}\in[1,l_{n'}]}$$

the family with the properties $(3, n')$, $(4, n')$ and $(5, n')$. By the equalities $(2, n')^{\#}$ and $(5, n')$, for any two fixed indices $h[n'+1, n]$ and $h'[n'+1, n]\in\mathscr{L}[n'+1, n]$, we have the relations

$$\begin{aligned}&Q(n',(h_{n'},h[n'+1]))P(n',(h_{n'},h[n'+1,n]),(h'_{n'},h[n'+1,n]))\\&\quad\times Q(n',(h'_{n'},h'[n'+1,n]))^{-1}\\&\quad=Q(n',(h''_{n'},h[n'+1,n]))P(n',(h''_{n'},h[n'+1,n]),(h'''_{n'},h'[n'+1,n]))\\&\quad\times Q(n',(h'''_{n'},h'[n'+1,n]))^{-1}\end{aligned}$$

in $D(r')^{n'-1}\times S_{n',h_{n'}h'_{n'}h''_{n'}h'''_{n'}}(r'_{n'})\times S^{[n'+1,n]}_{h[n'+1,n]h'[n'+1,n]}(r')$ for any $h_{n'}$, $h'_{n'}$, $h''_{n'}$ and $h'''_{n'}\in[1, l_{n'}]$. Hence, we can define a matricial function $P(n'+1, h[n'+1, n], h'[n'+1, n])$ in

$$D(r')^{n'-1}\times(D(r'_{n'})-\{0\})\times S^{[n'+1,n]}_{h[n'+1,n]h'[n'+1,n]}(r')$$

by

$$\begin{aligned}&P(n'+1,h[n'+1,n],h'[n'+1,n])\\&\quad=Q(n',h[n',n])P(n',h[n',n],h'[n',n])Q(n',h'[n',n])^{-1}\end{aligned}$$

in $D(r')^{n'-1}\times S_{n',h_{n'},h'_{n'}}(r'_{n'})\times S^{[n'+1,n]}_{h[n'+1,n]h'[n'+1,n]}(r')$ for $h_{n'}$, $h'_{n'}\in[1, l_{n'}]$. By Remark 1 in the first section, the function

$$P(n'+1,h[n'+1,n],h'[n'+1])$$

can be extended to a matricial function holomorphic in

$$D(r')^{n'}\times S^{[n'+1,n]}_{h[n'+1,n]h'[n'+1,n]}(r').$$

For simplicity, we write the extention by the same notation. Then, we can easily verify that, associated to the covering

$$D(r')^{n'}\times S^{[n'+1,n]}_{h[n'+1,n]}(r')_{h[n'+1,n]\in\mathscr{L}[n'+1,n]}$$

of $D(r')^{n'}\times(D(r'_{n'+1})-\{0\})\times\cdots\times(D(r'_n)-\{0\})$, the family

$$\{P(n'+1, h[n'+1, n], h'[n'+1, n])\}_{h[n'+1,n],h'[n'+1,n]\in\mathcal{L}[n'+1,n]}$$

satisfies the conditions $(0, n'+1)^{\#}$, $(1, n'+1)^{\#}$ and $(2, n'+1)^{\#}$.

Hence, this inductive procedure enables us to define a sequence of families of matricial functions

$$\{Q(n', h_{n'}, \cdots, h_n)\}_{(h_{n'},\cdots,h_n)\in\mathcal{L}[n',n]} \qquad (n'=1, \cdots, n)$$

with the properties $(3, n')$, $(4, n')$ and $(5, n')$. Now, we put

$$P_h=Q(n, h_n)\times\cdots\times Q(n', h_{n'}, \cdots, h_n)\times\cdots\times Q(1, h)$$

for all $h=(h_1, \cdots, h_n)\in\mathcal{L}[1, n]$. Then, we see easily that the family

$$\{P_h\}_{h\in\mathcal{L}[1,n]}$$

satisfies the conditions $(3, n+1)$ and $(5, n+1)$, from which we have

$$FA_J(P_h)=FA_J(P_{h'})$$

for any non-empty subset J of $[1, n]$ and any $h, h'\in\mathcal{L}[1, n]$. Therefore, we get

$$TA(P_h)_{\alpha_J}=TA(p_{h'})_{\alpha_J} \qquad \text{in } \prod_{i\in J^c} S_{i,h_i,h_i'}(r_i''),$$

and it follows that we can define a holomorphic function P_{α_J} in $\prod_{i\in J^c}(D(r_i'')-\{0\})$ by

$$P_{\alpha_J}=TA(P_h)_{\alpha_J} \quad \text{in } \prod_{i\in J^c} S_{i,h_i}(r_i''), \quad (h_i)_{i\in J^c}\in\prod_{i\in J^c}[1, l_i]$$

for any J and for any $\alpha_J\in N^J$. By Remark 1, P_{α_J} can be extended to a function holomorphic in $\prod_{i\in J^c} D(r_i'')$, and so P_{α_J} represents convergent power series. This implies $(4, n+1)$. Q.E.D.

In a similar way, we can prove the following theorem.

Theorem 4. *Let* $\{S_h(r)\}_{h\in\mathcal{L}}$ *be an open sectorial covering of the product of* n *punctured discs* $(D(r_1)-\{0\})\times\cdots\times(D(r_n)-\{0\})$, *and let* $\{U_{hh'}\}_{h,h'\in\mathcal{L}}$ *be a family of* m *by* m' *matrices such that*

$(0, 1)$ $U_{hh'}$ *is holomorphic in* $S_{hh'}(r)$,

$(1, 1)$ $U_{hh'}$ *is strongly asymptotically developable to* $0_{m,m'}$ *in* $S_{hh'}(r)$,

$(2, 1)$ $U_{hh'}+U_{h'h''}=U_{hh''}$ *in* $S_{hh'h''}(r)$ *for* h, h' *and* $h''\in\mathcal{L}$. *Then, there exists a family* $\{U_h\}_{h\in\mathcal{L}}$ *of* m *by* m' *matricial functions such that*

$(3, n+1)$ U_h *is holomorphic in* $S_h(r'')$, (4, $n+1$) *there exists a formal series $\hat{U}$ in $M(m, m'. \hat{\mathcal{O}}'_n(r''))$ such that U_h is strongly asymptotically developable to $\hat{U}$ in $S_h(r'')$ for all $h \in \mathscr{L}$*

(5, $n+1$) $U_h + U_{hh'} = U_{h'}$ *in $S_{hh'}(r'')$ for any $h, h' \in \mathscr{L}$, where $r'' \in (\boldsymbol{R}^+)^n$, $r'' \leqslant r$.*

§ 4. Theorems of Malgrange type.

In the following, we identify $(\boldsymbol{C}-\{0\})^n$ with $(\boldsymbol{R}/\mathrm{mod.}\ 2\pi\boldsymbol{Z})^n \times (\boldsymbol{R}^+)^n$ by the canonical mapping: for a point $x=(x_1, \cdots, x_n) \in (\boldsymbol{C}-\{0\})^n$, we associate with it $(\arg x_1, \cdots, \arg x_n, |x_1|, \cdots, |x_n|) \in (\boldsymbol{R}/\mathrm{mod.}\ 2\pi\boldsymbol{Z})^n \times (\boldsymbol{R}^+)^n$. We begin by constructing the *sheaf of germs of strongly asymptotically developable functions* over the real n-dimensional torus T^n homeomorphic to $(\boldsymbol{R}/\mathrm{mod.}\ 2\pi\boldsymbol{Z})^n$.

For any connected open subset $c=\prod_{i=1}^n (\tau_{-i}, \tau_{+i})$ of T^n and for any element $r=(r_1, \cdots, r_n)$ in $(\boldsymbol{R}^+)^n$, we denote by $A(c, r)$ the $\boldsymbol{C}$-algebra of functions holomorphic and strongly asymptotically developable in the open polysector $S(c, r)= \prod_{i=1}^n S(\tau_{-i}, \tau_{+i}, r_i)$ associated with (c, r) at the origin in $\boldsymbol{C}^n$. If $r, r' \in (\boldsymbol{R}^+)^n$ and $r<r'$, then we have the natural restriction mapping $i_{rr'}$ of $A(c, r')$ into $A(c, r)$. This implies that

$$\{A(c, r), i_{rr'}; r, r' \in (\boldsymbol{R}^+)^n\}$$

is an inductive system for any connected open subset c of the above-mentioned form of T^n. We put

$$A(c)=\mathrm{dir}\ \lim_{r\to 0} A(c, r).$$

For any two open subsets c, c' of the above-mentioned form of T^n, if $c\subset c'$, we have the natural restriction mapping $i_{cc'}$ of $A(c')$ into $A(c)$ so that the inductive system

$$\{A(c), i_{cc'}\}$$

is a presheaf which satisfies the sheaf conditions. We denote by $\mathscr{A}$ the associated sheaf, call it the *sheaf of germs of strongly asymptotically developable functions over T^n.*

By the usual notation, $GL(m, E)$ denotes the multiplicative group of invertible m by m matrices whose elements belong to $E=A(c, r)$, $\hat{\mathcal{O}}'_n$ or $\mathcal{O}_n$.

We denote by $GL(m, A(c, r))_{I_m}$ the subgroup of $GL(m, A(c, r))$ of matricial functions strongly asymptotically developable to the unit matrix I_m in the open polysector $S(c, r)$ associated with (c, r). In a way similar to the construction of $\mathscr{A}$, from $GL(m, A(c, r))_{I_m}$, we obtain the *sheaf $GL(m, \mathscr{A})_{I_m}$ of germs of matricial functions strongly asymptotically developable to I_m.*

For two germs (P), (P') in $GL(m, \hat{\mathcal{O}}'_n)$, we say that (P) is *equivalent* to (P') with respect to $GL(m, \mathcal{O}_n)$, if there exists a germ (Q) in $GL(m, \mathcal{O}_n)$ such that $(P)=(P')(Q)$

as germs in $GL(m, \hat{\mathcal{O}}'_n)$. We denote by $GL(m, \hat{\mathcal{O}}'_n)/GL(m, \mathcal{O}_n)$ the quotient set of $GL(m, \hat{\mathcal{O}}'_n)$ divided by the equivalence relation.

Combining Theorem of Sibuya type with Theorem of Borel-Ritt type, we have the following theorem.

Theorem 5 (*Theorem of Malgrange type*). *Let* $\{c_h\}_{h\in H}$ *be an open covering of* T^n, *where* $c_h=\prod_{i=1}^n(\tau_{-i}^h, \tau_{+i}^h)$ *for all* $h\in H$. *There exists a bijective mapping of* $GL(m, \hat{\mathcal{O}}'_n)/GL(m, \mathcal{O}_n)$ *into the first cohomology set of the covering* $\{c_h\}_{h\in H}$ *with coefficients in* $GL(m, \mathscr{A})_{I_m}$. *And hence,*

$$GL(m, \hat{\mathcal{O}}'_n)/GL(m, \mathcal{O}_n)\approx H^1(T^n, GL(m, \mathscr{A})_{I_m}).$$

Proof of Theorem 5. We can assume without loss of generality that, the given open covering is finite. For any $r=(r_1, \cdots, r_n)\in(\boldsymbol{R}^+)^n$, the product of n punctured discs $(D(r_1)-\{0\})\times\cdots\times(D(r_n)-\{0\})$ has the open sectorial covering $\{S(c_h, r)\}_{h\in H}$ associated with the covering $\{c_h\}_{h\in H}$ of T^n. Let $\hat{P}$ be an element in $GL(m, \hat{\mathcal{O}}'_n(r))$ for some $r\in(\boldsymbol{R}^+)^n$. By Theorem of Borel-Ritt type, i.e. the assertion (2) of Theorem 1, we obtain a family $\{P_h\}_{h\in H}$ of matricial functions such that P_h is holomorphic, invertible and strongly asymptotically developable to $\hat{P}$ in $S(c_h, r)$ for all $h\in H$. Then, $P_h^{-1}P_{h'}$ is holomorphic, invertible and strongly asymptotically developable to I_m in $S(c_h\cap c_{h'}, r)$ for all $h, h'\in H$, and the family $\{P_h^{-1}P_{h'}\}_{h,h'\in H}$ represents a one-cocyle of the covering $\{c_h\}_{h\in H}$ with coefficients in $GL(m, \mathscr{A})_{I_m}$. For an other element $\hat{P}'$ in $GL(m, \hat{\mathcal{O}}'_n(r'))$, if the germ $(\hat{P}')$ of $\hat{P}'$ is equivalent to the germ $(\hat{P})$ of $\hat{P}$ with respect to $GL(m, \mathcal{O}_n)$, the family $\{P_h'^{-1}P'_{h'}\}_{h,h'\in H}$ obtained from $\hat{P}'$ represents the same element as $\{P_h^{-1}P_{h'}\}_{h,h'\in H}$ in the first cohomology set $H^1(\{c_h\}_{h\in H}, GL(m, \mathscr{A})_{I_m})$ of the covering $\{c_h\}_{h\in H}$ with coefficients in $GL(m, \mathscr{A})_{I_m}$. Therefore, we can associate the equivalence class of the one-cocycle with the equivalence class of $(\hat{P})$, and we denote by b the mapping of $GL(m, \hat{\mathcal{O}}'_n)/GL(m, \mathcal{O}_n)$ into $H^1(\{c_h\}_{h\in H}, GL(m, \mathscr{A})_{I_m})$. By the definition of b and Remark 1, it is easily seen that b is injective. On the other side, Theorem of Sibuya type, i.e. Theorem 3 asserts that b is surjective. Q.E.D.

Consider now the additive group $M(m, m', A(c, r))$ of m by m' matrices whose elements belong to $A(c, r)$ for all open subset c of T^n and all r in $(\boldsymbol{R}^+)^n$. In a similar way, we can construct the sheaf $M(m, m', \mathscr{A})_{0_{m,m'}}$ of germs of matricial functions strongly asymptotically developable to zero matrix $0_{m,m'}$ over T^n. On the other side, from the additive group $M(m, m', \hat{\mathcal{O}}'_n)$ and its subgroup $M(m, m', \mathcal{O}_n)$, we obtain the quotient module $M(m, m', \hat{\mathcal{O}}'_n)/M(m, m', \mathcal{O}_n)$.

By the assertion (2) of Theorem 1 and Theorem 4, we can prove the following theorem.

Theorem 6 (*Theorem of Malgrange type*). *Let* $\{c_h\}_{h\in H}$ *be an open covering of* T^n, *where* $c_h=\prod_{i=1}^n(\tau_{-i}^h, \tau_{+i}^h)$ *for all* $h\in H$. *The first cohomology group* $H^1(\{c_h\}_{h\in H},$

$M(m, m', \mathscr{A})_{0_{m,m'}}$) of $\{c_h\}_{h \in H}$ with coefficients in $M(m, m', \mathscr{A})_{0_{m,m'}}$ is isomorphic to $M(m, m', \hat{\mathcal{O}}'_n)/M(m, m', \mathcal{O}_n)$. Therefore,

$$H^1(T^n, M(m, m', \mathscr{A})_{0_{m,m'}}) \approx M(m, m', \hat{\mathcal{O}}'_n)/M(m, m', \mathcal{O}_n).$$

This theorem can be proved in a way similar to that of Malgrange [4] (cf. Majima [3]).

§ 5. Strong asymptotic developability of functions with parameters.

Let $f(x, t)$ be a holomorphic function of (x, t) in the product of an open polysector $S = S_1 \times \cdots \times S_n$ at the origin in C^n and a demain T of C^m. The function $f(x, t)$ is said to be *strongly asymptotically developable with respect to x in S uniformly with respect to t in T*, there exists a family of functions

$$\{f(x_{J^c}; t; \alpha_J)\}_{J \subset [1,n], \alpha_J \in N^J} \tag{5.1}$$

with the properties (5.2) and (5.3)

(5.2) $\quad f(x_{J^c}; t; \alpha_J)$ is holomorphic in $S_{J^c} \times T$,

(5.3) for any $N_{J^c} \in N^{J^c}$ and for any closed subpolysector S'_{J^c} of S_{J^c}, there exists a positive constant $K_{N_{J^c}, S'_{J^c}}$ such that

$$|f(x_{J^c}; t; \alpha_J) - g(x_{J^c}; t; \alpha_J; N_{J^c})| \leqslant K_{N_{J^c}, S'_{J^c}} |x_{J^c}|^{N_{J^c}}$$

for any (x_{J^c}, t) in $S'_{J^c} \times T$, where $g(x_{J^c}; t; \alpha_J; N_{J^c})$ is defined by (1.6) for $J \neq \phi$ and (1.3) for $J \neq \phi$ (see § 1) with the functions of the family (5.1), and $f(x; t; \alpha_\phi) = f(x, t)$.

If $f(x, t)$ is strongly asymptotically developable with respect to x in S uniformly with respect to t in T, then the family (5.1) of functions is *uniquely* determined for $f(x, t)$. We call it the *total family of coefficients of uniform strong asymptotic expansion of* $f(x, t)$ and denote it by $TAU(f)$. For any non-empty subset J of $[1, n]$, we denote $f(x_{J^c}; t; \alpha_J)$ by $TAU(f)_{\alpha_J}$ and define the formal series by

$$FAU_J(f) = \textstyle\sum_{\alpha_J \in N^J} TAU(f)_{\alpha_J} x_J^{\alpha_J} \tag{5.4}$$

which is called the *formal series of uniform strong asymptotic expansion of* $f(x, t)$ *for* J. For simplicity, if $J = [1, n]$, we use $FAU(f)$ instead of $FAU_{[1,n]}(f)$.

If $FAU_J(f)$ is *convergent* series in S_{J^c} for all $J = \{i\}$, $i \in [1, n]$, we say that $f(x, t)$ is *strongly asymptotically developable to* $FAU(f)$ *with respect to x in S uniformly with respect to t in T.*

We define also the *approximate function of degree N of uniform strong asymptotic expansion*, the *error function of degree N of uniform strong asymptotic expansion*, the *set* $AU(S \times T)$ *of all functions holomorphic and strongly asymptotically developable with respect to x in S uniformly with respect to t in T*, etc. as in § 1. Then, *we obtain* *results similar to Propositions* 2 *and* 3, *Theorems* 1, 2, 3, 4, 5 *and* 6. We dare not to write the results and the proofs, because we obtain them with a little modification.

As other kinds of results, we have two propositions which are proved easily by using Cauchy's integral formula. (cf. W. Wasow. [9], Y. Sibuya [5]).

Proposition 4. *If* $f(x, t_1, \cdots, t_m)$ *is in* $AU(S\times T)$, *then, for every compact proper subset* T' *of* T,

(1) $(\partial/\partial t_k)f(x, t_1, \cdots, t_m) \in A(S\times T')$

and

(2) $TAU((\partial/\partial t_k)f)_{\alpha_J}=(\partial/\partial t_k)TAU(f)_{\alpha_J}$

for any non-empty subset J *of* $[1, n]$ *and any* $\alpha_J \in N^J$.

Let T be a *polydisc* at the origin in C^m and let $f(x, t)$ be in $AU(S\times T)$. In this case, f and $TAU(f)_{\alpha_J}$ have the Taylor's expansions with respect to t:

$$f(x, t)=\sum_{M\in N^n} f_M(x)t^M$$

and

$$TAU(f)_{\alpha_J}=\sum_{M\in N^n} f(x_{J^c}; \alpha_J)t^M$$

for any non-empty subset J of $[1, n]$ and any $\alpha_J \in N^J$. Then,

Proposition 5. *For any* $M \in N^m$,

i) $f_M \in A(S)$

and

ii) $TAU(f_M)_{\alpha_J}=f_M(x_{J^c}; \alpha_J)$

for all non-empty subset J *of* $[1, n]$ *and all* $\alpha_J \in N^J$.

Added in proof (for "Analogues of Cartan's decomposition theorem in asymptotic analysis" by H. Majima).

1. By using (1.16), we can deduce (1.9) from (1.12). Therefore, we can replace (1.9) by (1.12) in Definition 2.

2. The main theorems of this paper are reformulated in terms of exact sequences of sheaves and the derived exact cohomology sequences in [10], where the terminology is slightly changed.

3. Some applications of the main theorems are found in [11] and [12].

The details are written in [13].

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