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A Matrix Analogue of Atkinson's Oscillation Theorem

By

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Consider the nonlinear matrix differential equation

$$X''+X^{*k}Q(t)X^{k+1}=0, \tag{1}$$

where k is a positive integer, $Q(t)$ is a real valued $n\times n$ matrix function which is continuous, symmetric and positive definite on $[t_0, \infty)$, $t_0>0$, and X^* denotes the transpose of X.

By a solution of equation (1) we mean a real valued $n\times n$ matrix function $X(t)$ which exists on some ray $[T_x, \infty)$, satisfies (1) for all sufficiently large t and is not identically singular in any neighborhood of infinity. Any solution $X(t)$ of (1) satisfies the relation

$$X^*(t)X'(t)-X^{*\prime}(t)X(t)=C, \qquad t\in[T_x, \infty), \tag{2}$$

where C is a constant matrix. A solution $X(t)$ of (1) is called *prepared* if $C=0$ and *oscillatory* if the determinant of $X(t)$, det $X(t)$, has arbitrarily large zeros.

An oscillation criterion for (1) was first given by Tomastik [3] who showed that all prepared solutions of (1) are oscillatory if

$$\int_{t_0}^{t} Q(\tau)d\tau \qquad \text{is unbounded as } t\to\infty. \tag{3}$$

In view of the well-known oscillation theorem of Atkinson [1] for the scalar equation $x''+q(t)x^{2k+1}=0$, it is natural to ask what will happen for (1) if condition (3) is replaced by the following weaker one:

$$\int_{t_0}^{t} \tau Q(\tau)d\tau \qquad \text{is unbounded as } t\to\infty. \tag{4}$$

Below we show that condition (4) ensures that all *symmetric* prepared solutions of (1) are oscillatory and that, in case $Q(t)$ commutes $Q(s)$ for any values of $t, s\in[t_0, \infty)$, (4) is a necessary and sufficient condition for the oscillation of all symmetric prepared solutions of (1). Thus we are able to obtain a matrix analogue of Atkinson's oscillation theorem [1, Theorem 1].

Theorem 1. *If* (4) *holds, then all symmetric prepared solutions of* (1) *are oscillatory.*

Proof. Suppose to the contrary that there exists a symmetric prepared solution $X(t)$ of (1) such that $\det X(t)\neq 0$ on some interval $[t_1, \infty)$, $t_1\geqq t_0$. Then according to a lemma of [3], $\det X'(t)\neq 0$ on some interval $[t_2, \infty)$, $t_2\geqq t_1$, and the symmetric matrix $S(t)=X'(t)X^{-1}(t)$ satisfies

$$(5)\qquad S^{-1}(t)=S^{-1}(t_2)+\int_{t_2}^{t} S^{-1}(\tau)X^k(\tau)Q(\tau)X^k(\tau)S^{-1}(\tau)d\tau+(t-t_2)I$$

for $t\geqq t_2$, where I is the identity matrix. From (5) we see that $\lim_{t\to\infty}\lambda(S^{-1}(t))=\infty$, where $\lambda(S^{-1}(t))$ denotes the smallest eigenvalue of $S^{-1}(t)$. Therefore, $S(t)$ is positive definite on some interval $[T, \infty)$, $T\geqq t_2$. Using (1) and the symmetry of $X(t)$, we have

$$(tX^{-k}(t)X'(t)X^{-k-1}(t))'=-\frac{1}{2k}(X^{-2k}(t))'-(2k+1)tX^{-k}(t)S^2(t)X^{-k}(t)-tQ(t).$$

Integrating the above over $[T, t]$, $t\geqq T$, gives

$$\begin{aligned} &TX^{-k}(T)S(T)X^{-k}(T)+\frac{1}{2k}X^{-2k}(T)\\ &\quad=tX^{-k}(t)S(t)X^{-k}(t)+\frac{1}{2k}X^{-2k}(t)+(2k+1)\int_T^t \tau X^{-k}(\tau)S^2(\tau)X^{-k}(\tau)d\tau\\ &\qquad+\int_T^t \tau Q(\tau)d\tau,\end{aligned}$$

from which it follows that

$$(6)\qquad \Lambda\Big(TX^{-k}(T)S(T)X^{-k}(T)+\frac{1}{2k}X^{-2k}(T)\Big)\geqq\Lambda\Big(\int_T^t \tau Q(\tau)d\tau\Big),$$

where $\Lambda(M)$ denotes the largest eigenvalue of M. This implies that the right-hand side of (6) remains bounded as $t\to\infty$, which contradicts (4).

Theorem 2. *Suppose that*

$$(7)\qquad Q(t_1)Q(t_2)=Q(t_2)Q(t_1)\qquad \textit{for all } t_1, t_2\in[t_0, \infty).$$

Then equation (1) *has a prepared solution which is symmetric and nonsingular for all large t if the limit*

$$(8)\qquad \lim_{t\to\infty}\int_{t_0}^{t}\tau Q(\tau)d\tau$$

exists and is finite.

Proof. Choose $T>t_0$ so large that

$$(9) \qquad (1+n^{1/2})^{2k+1}\int_T^\infty (\tau-T)\,\|Q(\tau)\|\,d\tau\leqq 1,$$

where $\|Q(t)\|=(\sum_{i,j=1}^n q_{ij}^2(t))^{1/2}$, $Q(t)=(q_{ij}(t))_{i,j=1}^n$. This is possible because of (8). Let $\mathscr{C}$ denote the locally convex space of $n\times n$ matrix continuous functions on $[T,\infty)$ with the topology of uniform convergence on compact subintervals. Let $\mathscr{X}$ denote the set of functions $X\in\mathscr{C}$ such that

$$(10) \qquad \|X(t)-I\|\leqq 1,\quad X^*(t)=X(t),\quad Q(t_1)X(t_2)=X(t_2)Q(t_1)$$

for all $t, t_1, t_2\in[T,\infty)$. Then $\mathscr{X}$ is a nonempty convex subset of $\mathscr{C}$. For $X\in\mathscr{X}$, define the map F by

$$(11) \qquad (FX)(t)=I-\int_t^\infty (\tau-t)X^k(\tau)Q(\tau)X^{k+1}(\tau)d\tau,\qquad t\geqq T.$$

Using (7), (9) and (10), we see that F maps $\mathscr{X}$ into itself. The continuity of F is obvious. It is easy to verify that

$$\|(FX)(t)\|\leqq 1+n^{1/2}$$

and

$$\|(FX)(t_1)-(FX)(t_2)\|\leqq|t_1-t_2|\,(1+n^{1/2})^{2k+1}\int_T^\infty \|Q(\tau)\|\,d\tau$$

for all $X\in\mathscr{X}$ and for all $t, t_1, t_2\in[T,\infty)$. Therefore, $F(\mathscr{X})$ is precompact by the Ascoli-Arzela theorem. From the Schauder-Tychonoff fixed point theorem it follows that the map F has a fixed point $X\in\mathscr{X}$. In view of (11) this fixed point $X=X(t)$ is a symmetric nonsingular solution of equation (1) on some interval $[T_1,\infty)$, $T_1\geqq T$. That $X(t)$ is prepared follows from the fact that $\lim_{t\to\infty}X(t)=I$ and $\lim_{t\to\infty}X'(t)=0$.

Remark. From a result of Kartsatos and Walters [2, Theorem 1] it follows that condition (8) guarantees the existence of a nonsingular solution of (1). Theorem 2 asserts that under the additional condition (7) equation (1) possesses a *symmetric* nonsingular solution.

Combining Theorem 1 with Theorem 2 we have the following result which extends the oscillation theorem of Atkinson to the matrix differential equation (1).

Theorem 3. *Suppose that* (7) *holds. Then condition* (4) *is necessary and sufficient for all symmetric prepared solutions of* (1) *to be oscillatory.*

An example of matrices satisfying the conditions of Theorem 3 is

$$Q(t)=\begin{pmatrix} q_1(t) & q_2(t) \\ q_2(t) & q_1(t) \end{pmatrix} \qquad n=2$$

where $q_1(t)$ and $q_2(t)$ are continuous functions on $[t_0, \infty)$ such that

$$q_1(t)>0, \quad q_1(t)>|q_2(t)| \quad \text{and} \quad \int_{t_0}^{\infty} tq_1(t)dt=\infty.$$

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