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On Delay Differential Inequalities of First Order

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§ 1. Introduction.

The oscillatory and asymptotic behaviour of solutions of delay differential equations has been the subject of many recent investigations. Of particular importance however has been the study of oscillations which are caused by the retarded argument and which do not appear in the corresponding ordinary differential equation. For some contributions in this area the reader is referred to Kamenskii [1], Kusano and Onose [2], Ladas, Lakshmikantham and Papadakis [3], Ladas [4], Naito [5], Skubatsevskii [6] and Staikos and Stavroulakis [7].

In this paper we consider the first order delay differential inequalities

$$y'(t)+a(t)y(t)+p(t)y(t-\tau)\leqq 0, \tag{1}$$

$$y'(t)+a(t)y(t)+p(t)y(t-\tau)\geqq 0 \tag{2}$$

and the delay differential equation

$$y'(t)+a(t)y(t)+p(t)y(t-\tau)=0 \tag{3}$$

where $a(t)\geqq 0$ and continuous, $p(t)>0$ and continuous, $\tau>0$ and constant. We give sufficient conditions under which

(1) has no eventually positive solutions,
(2) has no eventually negative solutions, and
(3) has oscillatory solutions only.

We emphasize the fact that our conditions are the "best possible" in the sense that when $p>0$ and $a\geqq 0$ are constants the conditions reduce to $p\tau e^{a\tau}>1/e$ which is a necessary and sufficient condition. The above results are caused by the retarded argument and are not valid when $\tau=0$. For example

$$y'(t)+y(t)\leqq 0$$

has the positive solution $y(t)=e^{-2t}$ while

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$$y'(t)+y(t)\geqq 0$$

has the negative solution $y(t)=-e^{-2t}$.

Finally we apply the above mentioned results to obtain conditions for the existence of bounded oscillatory solutions of second order delay differential equations.

As it is customary, a solution is said to be oscillatory if it has arbitrarily large zeros.

§ 2. Main results.

Theorem 1. *Consider the delay differential inequality*

$$y'(t)+a(t)y(t)+p(t)y(t-\tau)\leqq 0 \tag{1}$$

where τ is a positive constant and $a(t)\geqq 0$, $p(t)>0$ are continuous functions for $t\in \boldsymbol{R}^+$. Assume that

$$\liminf_{t\to\infty}\int_{t-\tau}^{t} p(s)ds>\frac{1}{e}\exp\left(-\liminf_{t\to\infty}\int_{t-\tau}^{t} a(s)ds\right) \tag{4}$$

and

$$\liminf_{t\to\infty}\int_{t-\tau/2}^{t} p(s)ds>0. \tag{5}$$

Then (1) *has no eventually positive solutions.*

Proof. We will prove that the existence of an eventually positive solution leads to a contradiction. To this end suppose that $y(t)$ is a solution of (1) such that for t_0 sufficiently large

$$y(t)>0, \qquad t>t_0.$$

Then $y(t-\tau)>0$ for $t>t_0+\tau$ and, from (1), $y'(t)<0$ for $t>t_0+\tau$. Hence $y(t)<y(t-\tau)$ for $t>t_0+2\tau$. Set

$$w(t)=\frac{y(t-\tau)}{y(t)} \qquad \text{for } t>t_0+2\tau. \tag{6}$$

Then $w(t)>1$ and dividing both sides of (1) by $y(t)$, for $t>t_0+2\tau$, we obtain

$$\frac{y'(t)}{y(t)}+a(t)+p(t)w(t)\leqq 0, \qquad t>t_0+2\tau. \tag{7}$$

Integrating both sides of (7) from $t-\tau$ to t, for $t>t_0+3\tau$, we have

$$\log y(t)-\log y(t-\tau)+\int_{t-\tau}^{t} a(s)ds+\int_{t-\tau}^{t} p(s)w(s)ds\leqq 0, \qquad t>t_0+3\tau$$

which, in view of (6), yields

$$\log w(t) \geqq \int_{t-\tau}^{t} a(s)ds + \int_{t-\tau}^{t} p(s)w(s)ds, \qquad t > t_0 + 3\tau. \tag{8}$$

Now, integrating (1) from $t-\tau/2$ to t and using the fact that $y(t)$ is decreasing, we find

$$y(t) - y\left(t-\frac{\tau}{2}\right) + y(t)\int_{t-\tau/2}^{t} a(s)ds + y(t-\tau)\int_{t-\tau/2}^{t} p(s)ds \leqq 0, \qquad t > t_0 + \frac{\tau}{2}.$$

Dividing the last inequality first by $y(t)$ and then by $y(t-\tau/2)$ we obtain respectively:

$$1 - \frac{y(t-\tau/2)}{y(t)} + \int_{t-\tau/2}^{t} a(s)ds + \frac{y(t-\tau)}{y(t)}\int_{t-\tau/2}^{t} p(s)ds \leqq 0 \tag{9}$$

and

$$\frac{y(t)}{y(t-\tau/2)} - 1 + \frac{y(t)}{y(t-\tau/2)}\int_{t-\tau/2}^{t} a(s)ds + \frac{y(t-\tau)}{y(t-\tau/2)}\int_{t-\tau/2}^{t} p(s)ds \leqq 0. \tag{10}$$

Let

$$\liminf_{t\to\infty} w(t) = l.$$

Then $l \geqq 1$ and is finite or infinite. We consider the following two possible cases:

Case 1. l is finite. Taking limit inferiors on both sides of (8), we obtain

$$\log l \geqq \liminf_{t\to\infty} \int_{t-\tau}^{t} a(s)ds + l - \varepsilon \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds$$

where ε sufficiently small, and so

$$\log l - l \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds \geqq \liminf_{t\to\infty} \int_{t-\tau}^{t} a(s)ds.$$

Using the fact that

$$\max_{l \geqq 1} \left\{\log l - l \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds\right\} = -\log\left(\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds\right) - 1$$

the last inequality implies

$$\log\left(\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds\right) \leqq -1 - \liminf_{t\to\infty} \int_{t-\tau}^{t} a(s)ds$$

or

$$\liminf_{t\to\infty}\int_{t-\tau}^{t} p(s)ds \leqq \frac{1}{e}\exp\left(-\liminf_{t\to\infty}\int_{t-\tau}^{t} a(s)ds\right)$$

which contradicts hypothesis (4).

Case 2. l is infinite. That is,

$$\lim_{t\to\infty}\frac{y(t-\tau)}{y(t)}=+\infty.$$

In view of (5) and the fact that $a(t)\geqq 0$, inequality (9) implies

$$\lim_{t\to\infty}\frac{y(t-\tau/2)}{y(t)}=+\infty$$

and therefore

$$\lim_{t\to\infty}\frac{y(t-\tau)}{y(t-\tau/2)}=+\infty$$

which contradicts (10).

Since in both cases we are led to a contradiction the proof of the theorem is complete.

Theorem 2. *Consider the delay differential inequality*

$$y'(t)+a(t)y(t)+p(t)y(t-\tau)\geqq 0 \tag{2}$$

subject to the hypotheses of Theorem 1. *Then* (2) *has no eventually negative solutions.*

Proof. The result follows immediately from the observation that if $y(t)$ is a solution of (2) then $-y(t)$ is a solution of (1).

From Theorems 1 and 2 if follows that the delay differential equation (3) has no eventually positive or eventually negative solutions and therefore we are led to the following conclusion.

Corollary 1. *Consider the delay differential equation*

$$y'(t)+a(t)y(t)+p(t)y(t-\tau)=0 \tag{3}$$

subject to the hypotheses of Theorem 1. *Then every solution of* (3) *oscillates.*

Note that when $a(t)$ is a constant $a\geqq 0$ and $p(t)$ is a positive constant p that is, in the case of the delay differential inequalities

$$y'(t)+ay(t)+py(t-\tau)\leqq 0, \tag{1$'$}$$

(2)′ $$y'(t)+ay(t)+py(t-\tau)\geqq 0,$$

and the delay differential equation

(3)′ $$y'(t)+ay(t)+py(t-\tau)=0$$

the conditions (4) and (5) reduce to

(11) $$p\tau>\frac{1}{e}e^{-a\tau}, \qquad a\geqq 0.$$

Furthermore, (11) is a sharp condition. More precisely, the following result is true.

Theorem 3. *Assume that*

(12) $$p\tau\leqq\frac{1}{e}e^{-a\tau}, \qquad a\geqq 0.$$

Then

(1)′ *has eventually positive solutions,*
(2)′ *has eventually negative solutions,*

and

(3)′ *has nonoscillatory solutions.*

Proof. First we prove that (1)′ has eventually positive solutions. Looking for a solution of (1)′ of the form $y(t)=e^{\lambda t}$ it follows that

$$F(\lambda)\equiv\lambda+a+pe^{-\lambda\tau}\leqq 0.$$

But in view of (12)

$$F\left(-\frac{1}{\tau}-a\right)=-\frac{1}{\tau}-a+a+pe^{(1/\tau+a)\tau}\leqq\frac{1}{\tau}-\frac{1}{\tau}=0.$$

Hence there exists a λ, namely, $\lambda=-1/\tau-a$ such that $e^{\lambda t}$ is a positive solution of (1)′.

Following a similar procedure it follows that $-e^{\lambda t}$ with $\lambda=-1/\tau-a$ is a negative solution of (2)′.

Finally since $F(0)>0$ and $F(-1/\tau-a)\leqq 0$, there exists a λ in the interval $[-1/\tau-a, 0)$ such that $e^{\lambda t}$ is a nonoscillatory solution of (3)′.

The proof is complete.

In view of Theorems 1, 2 and 3 and Corollary 1 we conclude that the following holds:

Corollary 2. *The condition*

(11) $$p\tau>\frac{1}{e}e^{-a\tau}, \qquad a\geqq 0$$

is necessary and sufficient so that:

(1)′ *has no eventually positive solutions,*

(2)′ *has no eventually negative solutions,*

and

(3)′ *has oscillatory solutions only.*

Example 1. The delay differential equation

$$y'(t)+y\left(t-\frac{\pi}{2}\right)=0$$

has the oscillatory solutions $y_1(t)=\sin t$ and $y_2(t)=\cos t$. Furthermore, condition (11) is satisfied and therefore all solutions of this equation are oscillatory.

The delay differential equation

$$y'(t)+\frac{1}{\tau e}y(t-\tau)=0, \qquad \tau>0$$

has the nonoscillatory solution $y(t)=e^{-(1/\tau)t}$. As expected condition (11) is not satisfied for this equation.

Example 2. The delay differential inequality

$$y'(t)+e^{-t}y(t)+e^{-2t}y(t-\tau)\leqq 0$$

has for sufficiently large t the positive solution $y(t)=e^{-2t}$ and the delay differential inequality

$$y'(t)+e^{-t}y(t)+e^{-2t}y(t-\tau)\geqq 0$$

has for sufficiently large t the negative solution $y(t)=-e^{-2t}$. As expected condition (4) or (5) (or both) of Theorems 1 and 2 should be violated. In this case both are.

§ 3. Applications.

In this section we apply the above results on differential inequalities to establish sufficient conditions under which all bounded solutions of the second-order delay differential equations

$$y''(t)-a(t)y(t)-[p^2+q(t)]y(t-2\tau)=0 \tag{13}$$

and

$$y''(t)-a(t)y(t)-p^2(t)y(t-2\tau)=0 \tag{14}$$

are oscillatory.

Theorem 4. *Assume that* $a(t)\geqq 0$, $q(t)\geqq 0$ *are continuous functions for* $t\in \boldsymbol{R}^+$, p, τ *are positive constants and*

$$p\tau e>1. \tag{15}$$

Then all bounded solutions of the delay differential equation

$$y''(t)-a(t)y(t)-[p^2+q(t)]y(t-2\tau)=0 \tag{13}$$

are oscillatory.

Proof. Otherwise there exists a bounded solution $y(t)$ of (13) such that for t_0 sufficiently large

$$y(t)>0, \qquad t>t_0.$$

Then $y(t-2\tau)>0$ for $t>t_0+2\tau$ and, from (13), $y''(t)>0$ for $t>t_0+2\tau$. Since $y(t)$ is bounded, it follows that $y'(t)<0$.

Set

$$x(t)=y'(t)-py(t-\tau) \tag{16}$$

which, for sufficiently large t, is negative. Differentiating both sides of (16), we obtain

$$x'(t)=y''(t)-py'(t-\tau).$$

Now we have

$$\begin{aligned} x'(t)+px(t-\tau)&=y''(t)-py'(t-\tau)+py'(t-\tau)-p^2y(t-2\tau)\\ &=a(t)y(t)+q(t)y(t-2\tau)\geqq 0. \end{aligned}$$

That is

$$x'(t)+px(t-\tau)\geqq 0. \tag{17}$$

But $p\tau e>1$ which implies that every solution $x(t)$ of (17) is eventually positive. This contradicts (16) and the proof of Theorem 4 is complete.

Theorem 5. *Consider the delay differential equation*

$$y''(t)-a(t)y(t)-p^2(t)y(t-2\tau)=0 \tag{14}$$

where $a(t)\geqq 0$, $p(t)>0$ *are continuous functions for* $t\in \boldsymbol{R}^+$ *and* τ *is a positive constant. Assume that the conditions* (4) *and* (5) *of Theorem* 1 *are satisfied and furthermore*

$$p(t)[p(t)-p(t-\tau)]\geqq p'(t)\geqq 0. \tag{18}$$

Then every bounded solution of (14) *is oscillatory.*

Proof. As in the proof of Theorem 4, we assume that there exists a bounded solution $y(t)$ of (14) such that for t_0 sufficiently large

$$y(t)>0, \qquad t>t_0$$

and therefore $y'(t)<0$ for sufficiently large t.

Set

$$x(t)=y'(t)-p(t)y(t-\tau) \tag{19}$$

which for sufficiently large t is negative. Taking derivatives on both sides of (19), we obtain

$$x'(t)=y''(t)-p'(t)y(t-\tau)-p(t)y'(t-\tau).$$

We have

$$\begin{aligned} x'(t)+p(t)x(t-\tau) &= y''(t)-p'(t)y(t-\tau)-p(t)y'(t-\tau)+p(t)y'(t-\tau) \\ &\quad -p(t)p(t-\tau)y(t-2\tau) \\ &= a(t)y(t)+p^2(t)y(t-2\tau)-p'(t)y(t-\tau)-p(t)p(t-\tau)y(t-2\tau) \\ &= a(t)y(t)+p(t)[p(t)-p(t-\tau)]y(t-2\tau)-p'(t)y(t-\tau) \\ &\geqq p(t)[p(t)-p(t-\tau)]y(t-2\tau)-p'(t)y(t-\tau). \end{aligned}$$

Since $y(t)>0$ and $y'(t)<0$ for sufficiently large t we have

$$0<y(t-\tau)<y(t-2\tau)$$

and, in view of (18)

$$p(t)[p(t)-p(t-\tau)]y(t-2\tau)\geqq p'(t)y(t-\tau).$$

Therefore

$$x'(t)+p(t)x(t-\tau)\geqq 0$$

which, by virtue of (4) and (5), has eventually positive solutions. This contradicts (19) and the proof of Theorem 5 is complete.

Note. In the case where $a(t)=0$ and $p(t)$ is a positive constant p all the hypotheses of Theorem 5 are reduced to $p\tau e>1$.

Example 3. The second order delay differential equation

$$y''(t)-y(t-\pi)=0$$

has the bounded oscillatory solutions $y_1(t)=\sin t$ and $y_2(t)=\cos t$. Furthermore, condition (15) is satisfied and therefore all bounded solutions of this equation are oscillatory.

On the other hand the equation

$$y''(t)-\frac{9}{e^2}y\left(t-\frac{2}{3}\right)=0$$

admits the bounded nonoscillatory solution $y(t)=e^{-3t}$. As expected condition (15) is not satisfied for this equation.

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