Funkcialaj Ekvacioj, **25** (1982) 347-356

Uniform Stability for Delay-Differential Equations with Infinite Delays

By

George Seifert

(Iowa State University, U. S. A.)

In [1] sufficient conditions are given on systems of delay-differential equations with almost periodic (a.p. for short) time dependence for the existence of a.p. solutions. These conditions do not seem to imply any uniqueness or stability properties for such a.p. solutions. On the other hand for systems of ordinary differential equations, analogous conditions are not only sufficient for the existence of a.p. solutions, but also for their uniqueness and asymptotic stability; cf. [2].

In our first result we show that a natural additional condition does in fact yield the usual type of uniform stability for a.p. solutions of the delay-differential systems considered in [1]. Asymptotic stability, however, does not follow; in a sense, this is probably due to the fact that we consider systems with infinite delays not explicitely of fading memory type; cf. an example in [10].

Our next result is a Liapunov-Razumikhim theorem giving conditions for stability of solutions of systems with infinite delays also not explicitely of fading memory type. It is similar to a result due to Driver in [6] and implies our first result if the Euclidean norm in R^n is used. An application of this theorem to systems of the type considered in [1] is given.

For our first theorem we need the following two lemmas. Their proofs are fairly simple, but we include them for the sake of clarity and completeness.

Lemma 1. *Let $r(t)$ be real-valued, continuous on the closed bounded interval $[a, b]$, and continuously differentiable on (a, b). Let $r(a)=0$, $r(b)>0$, and define*

$$m(t)=\sup\{r(s): a\leq s\leq t\} \qquad \text{for } t\in[a, b]$$

then there exists an open interval $I\subset[a, b]$ in which $m(t)$ is increasing.

Proof. First, there exists a $t_1\in(a, b)$ such that $D^-m(t_1)$, the upper left derivative of m is positive there; if not $D^-m(t)\leq 0$ for $t\in(a, b)$ and by a standard argument (cf. for example, [11]: p. 354) we would have $m(b)\leq m(a)=0$, a contradiction. It follows that $r(t_1)=m(t_1)$; if not, $m(t_1)=r(t_0)$ for some $t_0\in[a, t_1)$ and hence $m(t)=m(t_0)=m(t_1)$ for $t\in[t_0, t_1]$ which contradicts the fact that $D^-m(t_1)>0$.

Let $h<0$ and $|h|$ be sufficiently small; then since $m(t_1+h)\geq r(t_1+h)$, it follows that

$$(m(t_1+h)-m(t_1))h^{-1} \leq (r(t_1+h)-r(t_1))h^{-1},$$

and letting $h \to 0$ we get $D^- m(t_1) \leq r'(t_1)$; i.e., $r'(t_1) > 0$. But then $r'(t) > 0$ in some open interval $I=(t_1, t_2) \subset [a, b]$, and this, with $m(t_1)=r(t_1)$, clearly implies $m(t)=r(t)$, and is therefore also increasing on I; q.e.d.

Remark. The condition that $r(t)$ be continuously differentiable seems crucial for Lemma 1; there are simple examples of functions with positive derivative at a point, differentiable in an interval containing this point, and yet not nondecreasing in such an interval. In fact, there exist non-trivial examples of functions continuous on an interval and yet not monotonic on any subinterval; cf [3], p. 29.

The next lemma uses a method due essentially to Medvedev [4]. In what follows, R^n will denote the set of real n-vectors and $|x|$ a norm for $x \in R^n$ such that $|x(t)|$ is continuously differentiable whenever $x(t)$ is and $x(t) \neq 0$.

Lemma 2. *Let $z(t)$: $[t_1, b) \to R^n$, $b \leq \infty$, and suppose on $[t_1, b)$:*

(i) *the derivative $z'(t)$ exists, in a right hand sense at t_1;*

(ii) *there exist $p>0$ and $h>0$, $ph<1$, such that*

$$|z(t)+hz'(t)| \leq (1-ph)|z(t)|.$$

Then $|z(t)| \leq |z(t_1)| \exp[-p(t-t_1)]$, $t \in [t_1, b)$.

Proof. In what follows, $t \in [t_1, b)$ and $z=z(t)$, $z'=z'(t)$. Using (ii) we get

$$\begin{aligned} |hz'+(1+ph)z| &\leq |hz'+z|+ph\,|z| \\ &\leq (1-ph)|z|+ph|z|=|z|. \end{aligned}$$

Put $\alpha=(1-ph)/h$, multiply by $h^{-1}\exp[\alpha(t-t_1)]$, we obtain, eventually

$$(\exp[\alpha(t-t_1)]\,|z(t)|)' \leq \exp[\alpha(t-t_1)]\,|z(t)|/h,$$

from which we easily get the desired conclusion; q.e.d.

Remarks. If f: $R \times R^n \to R^n$ is continuous and there exist positive numbers p, h, and r with $ph<1$, such that

(ii)′ $|x-y+h(f(t, x)-f(t, y))| \leq (1-ph)|x-y|$

holds for $|x| \leq r$, $|y| \leq r$, $t \in [t_1, b)$, $b \leq \infty$, then if $x(t)$ and $y(t)$ are solutions of

$$x'=f(t, x) \tag{1.0}$$

such that $|x(t)| \leq r$, $|y(t)| \leq r$ for $t \in [t_1, b)$, then by using Lemma 2 with $z(t)=x(t)-y(t)$, we get easily that

$$|x(t)-y(t)| \leq |x(t_1)-y(t_1)| \exp[-p(t-t_1)], \qquad t \in [t_1, b).$$

If also $|f(t, 0)| \leq M$ for $t \in [t_1, b)$ where $M < pr$, it can be shown that if $x(t)$ solves (1.0) on $[t_1, b)$, and $|x(t_1)| < r$, then $|x(t)| < r$ on this interval; for a proof, cf. [4].

The following is a simple generalization of Lemma 2:

Lemma 3. *Let* (i) *of Lemma 2 hold, and suppose there exists a realvalued function $p(t)$ locally integrable on $[t_1, b)$ and a constant $h \neq 0$ such that $hp(t) < 1$ and* (ii) *holds with $p = p(t)$. Then*

$$|z(t)| \leq |z(t_1)| \exp\left[-\int_{t_1}^{t} p(s)ds\right], \qquad t \in [t_1, b).$$

A proof of this can easily be obtained following the proof of Lemma 2.

We also note that condition (ii) in Lemma 2 is implied by the limit condition: there exists $p > 0$ such that

$$\overline{\lim_{h \to +0}} \, (|z(t) + hz'(t)| - |z(t)|)(h|z(t)|)^{-1} \leq -p$$

uniformly for $t \in [t_1, b)$, $|z(t)| \neq 0$.

We introduce the following notation and definitions. CB is the set of R^n-valued functions continuous and bounded on $(-\infty, 0]$; with norm $\|\phi\| = \sup\{|\phi(t)|: t \leq 0\}$ for $\phi \in CB$, $\{CB, \|\ \ \|\}$ is a real Banach space. For $r > 0$ define $CB_r = \{\phi \in CB: \|\phi\| \leq r\}$.

If $x(t): (-\infty, b) \to R^n$, $b \leq \infty$, then for any $t < b$ we denote x_t the function $x(t+s): s \leq 0$. Thus if $x(t)$ is continuous and bounded on $(-\infty, b)$, then $x_t \in CB$ for $t \in (-\infty, b)$.

Let $f(t, \phi): R \times CB \to R^n$ and $-\infty < t_0 < b \leq \infty$. The function $x(t): (-\infty, b) \to R^n$ is said to be a solution of

$$(1) \qquad x'(t) = f(t, x_t)$$

for $t \in [t_0, b)$ if $x_t \in CB$ for $t < b$ and the derivative $x'(t)$ exists on $[t_0, b)$, in a right hand sense at t_0, and satisfies (1) on $[t_0, b)$.

The following stability definition is standard: A solution $\bar{x}(t)$ of (1) on $[t_0, \infty)$ is uniformly stable (on $[t_0, \infty)$) if given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for $t_1 \geq t_0$ and any solution $x(t)$ of (1) on $[t_1, b)$ with $\|x_t - \bar{x}_{t_1}\| < \delta_\varepsilon$, it follows that $x(t)$ exists for $t \geq t_1$ and satisfies $|x(t) - \bar{x}(t)| < \varepsilon$ for $t \geq t_1$.

In the above definitions, we also allow $t_0 = -\infty$; in that case we simply replace $[t_0, b)$ by $(-\infty, b]$.

The question of whether, given $\phi \in CB$, and $t_0 \in R$, there exsts a $b \leq \infty$ and a solution $x(t)$ of (1) on $[t_0, b)$ such that $x_t = \phi$, is the socalled initial value problem (i.v.p. for short) for (1). Conditions on f sufficient for its solution must be stronger than continuity on $R \times CB$ and locally Lipschitz in ϕ, such an i.v.p. may have no solutions; cf. [5]. On the other hand, if one restricts the class of initial functions ϕ

to certain subspaces of $\{CB, \|\ \|\}$; i.e., for example, $\{CBU, \|\ \|\}$ where CBU consisting of functions of CB uniformly continuous on $(-\infty, 0]$, and $\|\ \|$ is as before, then a Peano existence theorem for the i.v.p. can be established; i.e., continuity of f on $R\times CB$ is sufficient. Also under the assumption that the composite $f(t, x_t)$ is continuous on $[t_0, \infty)$ for any $x(t)$, $R\to R^n$ with $x_t \in CB$ for $t\in R$, Peano type existence theorems follow; for early work on such problems, cf. Driver [6], and for more recent work involving very general initial value (or state) spaces, cf. Hale and Kato [7] and Kappel and Schappacher [8]. These references are also recommended for conditions on f under which $b=\infty$ for solutions of the i.v.p.. For some stability results; cf. [6] and [9].

If f is almost periodic (a.p. for short) in t for each $\phi\in CB$ in a sufficiently uniform sense (cf. [1]) and one is concerned with existence of a.p. solutions of (1) on $(-\infty, \infty)$, then since $x(t)$: $R\to R^n$ and a.p. implies x_t is in CBU for any $t\in R$, one need not impose such strong conditions on f as indicated above to get the existence of a.p. solutions. However, in order to establish stability of such a.p. solutions in the sense of our definition, one must be concerned with the i.v.p. for general $\phi\in CB$, and hence a condition such as in (H_1) below seems quite natural. In fact, we will use the following hypotheses:

(H_1) For any function $x(t)$: $R\to R^n$ such that $x_t\in CB$ for $t\in R$, $f(t, x_t)$ is continuous.

(H_2) There exist positive numbers r, p, and h such that $ph<1$, and if $x(t)$ and $y(t)$ are functions on R to R^n such that $x_t\in CB_r$, $y_t\in CB_r$ for $t\in R$, then

$$|x(t)-y(t)+h(f(t, x_t)-f(t, y_t))|\leq(1-ph)\|x_t-y_t\|, \tag{2}$$

for $t\in R$.

Remark. It is not difficult to show that (H_2) implies that f is globally Lipschitz in ϕ on CB_r; it follows therefore from known results ([6], [7], [8]) that (H_1) and (H_2) imply that the i.v.p. for (1) has a solution for any $\phi\in CB_r$.

Theorem 1. *Let $\bar{x}(t)$ be a solution of* (1) *on $[t_0, \infty)$ such that $|\bar{x}(t)|\leq r_1<r$ for $t\in R$, and (H_1) and (H_2) hold. Then $\bar{x}(t)$ is uniformly stable on $[t_0, \infty)$. This result also holds if $t_0=-\infty$.*

Proof. Fix δ_0, $0<\delta_0<r-r_1$. Then $\sup\{|x-\bar{x}(t)|: t\in R\}\leq\delta_0$ implies $|x|<r$. Let $\phi\in CB$ be such that $\|\phi-\bar{x}_{t_1}\|<\delta_0$ for $t_1\geq t_0$. Let $x(t)=x(t, \phi, t_1)$ be a solution of (1) such that $x_{t_1}=\phi$. Clearly $x_{t_1}\in CB_r$. Suppose there exists $t_2>t_1$ such that $\delta_0<|x(t_2)-\bar{x}(t_2)|$. Then there exists $\bar{t}\in(t_1, t_2)$ such that $|x(\bar{t})-\bar{x}(\bar{t})|=\delta_0$ and $|x(t)-\bar{x}(t)|<\delta_0$ for $t_1<t<\bar{t}$. We may suppose t_2 to be such that $|x(t)-\bar{x}(t)|>0$ for $\bar{t}\leq t\leq t_2$. We now define $r(t)=|x(t)-\bar{x}(t)|-\delta_0$ and if $a=\bar{t}$, $b=t_2$, $r(t)$ satisfies the hypotheses of Lemma 1 on $[\bar{t}, t_2]$; note that (H_1) implies that any solution of (1) is continuously differentiable for $t \geq t_1$ whereever it is defined, and so $r(t)$ is on $(\bar{t}, t_2)$. Let $m(t)$ and the open interval $I \subset [\bar{t}, t_2]$ be as defined in Lemma 1; clearly $m(t) > 0$ for $t \in I$, and from the definition of $m(t)$, it follows that for $t \in I$, $r(s) < r(t)$ for $\bar{t} \leq s < t$; i.e., $|x(s) - \bar{x}(s)| < |x(t) - \bar{x}(t)|$ for such s. But since $|x(s) - \bar{x}(s)| = \delta_0$ for $s = \bar{t}$, and $|x(s) - \bar{x}(s)| < \delta_0$ for $s < \bar{t}$, it follows that

$$|x(s) - \bar{x}(s)| < |x(t) - \bar{x}(t)| \qquad \text{for } s < t,\ t \in I;$$

i.e.,

$$\|x_t - \bar{x}_t\| = |x(t) - \bar{x}(t)|, \qquad t \in I. \tag{3}$$

Using (2) in (H_2) we get

$$|x(t) - \bar{x}(t) + h(f(t, x_t) - f(t, \bar{x}_t))| \leq (1 - ph)|x(t) - \bar{x}(t)|;$$

i.e.,

$$|x(t) - \bar{x}(t) + h(x'(t) - \bar{x}'(t))| \leq (1 - ph)|x(t) - \bar{x}(t)|, \qquad t \in I. \tag{4}$$

Using Lemma 2 with $z(t) = x(t) - y(t)$, $t_1 = t' \in I$, and $b \in I$, $b > t'$, it follows that

$$|x(t) - \bar{x}(t)| \leq |x(t') - \bar{x}(t')| \exp[-p(t - t')] \qquad \text{for } t \in [t', b) \subset I,$$

which contradicts the fact that $|x(t) - \bar{x}(t)|$ is increasing on I. We therefore conclude that $|x(t) - \bar{x}(t)| \leq \delta_0$ for all $t \geq t_1$ for which $x(t)$ is defined. By a standard argument on continuation of solutions, it follows that $x(t)$ is a solution of (1) for all $t > t_1$. Thus $\bar{x}(t)$ is stable, and the proof is complete.

Remarks. As was the case with condition (ii) in Lemma 2, (H_2) is implied by the following limit condition:

$(H_2)'$ There exist positive numbers p and r such that

$$\overline{\lim_{h \to 0+}} \, (|x(t) - y(t) + h(f(t, x_t) - f(t, y_t))| - \|x_t - y_t\|)(h\|x_t - y_t\|)^{-1} \leq -p$$

uniformly for $t \in R$ and functions $x(t)$, $y(t)$ with $x_t \in CB_r$, $y_t \in CB_r$, $x_t \neq y_t$, $t \in R$.

Also, it is clear that if $t_0 > -\infty$, conditions (H_1) and (H_2) need hold only for $t \in [t_0, \infty)$, and the constants p, h, and r in (H_2) may depend on t_0.

As remarked earlier, sufficient conditions for the existence of an a.p. solution for delay-differential systems with a.p. time dependence of the form

$$x'(t) = F(t, x(t), x_t) \tag{5}$$

are given by Theorem 2 in [1]. One of these, specifically (ii) of (H_4) in this result, implies our (H_2) with $F(t, x(t), x_t) = f(t, x_t)$. Thus if we add to the hypotheses of this theorem the condition that for any $x(t)\colon R\to R^n$ such that $x_t \in CB$ for $t \in R$, $F(t, x(t), x_t)$ is continuous on R, then by our Theorem 1 with $t_0=-\infty$, the a.p. solution of (5) is uniformly stable on $(-\infty, \infty)$. It should be observed that even under the additional condition (H_1), no uniqueness for the a.p. solution is implied.

We now state and prove a Liapunov-Razumikhin stability result which can also be used to establish stability conditions for a.p. solutions of (1). In fact, if the Euclidean norm on R^n is used in Theorem 1, this theorem is a corollary of this result. This result is also related to Theorem 5 in [6] and in fact uses the important Lemma 1 in this paper, which we state below as Lemma 4.

For other stability results for systems with infinite delays using Liapunov-Razumikhin conditions, cf. [9].

Lemma 4 (cf. Lemma 1 in [6]). *Suppose that if the real-valued function $w(t)$, continuous for $t<b$ and satisfying $w(s)\le w(t)$ for $s\le t<b$, t fixed, we have*

$$\overline{\lim_{h\to 0+}}\,(w(t+h)-w(t))/h\le 0, \qquad t_0\le t<b;$$

then $w(t)\le \sup\{w(s)\colon s\le t_0\}$ for $t_0<t<b$.

Theorem 2. *Let $V(t, x)\colon R\times R^n\to R$ be continuous and satisfy*

(i) *$u(|x|)\le V(t, x)\le v(|x|)$ on $R\times R^n$, where $u(r)$ and $v(r)$ are continuous and increasing on $[0, \infty)$ and $u(0)=v(0)=0$, and*

(ii) *there exists a number $r_0>0$ such that if $x(t)$ is a solution of (1) on $[t_0, \infty)$ for which $|x(t)|\le r_0$, $t \in R$ and*

$$V(t, x(t))\ge V(s, x(s)) \qquad \text{for } s\le t,\ t\ge t_0,$$

then

$$\dot V(t, x(t))=\overline{\lim_{h\to 0+}}\,(V(t+h, x(t+h))-V(t, x(t)))/h\le 0 \qquad \text{for } t\ge t_0.$$

Finally, let f be such that the i.v.p. for (1) has a solution for any $\phi \in CB_{r_0}$ and $f(t, 0)=0$ for $t \in R$.

Then $x(t)=0$ is uniformly stable on $[t_0, \infty)$.

Remark. The theorem also holds if $t_0=-\infty$; again we replace $[t_0, \infty)$ by $(-\infty, \infty)$ in this case, and the proof is as below.

Proof of Theorem 2. Let ε, $0<\varepsilon<r_0$ be given; there exists $\delta_1(\varepsilon)>0$ such that $0\le r<\delta_1(\varepsilon)$ implies $u^{-1}(r)<\varepsilon$, where u^{-1} denotes the inverse of u. Thus $V(t, x(t))<\delta_1(\varepsilon)$ implies $u(|x(t)|)<\delta_1(\varepsilon)$, which in turn implies $|x(t)|<\varepsilon$. Fix $\delta_2(\varepsilon)<r_0$ and that $0\le r<\delta_2(\varepsilon)$ implies $v(r)<\delta_1(\varepsilon)$. Now consider a solution $x(t)$ of (1) on $[t_1, b)$, such that $|x(t)|<\delta_2(\varepsilon)$ for $t\le t_1$; here $t_1\ge t_0$. Then $V(t, x(t))<v(|x(t)|)<\delta_1(\varepsilon)$ for $t\le t_1$.

With $w(t)=V(t, x(t))$, condition (ii) and Lemma 4 imply $V(t, x(t))\leq\delta_1(\varepsilon)$ for $t\in[t_1, b)$; i.e. $|x(t)|<\varepsilon$ for $t\in[t_1, b)$. Since the i.v.p. for (1) and any initial function $\phi\in CB_{r_0}$ has a local solution for any initial time, a standard continuation argument shows that $x(t)$ exists for $t>t_1$ and satisfies $|x(t)|<\varepsilon$ there.

This proves the theorem.

In the following application of Theorem 2 to systems like (5), we use the inner product $x\cdot y=\sum_{j=1}^{n} x_j y_j$ for $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n)$, and the Euclidean norm $|x|=(x\cdot x)^{1/2}$ for $x\in R^n$, $y\in R^n$.

Theorem 3. *Let $F\colon R\times R^n\times CB\to R^n$ be continuous, locally Lipschitz in $(x, \phi)\in R^n\times CB$, and satisfy* ($H_1$) *with $f(t, \phi)=F(t, \phi(0), \phi)$. Let $\bar{x}(t)$ be a solution of* (5) *on $[t_0, \infty)$ such that for $|x|\leq r$, $\|\phi\|\leq r$, and $[t_0, \infty)$, we have*

$$(\mathrm{H}_3)\qquad x\cdot(F(t, x+\bar{x}(t), \bar{x}_t)-F(t, \bar{x}(t), \bar{x}_t))\leq-\alpha(t)|x|^2,$$

and

$$(\mathrm{H}_4)\qquad |F(t, x+\bar{x}(t), \phi+\bar{x}_t)-F(t, x+\bar{x}(t), \bar{x}_t)|\leq\beta(t, x)\|\phi\|.$$

Suppose that there exists a $\mu>0$ such that $\alpha(t)-\beta_0(t)\geq\mu$ for $t\geq t_0$, where

$$\beta_0(t)=\lim_{|x|\to 0}\beta(t, x),$$

the limit being uniform for $t\geq t_0$, and that $\beta_0(t)$ is bounded on $[t_0, \infty)$. Then $\bar{x}(t)$ is uniformly stable on $[t_0, \infty)$. This also holds for $t_0=-\infty$; i.e., we replace $[t_0, \infty)$ by $(-\infty, \infty)$.

Proof. Define $h(t, x, \phi)=F(t, x+\bar{x}(t), \phi+\bar{x}_t)-F(t, \bar{x}(t), \bar{x}_t)$; we apply Theorem 2 to (1) with $f(t, \phi)=h(t, \phi(0), \phi)$. From known results, the conditions on F imply the existence of solutions for the i.v.p. for (1) with f as above.

It is easily seen that (H_3) is equivalent to

$$x\cdot h(t, x, 0)\leq-\alpha(t)|x|^2$$

and (H_4) to

$$|h(t, x, \phi)-h(t, x, 0)|\leq\beta(t, x)\|\phi\|.$$

Clearly, there exists $r_0>0$ such that

$$\alpha(t)-\beta(t, x)\geq 0\qquad\text{for } |x|\leq r_0,\ t\geq t_0.$$

With this r_0 and $V(t, x)=|x|^2/2$, $u(r)=v(r)=r^2/2$, we apply Theorem 2 to (1) where $f(t, \phi)=h(t, \phi(0), \phi)$, i.e. to

$$(6)\qquad x'(t)=h(t, x(t), x_t).$$

Let $x(t)$ solve this equation, $|x(t)|\leq r_0$, $t \in R$, and satisfy $V(t, x(t)) > V(s, x(s))$, $s \leq t$, $t \geq t_0$; i.e.,

$$|x(t)| \geq \|x_t\|.$$

Then

$$\begin{aligned}\dot{V}(t, x(t)) &= x(t)\cdot x'(t)\\ &= x(t)\cdot h(t, x(t), 0) + x(t)\cdot (h(t, x(t), x_t) - h(t, x(t), 0))\\ &\leq -\alpha(t)\,|x(t)|^2 + \beta(t, x(t))\,|x(t)|\,\|x_t\|\\ &\leq -\alpha(t)\,|x(t)|^2 + \beta(t, x(t))\,|x(t)|^2\\ &\leq 0, \qquad t \geq t_0.\end{aligned}$$

Thus by Theorem 2, $x(t)\equiv 0$ is a solution of (6) uniformly stable on $[t_0, \infty)$ and hence the solution $\bar{x}(t)$ of (5) is also; q.e.d.

We note finally that if $\bar{x}(t)$ is a.p. and F is a.p. in t uniformly for (x, ϕ) in $R\times CB$ on suitable subsets thereof, then $h(t, x, \phi)$ is also. In this case $\alpha(t)$ and $\beta(t, x)$ could clearly be assumed a.p. in t.

Also if F in (5) is continuously differentiable with respect to each component of x, the left side of the inequality of condition (H_3) can be replaced by

$$\frac{\partial F}{\partial x}(t, \bar{x}(t), \bar{x}_t)x\cdot x$$

where $\partial F/\partial x$ denotes the Jacobian matrix of F with respect to x; i.e.,

$$\frac{\partial F}{\partial x} = \left(\frac{\partial F_i}{\partial x_j}\right).$$

In case F is T-periodic in t and $\bar{x}(t)$ is a T-periodic solution of (5), Floquet theory can be applied, and the condition (H_3) then clearly requires that the so called characteristic multipliers corresponding to the linear system

$$x' = \frac{\partial F}{\partial x}(t, \bar{x}(t), \bar{x}_t)x \tag{7}$$

must be less than one in absolute value. For the non-periodic a.p. case, we do not have a general Floquet theory, however.

Some final comments concerning Theorem 2 are in order. First, as was previously remarked, it is related to Theorem 5 in [6]. While this result in [6] assumes growth and derivative conditions on $V(t, x)$ less restrictive than ours, it assumes V to be locally Lipschitz in x and yields only stability and not uniform stability of the trivial solution.

Finally, if the Euclidean norm and associated inner product is used in R^n, Theorem 1 follows from Theorem 2; we indicate some of the details of the proof. Let $\bar{x}(t)$ be the solution of (1) on $[t_0, \infty)$ satisfying the conditions of Theorem 1. Define $g(t, \phi)=f(t, \bar{x}_t+\phi)-f(t, \bar{x}_t)$. Let $x(t)$ be another solution of (1) such that $|x(t)-\bar{x}(t)|$ is suitably bounded on R. If $z(t)=x(t)-\bar{x}(t)$, then $z(t)$ solves

$$z'(t)=g(t, z_t). \tag{8}$$

Using (H_2), we have

$$|z(t)+h(f(t, \bar{x}_t+z_t)-f(t, \bar{x}_t))|\leq(1-ph)\,\|z_t\|;$$

i.e.

$$(|z(t)|^2+2hg(t, z_t)\cdot z(t)+h^2|g(t, z_t)|^2)^{1/2}\leq(1-ph)\sup\{|z(s)|: s\leq t\}. \tag{9}$$

Choose $V(t, z)=|z|^2/2$; let $z(t)$ solve (8) on $[t_0, \infty)$ and satisfy $V(t, z(t))\geq V(s, z(s))$ for $t\geq s$, $t\geq t_0$; i.e., $|z(t)|\geq|z(s)|$ for such t and s. Then using (9) it follows easily that

$$g(t, z_t)\cdot z(t)+h|g(t, z_t)|^2\leq -p|z(t)|^2/2;$$

i.e.

$$g(t, z_t)\cdot z(t)\leq 0 \qquad \text{for } t\geq t_0.$$

But this is just $\dot{V}(t, z(t))\leq 0$, and by Theorem 2 we conclude that the trivial solution $z\equiv 0$ of (8) is uniformly stable; i.e., that $\bar{x}(t)$ is a uniformly stable solution of (1) on $[t_0, \infty)$.

The author thanks the referee for some very helpful comments and suggestions.

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nuna adreso:
Department of Mathematics
Iowa State University
Ames, Iowa 50011
U. S. A.

(Ricevita la 5-an de marto, 1982)
(Reviziita la 29-an de marto, 1982)
(Reviziita la 1-an de aprilo, 1982)
(Reviziita la 19-an de julio, 1982)