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On the Three Circles Theorem for Solutions of Elliptic Equations with Two Variables

By

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1. Let Ω be a bounded domain in R^2 with coordinates (x, y). Let κ and μ be real valued functions in $C^1(\bar{\Omega})$. We shall say that the pair $\{\kappa, \mu\}$ has property (C), if it satisfies in $\bar{\Omega}$

$$\text{(C)}\qquad \tfrac{1}{2}<\kappa^2+\mu^2<2, \qquad 9\mu^2<(2-\kappa^2-\mu^2)(2(\kappa^2+\mu^2)-1).$$

This condition means that $\kappa\neq 0$ in $\bar{\Omega}$. The pair $\{\kappa, 0\}$ always satisfies (C) for κ with $\frac{1}{2}<\kappa^2<2$.

A subset F of Ω is said to satisfy (D_γ) for a real number γ, if it holds

$$(\mathrm{D}_\gamma)\qquad \iint_{\Omega-F}[\mathrm{dis}((x, y), F)]^{-\gamma}dxdy<\infty,$$

where dis $((x, y), F)$ is the distance from the point (x, y) to F.

Hereafter we denote by $|\ \ |_2$ the two-dimensional Lebesgue measure. In the final part of this section we shall give an example of a closed subset F of Ω with $|F|_2 \neq 0$ such that F satisfies (D_γ) for any given γ $(0<\gamma<1)$ and F has no interior point.

We assume that each pair $\{\kappa_j, \mu_j\}$ $(1\leqq j\leqq m)$ satisfies (C). Then we consider the system

$$(1.1)\qquad [\partial_y+(\mu_j+i\kappa_j)\partial_x]u_j=\sum_{l=1}^{m} b_{jl}u_l, \quad j=1, 2, \cdots, m, \quad i=\sqrt{-1}\ ^{1)},$$

where b_{jl} is in $L^\infty(\Omega)$. It is well known that single elliptic equations of simple characteristics are reduced to the form (1.1).

It is easy to see that there is a real number α with $0<\alpha<1$ such that

$$(1.2)\qquad \begin{aligned}&\frac{1}{1+\alpha}<\kappa_j^2+\mu_j^2<1+\alpha\\ &(2+\alpha)^2\mu_j^2<(1+\alpha-\kappa_j^2-\mu_j^2)((1+\alpha)(\kappa_j^2+\mu_j^2)-1), \qquad j=1, 2, \cdots, m.\end{aligned}$$

From now on let α be the same as in (1.2) and let us write simply $\partial^{n_1+n_2}=\partial_x^{n_1}\partial_y^{n_2}$.

Our object is to prove

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1) We write simply $\partial_x=\partial/\partial x$ and $\partial_y=\partial/\partial y$.

Theorem. *Suppose that F is a closed subset of Ω with $|F|_2 \neq 0$ satisfying* ($D_{\alpha'}$) *for some α' with $\alpha < \alpha' < 1$. Then there are an open subset G of Ω and positive constants C, d, k with $d, k < 1$ such that the following holds*:

Let ε and K be any given real numbers satisfying $0 \leqq \varepsilon \leqq dK$ and $K \leqq 1$. Let $\{u_j\}$ be in $C^2(\Omega)$ and solutions of (1.1) *in Ω. If $|\partial^n u_j| \leqq K$ in Ω ($n \leqq 2$) and $|u_j|$, $|\partial u_j| \leqq \varepsilon$ in F, then we have*

$$\left(\iint_G |u_j|^2 dxdy\right)^{1/2} \leqq C\varepsilon^k K^{1-k},$$

where C, d, k and G depend only on $\{\kappa_j, \mu_j\}$, $\{b_{jl}\}$ and F.

The proof is given in the final section.

This theorem corresponds to Hadamard's three circles theorem in the theory of functions with one complex variable, where $F(G)$ is called "the inner (middle) circle", respectively.

When the inner circle is an open set, there are several results for elliptic operators (see e.g. [1], [5], [7]). The case of F with no interior point appears in the book of M. M. Lavrentiev [4], where the three circles theorem is stated with the maximum norm for analytic functions. Recently, this case has been treated for elliptic operators with analytic coefficients by N. S. Nadirashvili [6]. Our method is to prove an L^2-estimate of Carleman's type as in the book of L. Hörmander (Chapter 8, [2]) and to use the idea of F. John [3] with respect to the incorrectly posed problem in R^n, where the Cauchy surface is an open subset of an $(n-1)$-dimensional hypersurface.

Now we give an example of a closed subset F with $|F|_2 \neq 0$ such that F has not any interior point and satisfies (D_γ) for any given fixed γ with $0 < \gamma < 1$.

Example. Let $G = \{(x, y) \mid |x|, |y| < a\}$ $(a < 1)$. We first show

$$\iint_G [\text{dis}((x, y), G^c)]^{-\gamma} dxdy \leqq Ca^{2-\gamma}, \tag{1.3}$$

where C is independent of a. In fact, denoting by G' the triangular $\{(x, y) \mid 0 \leqq x \leqq y \leqq a\}$, we have

$$\begin{aligned}\iint_G &[\text{dis}((x, y), G^c)]^{-\gamma} dxdy \\ &= 8\iint_{G'} (a-y)^{-\gamma} dxdy \\ &= 8(1-\gamma)^{-1}(2-\gamma)^{-1} a^{2-\gamma}\end{aligned}$$

(see Figure 1). Hence (1.3) is valid.

Secondly we take a monotone decreasing sequence $\{a_n\}$ of positive numbers for any given c with $0 < c < 1$ in such a way that

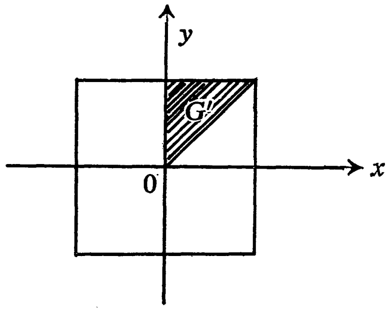

Figure 1

$$a_1 < c, \quad a_n < 2^{1-n}\left(c - \sum_{j=1}^{n-1} 2^{j-1} a_j\right) \quad (n \geqq 2)$$

and

$$\sum_{n=1}^{\infty} 2^{3n} a_n^2 < 8c^2.$$

Denoting by J the closed square $\{(x, y) \mid 0 \leqq x, y \leqq c\}$, we first eliminate from J an open square whose sides have length a_1 and whose center is identical with that of J. That is, the square eliminated from J is the set $\{(x, y) \mid \frac{1}{2}(c - a_1) < x, y < \frac{1}{2}(c + a_1)\}$. The rest consists of eight pieces of rectangles, from which we eliminate respectively

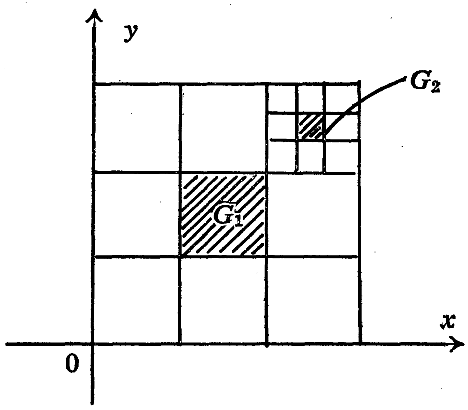

Figure 2

an open new square of its length of side a_2 from each center (see Figure 2). Continuing such operation eternally, we obtain the remainder, which is denoted by F. It is clear that F has not any interior point and

$$|F|_2 = c^2 - \sum_{n=1}^{\infty} 2^{3(n-1)} a_n^2 > 0.$$

Giving a number in turn to each eliminated open square, we write these by G_1, $G_2, \cdots$. It follows

$$\iint_{J-F}[\text{dis}((x, y), F)]^{-\gamma}dxdy$$
$$\leqq\sum_{j=1}^{\infty}\iint_{G_j}[\text{dis}((x, y), G_j^c)]^{-\gamma}dxdy$$
$$\leqq C\sum_{n=1}^{\infty}2^{3(n-1)}a_n^{2-\gamma} \qquad \text{(by (1.3))}.$$

Hence we have

$$\iint_{J-F}[\text{dis}((x, y), F)]^{-\gamma}dxdy<\infty.$$

If Ω is a bounded domain containing J, we see

$$\iint_{\Omega-F}[\text{dis}((x, y), F)]^{-\gamma}dxdy$$
$$\leqq\iint_{\Omega-J}[\text{dis}((x, y), J)]^{-\gamma}dxdy$$
$$+\iint_{J-F}[\text{dis}((x, y), F)]^{-\gamma}dxdy.$$

The first integral on the right-hand side is obviously finite. Thus F satisfies (D_γ).

2. Let $\{\kappa_j, \mu_j\}$ be the same as in the previous section and let us write $r=(x^2+y^2)^{1/2}$. We denote $\{\kappa_j, \mu_j\}$ simply by $\{\kappa, \mu\}$ for any fixed j. We define

$$M=\partial_y^2+2\mu\partial_x\partial_y+(\kappa^2+\mu^2)\partial_x^2.$$

Then we have:

Lemma 1. *There is a positive constant c_0 such that*

$$Mr^{-\alpha}\geqq c_0 r^{-2-\alpha} \qquad \text{in } \bar{\Omega}.$$

Proof. An easy computation shows

$$Mr^{-\alpha}=\alpha r^{-4-\alpha}[(1+\alpha-\kappa^2-\mu^2)y^2+2(2+\alpha)\mu xy+((1+\alpha)(\kappa^2+\mu^2)-1)x^2].$$

In virtue of (1.2) this completes the proof.

From now on let F be such a set as in our theorem. We set

$$\psi_F(x, y)=\iint_F((x-s)^2+(y-t)^2)^{-\alpha/2}dsdt. \tag{2.1}$$

Then we see:

Lemma 2. *The function ψ_F is in $C^1(R^2)$, and for any fixed α' with $\alpha<\alpha'<1$, $[\text{dis}((x, y), F)]^{\alpha'}\partial^2\psi_F(x, y)$ is bounded in R^2-F.*

Proof. The proof is conventional in the potential theory. It is evident that there are functions $f_\delta(r)$ for any $\delta>0$ such that

$$f_\delta(r) \in C^1(R^2_{x,y}), \quad |\partial f_\delta|\leqq 1 \qquad \text{in } R^2,$$
$$f_\delta(r)=r \qquad (r\geqq\delta)$$

and

$$\delta/2\leqq f_\delta(r)\leqq\delta \qquad (r\leqq\delta).$$

We define

$$\psi_F^{(\delta)}(x, y)=\iint_F [f_\delta(((x-s)^2+(y-t)^2)^{1/2})]^{-\alpha}dsdt.$$

Then it is clear that $\psi_F^{(\delta)} \in C^1(R^2)$.

For any fixed point (x, y) we put

$$D_\delta=\{(s, t)\,|\,(x-s)^2+(y-t)^2\leqq\delta^2\}.$$

It follows

$$|\psi_F^{(\delta)}(x, y)-\psi_F(x, y)|\leqq\iint_{F\cap D_\delta} [f_\delta(((x-s)^2+(y-t)^2)^{1/2})]^{-\alpha}dsdt$$
$$+\iint_{F\cap D_\delta} ((x-s)^2+(y-t)^2)^{-\alpha/2}dsdt.$$

Hence the sequence $\{\psi_F^{(\delta)}\}$ converges to ψ_F uniformly in R^2. Putting

$$h(x, y)=\iint_F \partial[((x-s)^2+(y-t)^2)^{-\alpha/2}]dsdt,$$

we see similarly that $\{\partial\psi_F^{(\delta)}\}$ converges uniformly to h by virtue of $|\partial f_\delta|\leqq 1$. Therefore ψ_F is in $C^1(R^2)$.

Secondly it is evident that

$$|\partial^2\psi_F(x, y)|\leqq C\iint_F ((x-s)^2+(y-t)^2)^{-(2+\alpha)/2}dsdt$$

and

$$\text{dis}((x, y), F)\leqq((x-s)^2+(y-t)^2)^{1/2}$$

for any $(s, t) \in F$. Hence we get

$$(\text{dis}((x, y), F))^{\alpha'}|\partial^2\psi_F(x, y)|\leqq C\iint_F ((x-s)^2+(y-t)^2)^{(\alpha'-\alpha-2)/2}dsdt.$$

Since the integral on the right-hand side is finite, the proof is complete.

Further we have:

Lemma 3. *For any given* $\rho>0$, *there are positive numbers* c_1, c_2 *and subsets* F', $\Omega_1, \Omega_2, \Omega'$ *having the following properties*:

(i) $F'\subset\Omega_1\Subset\Omega_2\Subset\Omega'$.

(ii) F' *is a closed subset of* F, *and it satisfies* $(D_{\alpha'})$ *for the same* α' *as in our theorem.*

(iii) Ω_1, Ω_2 *and* Ω' *are subdomains of* Ω.

(iv) *It holds that*

$$\psi_{F'}, |\partial\psi_{F'}|\leqq\rho M\psi_{F'} \qquad \text{in } \Omega'-F',$$

where $\psi_{F'}$ *is the potential* (2.1) *substituted for* F'.

(v) $c_2<c_1$.

(vi) $\psi_{F'}\geqq c_1$ *in* Ω_1 *and* $\psi_{F'}\leqq c_2$ *in* $\Omega'-\Omega_2$.

Proof. We may assume that the origin is the Lebesgue density point of F. If we write $r_{st}=((x-s)+(y-t)^2)^{1/2}$ for any fixed point (s, t), it follows that $Mr_{st}^{-\alpha}\geqq c_0r_{st}^{-2-\alpha}$ by virtue of Lemma 1. And we easily see that $r_{st}^{-\alpha}$, $|\partial r_{st}^{-\alpha}|\leqq Cr_{st}^{-\alpha-1}$, when $|s|, |t|\leqq 1$ and $(x, y)\in\Omega$. On the other hand, there are a positive number $\delta\,(<1)$ and a subdomain Ω' containing the origin such that if $|s|, |t|\leqq\delta$ and $(x, y)\in\Omega'$, we have $Cr_{st}^{-1-\alpha}\leqq\rho c_0 r_{st}^{-2-\alpha}$. Hence we have

$$r_{st}^{-\alpha}, \qquad |\partial r_{st}^{-\alpha}|\leqq\rho Mr_{st}^{-\alpha},$$

if $|s|$, $|t|\leqq\delta$ and $(x, y)\in\Omega'$.

Secondly we take subdomains Ω_1, Ω_2 and positive numbers a_1, a_2 in such a way that $(0, 0)\in\Omega_1\Subset\Omega_2\Subset\Omega'$, $a_1<a_2$ and $r\leqq a_1$ in Ω_1, $r\geqq a_2$ in Ω_2^c. Then taking δ smaller if necessary, we see that $r_{st}\leqq b_1$ in Ω_1 and $r_{st}\geqq b_2$ in Ω_2^c for $|s|, |t|\leqq\delta$, where b_1, b_2 are some positive numbers with $b_1<b_2$.

Since the origin is a density point of F, there is a closed square J containing the origin such that J is in $\{|x|, |y|<\delta\}$ and $|F\cap J|_2\neq 0$. Denoting $F'=F\cap J$, we see that F' also satisfies $(D_{\alpha'})$. In fact, it follows that

$$\begin{aligned}&\iint_{\Omega-F'}[\mathrm{dis}\,((x, y), F')]^{-\alpha'}dxdy\\ &\quad\leqq\iint_{\Omega-F}[\mathrm{dis}\,((x, y), F)]^{-\alpha'}dxdy\\ &\qquad+\iint_{F-F'}[\mathrm{dis}\,((x, y), F')]^{-\alpha'}dxdy.\end{aligned}$$

The first integral on the right-hand side is finite by our hypothesis. The second integral there is estimated from the above by

$$\iint_{F'-J} [\text{dis}\,((x, y), J)]^{-\alpha'} dxdy,$$

which is obviously finite.

If we put $c_j = b_j^{-\alpha} |F'|_2\ (j=1, 2)$, we complete the proof, because $\psi_{F'} \geqq b_1^{-\alpha} |F'|_2$ in Ω_1 and $\psi_{F'} \leqq b_2^{-\alpha} |F'|_2$ in $\Omega - \Omega_2$.

3. In this section let F', Ω_1, Ω_2, Ω' and c_1, c_2 be those such as in Lemma 3. The positive ρ will be determined later. Let c_3 be any fixed number such that $c_1 < c_3$ and $\psi_{F'} \leqq c_3$ in Ω'. Lemma 2 holds naturally for $\psi_{F'}$. We define

$$L = \partial_y + (\mu + i\kappa)\partial_x.$$

Then we have:

Lemma 4. *Let v be a function in $C_0^2(\Omega')$*[2] *such that $|\partial^n v| \leqq K (\leqq 1)$ in Ω' for $n \leqq 2$ and $|v|$, $|\partial v| \leqq \varepsilon\ (\leqq K)$ in F'. Then the inequality*

$$\tau \iint_{F'^c} |v|^2 \exp(2\tau\psi_{F'}) dxdy$$
$$\leqq C \left[\varepsilon K e^{3c_3\tau} + \iint_{F'^c} |Lv|^2 \exp(2\tau\psi_{F'}) dxdy \right]$$

is valid for any $\tau > 1$, where C is independent of v and τ.

Proof. Writing $\psi_{F'}$ simply by ψ, we set $w = ve^{\tau\psi}$ and $g = L\psi$. For any fixed η, we denote by $[a_0, b_0]$ the minimum closed interval containing the set $F' \cap \{y=\eta\}$. Let us write $F'_\eta = F' \cap \{y=\eta\}$. The complement of F'_η in R^1 is written as

$$F'^c_\eta = F'^c \cap \{y=\eta\} = (-\infty, a_0) \cup (b_0, \infty) \cup \left(\bigcup_{n=1}^{\infty} (a_n, b_n) \right).$$

By virtue of Lemma 2, $w \in C_0^1(\Omega')$, $g \in C^0(\bar{\Omega})$ and $|\partial g| \leqq C [\text{dis}\,((x, y), F')]^{-\alpha'}$ in $F'^c \cap \bar{\Omega}$. From (ii) of Lemma 3 this implies that $\bar{w} \cdot \partial(gw)$ is in $L^1(F'^c)$, that is, it is in $L_x^1(F'^c_\eta)$ for almost all η. If f is a fixed function in $C^1(\bar{\Omega}')$, we have the following for such η:

$$
\begin{aligned}
&(f\partial_x w, gw)_{F'^c_\eta} + (w, \partial_x(\bar{f}gw))_{F'^c_\eta} \\
&= [(f\bar{g}|w|^2)(a_0, \eta) - (f\bar{g}|w|^2)(-\infty, \eta)] \\
&\quad + [(f\bar{g}|w|^2)(\infty, \eta) - (f\bar{g}|w|^2)(b_0, \eta)] \\
&\quad + \sum_{n=1}^{\infty} [(f\bar{g}|w|^2)(b_n, \eta) - (f\bar{g}|w|^2)(a_n, \eta)],
\end{aligned}
\tag{3.1}
$$

where $(\ ,\ )_{F'^c_\eta}$ means the inner product in $L^2(F'^c_\eta)$.

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[2] By $C_0^k(\Omega')$ we denote the set of all functions in $C^k(\Omega')$ with compact support in Ω'.

On the other hand we have

$$\begin{aligned}(f\bar{g}|w|^2)(b_n, \eta)-(f\bar{g}|w|^2)(a_n, \eta)\\ &=(f\bar{g}w)(b_n, \eta)(\bar{w}(b_n, \eta)-\bar{w}(a_n, \eta))\\ &\quad+(f\bar{g})(b_n, \eta)\bar{w}(a_n, \eta)(w(b_n, \eta)-w(a_n, \eta))\\ &\quad+((f\bar{g})(b_n, \eta)-(f\bar{g})(a_n, \eta))|w(a_n, \eta)|^2.\end{aligned}$$

This implies that

$$\begin{aligned}&|(f\bar{g}|w|^2)(b_n, \eta)-(f\bar{g}|w|^2)(a_n, \eta)|\\ &\leqq|(f\bar{g}w)(b_n, \eta)|\int_{a_n}^{b_n}|\partial_x w(x, \eta)|\,dx\\ &\quad+|(f\bar{g})(b_n, \eta)|\,|w(a_n, \eta)|\int_{a_n}^{b_n}|\partial_x w(x, \eta)|\,dx\\ &\quad+|w(a_n, \eta)|^2\int_{a_n}^{b_n}(|\partial_x f(x, \eta)|\,|g(x, \eta)|+|f(x, \eta)|\,|\partial_x g(x, \eta)|)dx.\end{aligned}$$

We obtain similar inequalities for

$$|(f\bar{g}|w|^2)(a_0, \eta)-(f\bar{g}|w|^2)(-\infty, \eta)| \quad \text{and} \quad |(f\bar{g}|w|^2)(\infty, \eta)-(f\bar{g}|w|^2)(b_0, \eta)|.$$

Since the points (a_n, η) and (b_n, η) belong to F', we see

$$|w(a_n, \eta)|, \quad |w(b_n, \eta)|\leqq\varepsilon e^{c_3\tau}.$$

For the sake of simplicity we put $E=F'^c$. We denote by $\|\ \|_E$ $((\ ,\ )_E)$ the norm (the inner product) in $L^2(E)$, respectively. Integrating the both sides of (3.1), we obtain from the above

$$\begin{aligned}&|(f\partial_x w, gw)_E+(w, \partial_x(\bar{f}gw))_E|\\ &\leqq C\left[\varepsilon e^{c_3\tau}\iint_E|\partial w|\,dxdy+\varepsilon^2 e^{2c_3\tau}\iint_{\Omega'-F'}(1+|\partial g|)\,dxdy\right].\end{aligned}$$

Since F' satisfies $(\mathrm{D}_{\alpha'})$, the last integral is finite. Obviously $|\partial w|\leqq 2\tau Ke^{c_3\tau}$. Accordingly, it follows that

$$|(f\partial_x w, gw)_E+(w, \partial_x(\bar{f}gw))_E|\leqq C\varepsilon\tau Ke^{2c_3\tau}.$$

The same inequality holds also for $|(f\partial_y w, gw)_E+(w, \partial_y(\bar{f}gw))_E|$. Thus we get

$$(3.2) \qquad |(\kappa^{-1}Lw, gw)_E-(w, L^*(\kappa^{-1}gw))_E|\leqq C\tau\varepsilon Ke^{2c_3\tau}.$$

From now on we assume that $\kappa>0$ in $\bar{\Omega}$. If $\kappa<0$, we replace κ by $|\kappa|$. We define the quantity R as follows:

$$(\kappa^{-1}Lw, gw)_E+(gw, \kappa^{-1}Lw)_E$$
$$=(w, \kappa^{-1}(gL^*w+\bar{g}Lw))_E+(w, \kappa^{-1}wL^*g)_E$$
$$+(w, gwL^*\kappa^{-1})_E+2(w, \kappa^{-1}\partial_x(\mu-i\kappa)\cdot gw)_E+R.$$

By (3.2) we see

$$|R|\leqq C\tau\varepsilon Ke^{2c_3\tau}.$$

Let us denote by $Q_1, Q_2, \cdots$ the differential operators of first order. We can write

$$L^*g=-M\psi+Q_1\psi$$

and

$$gL^*w+\bar{g}Lw=-g\bar{L}w+\bar{g}Lw+wQ_2\psi,$$

where $\bar{L}=\partial_y+(\mu-i\kappa)\partial_x$. Hence it follows that

$$-\tau[(\kappa^{-1}Lw, gw)_E+(gw, \kappa^{-1}Lw)_E]$$
$$=\tau(w, \kappa^{-1}wM\psi)_E-\tau(w, \kappa^{-1}(\bar{g}Lw-g\bar{L}w))_E+\tau(w, wQ_3\psi)_E-\tau R.$$

In Lemma 3 we chose previously ρ in such a way that $\kappa|Q_3\psi|\leqq\frac{1}{2}M\psi$ in $\Omega'-F'$.
On the other hand it holds

$$\|\kappa^{-1/2}e^{\tau\psi}Lv\|_E^2=\|\kappa^{-1/2}Lw\|_E^2+\tau^2\|\kappa^{-1/2}gw\|_E^2$$
$$-\tau[(\kappa^{-1}Lw, gw)_E+(gw, \kappa^{-1}Lw)_E],$$

because $e^{\tau\psi}Lv=Lw-\tau gw$. And we see

$$\tau|(w, \kappa^{-1}(\bar{g}Lw-g\bar{L}w))_E|\leqq\tau^2\|\kappa^{-1/2}gw\|_E^2+\tfrac{1}{4}\|\kappa^{-1/2}((\bar{g}/g)Lw-\bar{L}w)\|_E^2$$
$$\leqq\tau^2\|\kappa^{-1/2}gw\|_E^2+\tfrac{1}{2}(\|\kappa^{-1/2}Lw\|_E^2+\|\kappa^{-1/2}\bar{L}w\|_E^2).$$

Combining the above inequalities, we obtain

$$\frac{1}{2}(\|\kappa^{-1/2}Lw\|_E^2-\|\kappa^{-1/2}\bar{L}w\|_E^2)+\frac{\tau}{2}(w, \kappa^{-1}wM\psi)_E$$
$$\leqq C\tau^2\varepsilon Ke^{2c_3\tau}+\|\kappa^{-1/2}e^{\tau\psi}Lv\|_E^2.$$

In virtue of Lemma 1, this implies that

$$\tau\iint_E|v|^2e^{2\tau\psi}dxdy$$
$$\leqq C\left[\varepsilon Ke^{3c_3\tau}+\iint_E|Lv|^2e^{2\tau\psi}dxdy+(\|\kappa^{-1/2}\bar{L}w\|_E^2-\|\kappa^{-1/2}Lw\|_E^2)\right].$$

Hence if we can show

$$|\|\kappa^{-1/2}Lw\|_E^2-\|\kappa^{-1/2}\bar{L}w\|_E^2|\leqq C\varepsilon Ke^{3c_3\tau}, \tag{3.3}$$

our proof is complete.

We see immediately

$$\|\kappa^{-1/2}Lw\|_E^2-\|\kappa^{-1/2}\bar{L}w\|_E^2=2i((\partial_x w, \partial_y w)_E-(\partial_y w, \partial_x w)_E).$$

By virtue of (ii) of Lemma 3, $\partial^2 w\cdot\bar{w}$ is in $L^1(F'^c)$. Hence we can proceed simiiarly as in (3.1), that is

$$\begin{aligned}(\partial_y w, \partial_x w)_{F'^c_\eta}&+(\partial_x\partial_y w, w)_{F'^c_\eta}\\ &=(\partial_y w)(\infty, \eta)\bar{w}(\infty, \eta)-(\partial_y w)(b_0, \eta)\bar{w}(b_0, \eta)\\ &\quad+(\partial_y w)(a_0, \eta)\bar{w}(a_0, \eta)-(\partial_y w)(-\infty, \eta)w(-\infty, \eta)\\ &\quad+\sum_{n=1}^{\infty}((\partial_y w)(b_n, \eta)\bar{w}(b_n, \eta)-(\partial_y w)(a_n, \eta)\bar{w}(a_n, \eta)).\end{aligned}$$

And we see

$$\begin{aligned}(\partial_y w)&(b_n, \eta)\bar{w}(b_n, \eta)-(\partial_y w)(a_n, \eta)\bar{w}(a_n, \eta)\\ &=(\partial_y w)(b_n, \eta)(\bar{w}(b_n, \eta)-\bar{w}(a_n, \eta))+\bar{w}(a_n, \eta)((\partial_y w)(b_n, \eta)-(\partial_y w)(a_n, \eta))\\ &=(\partial_y w)(b_n, \eta)\int_{a_n}^{b_n}(\overline{\partial_x w})(x, \eta)dx+\bar{w}(a_n, \eta)\int_{a_n}^{b_n}(\partial_x\partial_y w)(x, \eta)dx.\end{aligned}$$

Since (a_n, η) and $(b_n, \eta)\in F'$, we have

$$|w(a_n, \eta)|,\quad |w(b_n, \eta)|,\quad |(\partial w)(a_n, \eta)|,\quad |(\partial w)(b_n, \eta)|\leqq C\tau\varepsilon e^{c_3\tau}.$$

Thus we obtain

$$\begin{aligned}|(\partial_y w, \partial_x w)_E&+(\partial_x\partial_y w, w)_E|\\ &\leqq C\tau\varepsilon e^{c_3\tau}\iint_E(|(\partial w)(x, \eta)|+|(\partial^2 w)(x, \eta)|)dxd\eta\\ &\leqq C\tau^3\varepsilon Ke^{2c_3\tau}\iint_{\Omega'-F'}(1+|\partial^2\psi|)dxd\eta\\ &\leqq C\tau^3\varepsilon Ke^{2c_3\tau}.\end{aligned}$$

Similarly we get

$$|(\partial_x w, \partial_y w)_E+(\partial_y\partial_x w, w)_E|\leqq C\tau^3\varepsilon Ke^{2c_3\tau}.$$

Therefore it follows that

$$|(\partial_y w, \partial_x w)_E-(\partial_x w, \partial_y w)_E|\leqq C\tau^3\varepsilon Ke^{2c_3\tau}.$$

This implies (3.3). Thus we complete the proof.

4. Proof of our theorem.

Taking a function $\zeta \in C_0^\infty(\Omega')$ such that $\zeta=1$ in Ω_2. If we set $v_j=\zeta u_j$ and $L_j=\partial_y+(\mu_j+i\kappa_j)\partial_x$, the following inequality holds from Lemma 4:

$$\tau\iint_{F'^c}\left(\sum_{j=1}^m |v_j|^2\right)\exp(2\tau\psi_{F'})dxdy$$
$$\leqq C\left[\varepsilon K e^{3c_3\tau}+\iint_{F'^c}\left(\sum_{j=1}^m |L_j v_j|^2\right)\exp(2\tau\psi_{F'})dxdy\right].$$

On the other hand we see

$$\begin{aligned} L_j v_j &= u_j L_j\zeta+\zeta L_j u_j \\ &= u_j L_j\zeta+\sum_{l=1}^m b_{jl} v_l. \end{aligned}$$

Hence there is a positive τ_0 such that for $\tau\geqq\tau_0$ the inequality is valid

$$\iint_{F'^c}\left(\sum_{j=1}^m |v_j|^2\right)\exp(2\tau\psi_{F'})dxdy$$
$$\leqq C\left[\varepsilon K e^{3c_3\tau}+\iint_{F'^c}\left(\sum_{j=1}^m |u_j L_j\zeta|^2\right)\exp(2\tau\psi_{F'})dxdy\right].$$

Since $L_j\zeta=0$ in Ω_2, we obtain

$$\iint_{\Omega_1-F'}\left(\sum_{j=1}^m |u_j|^2\right)\exp(2\tau\psi_{F'})dxdy$$
$$\leqq C\left[\varepsilon K e^{3c_3\tau}+\iint_{\Omega'-\Omega_2}\left(\sum_{j=1}^m |u_j|^2\right)\exp(2\tau\psi_{F'})dxdy\right].$$

In virtue of Lemma 3 this implies

$$e^{2c_1\tau}\iint_{\Omega_1-F'}\left(\sum_{j=1}^m |u_j|^2\right)dxdy\leqq CK(\varepsilon e^{3c_3\tau}+Ke^{2c_2\tau}).$$

Dividing the both sides by $e^{2c_1\tau}$, we put $h=\frac{1}{2}(3c_3-2c_1)^{-1}$ and $\tau=\log(K/\varepsilon)^h$, where we can assume that $\varepsilon\neq 0$ without loss of generality. Then the following estimate holds:

$$\iint_{\Omega_1-F'}\left(\sum_{j=1}^m |u_j|^2\right)dxdy\leqq C(\varepsilon^{1/2}K^{3/2}+\varepsilon^{2h(c_1-c_2)}K^{2+2h(c_2-c_1)}).$$

On the other hand it is trivial that

$$\iint_{F'}\left(\sum_{j=1}^m |u_j|^2\right)dxdy\leqq C\varepsilon^2$$

from our assumption. Therefore, taking $d=e^{-\tau_0/h}$, $k=\min(1/4, h(c_1-c_2))$ and $G=\Omega_1$, we complete the proof of our theorem.

Remark. Let E be a closed set on $\Omega\cap\{y=0\}$. We assume that the one-dimensional measure of E is positive and we set

$$\varphi_E(x, y)=\int_E ((x-s)^2+y^2)^{-\alpha/2}ds,$$

where α is a number such as in (1.2). Then, in parallel with Lemma 2 we can show that $\varphi_E \in C^0(R^2)$ and $[\mathrm{dis}\,((x, y), E)]^{\alpha'}\partial\varphi_E$, $[\mathrm{dis}\,((x, y), E)]^{1+\alpha'}\partial^2\varphi_E$ are bounded in $\Omega-E$ for any fixed α' with $\alpha<\alpha'<1$.

And there are a subdomain Ω'' of Ω, a closed subset E' of E and a constant C such that the inequality

$$\tau\iint |v|^2 \exp(2\tau\varphi_{E'})\,dxdy \leqq C\iint |Lv|^2 \exp(2\tau\varphi_{E'})dxdy$$

is valid for any $\tau\geqq 1$ and any $v \in C_0^1(\Omega'')$. Its proof is similar to that of Lemma 4. Further the following statement is quite similarly verified:

Let $\{u_j\}$ be in $C^2(\Omega)$ and solutions of (1.1) in Ω. If $u_j=0$ on E, then u_j vanish identically in Ω.

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