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On the Fundamental System of Solutions of $y^{(n)}+py=0$

By

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§ 1. Introduction.

The differential equation to be considered is of the form

$$y^{(n)}+py=0, \tag{E}$$

where p is continuous and of constant sign on an interval $[a, \infty)$. A nontrivial solution of (E) is said to be *oscillatory* if it has an infinite number of zeros on $[a, \infty)$. Unless the contrary is stated, the term "solution" will be used as an abbreviation for "nontrivial solution." A solution of (E) which is not oscillatory is called *nonoscillatory*. Equation (E) is said to be *oscillatory* if it has at least one oscillatory solution; otherwise, it is said to be *nonoscillatory*.

Various aspects of (E) have been investigated by many authors [1–3, 6, 7, 9–11, 14–20]. As a result of these investigations, it is known that the asymptotic and growth properties of solutions of (E) depend on the parity of n and the sign of p [9, 15, 18, 19]:

(i)	n even,	$p>0$,
(ii)	n odd,	$p>0$,
(iii)	n even,	$p<0$,
(iv)	n odd,	$p<0$.

Equation (E) satisfying condition (i), for example, is denoted by (E_i); (E_{ii}), (E_{iii}), and (E_{iv}) are similarly defined.

The set N of all nonoscillatory solutions of (E) may be partitioned with the help of the following lemma.

Lemma 1 [9]. *Suppose that y is a nonoscillatory solution of* (E) *such that $y\geq 0$ on $[b, \infty)$ for some $b\geq a$ and that $p\not\equiv 0$ on $[b_1, \infty)$ for every $b_1\geq a$. Let $[C]$ be the greatest integer less than or equal to C.*

If y is a solution of (E_i) *or* (E_{iv}), *there exists an integer j, $0\leq j\leq[(n-1)/2]$, such that*

$$y^{(i)}>0, \qquad i=0, 1, \cdots, 2j, \tag{1}$$

on $[b_2, \infty)$ *for some* $b_2 \geq b$,

(2) $$(-1)^{i+1}y^{(i)}>0, \qquad i=2j+1, \cdots, n-1,$$

on $[b, \infty)$, *and* $y^{(i)}(x) \to 0$ *as* $x \to \infty$, $i=2j+2, \cdots, n-1$.

If y *is a solution of* (E_{ii}) *or* (E_{iii}), *there exists an integer* j, $0 \leq j \leq [n/2]$, *such that*

(3) $$y^{(i)}>0, \qquad i=0, 1, \cdots, 2j-1,$$

on $[b_2, \infty)$ *for some* $b_2 \geq b$,

(4) $$(-1)^{i}y^{(i)}>0, \qquad i=2j, \cdots, n-1,$$

on $[b, \infty)$ *and* $y^{(i)}(x) \to 0$ *as* $x \to \infty$, $i=2j+1, \cdots, n-1$.

For (E_i) and (E_{iv}), we define

$$A_j=\{y \mid y \text{ or } -y \text{ satisfies (1) and (2)}\}, \qquad 0 \leq j \leq [(n-1)/2].$$

Similarly, define for (E_{ii}) and (E_{iii}),

$$A_j=\{y \mid y \text{ or } -y \text{ satisfies (3) and (4)}\}, \qquad 0 \leq j \leq [n/2].$$

Lemma 1 may now be restated in terms of these classes A_j:

$$A_j \cap A_k = \emptyset, \qquad j \neq k,$$

and the set N has the representation

$$N=\begin{cases} A_0 \cup A_1 \cup \cdots \cup A_{(n-2)/2} & \text{for } (E_i), \\ A_0 \cup A_1 \cup \cdots \cup A_{(n-1)/2} & \text{for } (E_{ii}) \text{ and } (E_{iv}), \\ A_0 \cup A_1 \cup \cdots \cup A_{n/2} & \text{for } (E_{iii}). \end{cases}$$

The author studied the problem of determining the number of solutions in class A_j and proved, for example, that for (E_i) the maximum number of solutions belonging to A_j, of which every nontrivial linear combination again belongs to A_j, is 0 or 2 [9, 10].

Studying the special equation

(5) $$y^{iv}+py=0, \qquad p<0,$$

Hastings and Lazer [6] obtained results on the existence of bounded oscillatory solutions, while Ahmad [1] proved that the equation has three linearly independent oscillatory solutions if it is oscillatory. On the other hand, Leighton and Nehari [14] showed that every solution of

$$y^{iv}+py=0, \qquad p>0,$$

is oscillatory if the equation is oscillatory. Johnson [7], Read [20], Lovelady [17], and Etgen and Taylor [3] investigated Equation (E_{iii}). In particular, Lovelady [17] proved that (E_{iii}) has $n-1$ linearly independent oscillatory solutions if $A_1 \cup A_2 \cup \cdots \cup A_{(n-2)/2} = \emptyset$, i.e., every nonoscillatory solution belongs to $A_0 \cup A_{n/2}$.

We shall study the structure of the solution space of (E) in a series of lemmas and theorems, and generalize some of the above results. For example, it will be shown in Theorem 5 that if (E_{iii}) with $n \geq 4$ is oscillatory, it has at least $(n+2)/2$ or $(n+4)/2$ linearly independent oscillatory solutions according as $n/2$ is even or odd. Also motivated by the notion of principal solutions [4, 13], we introduce the concepts of *small* and *large solutions* and prove their existence in Theorems 1 and 2. We make use of asymptotic properties of nonoscillatory solutions such as proved in [9, 10] and others to follow, to investigate the behavior of oscillatory solutions.

§2. Preliminary results.

In this section we collect and prove results which will be used repeatedly in the discussion of our main theorems.

Remark 1 [10]. If $u \in A_i$ and $v \in A_{i+k}$, $k \geq 1$, then $v + Cu \in A_{i+k}$ for any constant C.

Lemma 2. *Suppose that $u \in A_i$, $v \in A_{i+k}$, $k \geq 1$, and u and v are eventually positive. Then*

$$\lim_{x\to\infty} \frac{v^{(r)}(x)}{u^{(r)}(x)} = \begin{cases} \infty, & 0 \leq r \leq 2i, \\ (-1)^{r+1}\infty, & 2i+1 \leq r \leq 2i+2k, \\ \infty, & 2i+2k+1 \leq r \leq n, \end{cases}$$

for (E_i) *and* (E_{iv}); *and*

$$\lim_{x\to\infty} \frac{v^{(r)}(x)}{u^{(r)}(x)} = \begin{cases} \infty, & 0 \leq r \leq 2i-1, \\ (-1)^{r}\infty, & 2i \leq r \leq 2i+2k-1, \\ \infty, & 2i+2k \leq r \leq n, \end{cases}$$

for (E_{ii}) *and* (E_{iii}).

Proof. Consider (E_{ii}) and (E_{iii}). Assume that $u^{(r)} > 0$, $r = 0, 1, \cdots, 2i-1$, and $v^{(r)} > 0$, $r = 0, 1, \cdots, 2i+2k-1$, on $[b_2, \infty)$ for some $b_2 \geq a$ (Cf. Lemma 1). Since $v^{(2i+2k-2)}(x) \to \infty$ and $u^{(2i+2k-2)}(x)$ remains bounded as $x \to \infty$,

$$v^{(2i+2k-2)}(x) - Cu^{(2i+2k-2)}(x) \to \infty \qquad \text{as } x \to \infty$$

for any constant C; and this in turn implies

(6) $$v^{(r)}(x)-Cu^{(r)}(x)\to\infty \qquad \text{as } x\to\infty,$$

$r=0, 1, \cdots, 2i+2k-2$, for any constant C. Furthermore, for any fixed constant C, there exists a point α such that

(7) $$v^{(2i+2k-1)}(x)-Cu^{(2i+2k-1)}(x)>0, \qquad x\in[\alpha, \infty),$$

since $v^{(2i+2k-1)}$ is eventually positive and increasing, while $u^{(2i+2k-1)}(x)\to 0$ as $x\to\infty$ by Lemma 1.

It is easily seen that

$$\limsup_{x\to\infty}\frac{v^{(r)}(x)}{u^{(r)}(x)}=\liminf_{x\to\infty}\frac{v^{(r)}(x)}{u^{(r)}(x)},$$

$r=0, 1, \cdots, n$. If this were not the case, there would be a constant C_j such that $v^{(j)}-C_ju^{(j)}$ for some j, $0\le j\le n$, has an infinity of zeros on $[a, \infty)$. But this is contrary to Lemma 1 because $v-C_ju\in A_{i+k}$ by Remark 1. Consequently,

$$\lim_{x\to\infty}\{v^{(r)}(x)/u^{(r)}(x)\}, \qquad r=0, 1, \cdots, n,$$

exist. We claim that these limits cannot be finite. If $\lim_{x\to\infty}\{v^{(j)}(x)/u^{(j)}(x)\}$ is finite for some j, $0\le j\le 2i+2k-1$, then there exist B_1, B_2 and $a_1\ge b_2$ such that

$$B_1<\frac{v^{(j)}(x)}{u^{(j)}(x)}<B_2, \qquad x\in[a_1, \infty).$$

If $u^{(j)}>0$ on $[a_1, \infty)$, then $v^{(j)}-B_2u^{(j)}<0$ on $[a_1, \infty)$, contradicting (6) or (7). Thus,

(8) $$\lim_{x\to\infty}\frac{v^{(r)}(x)}{u^{(r)}(x)}=\infty, \qquad 0\le r\le 2i+2k-1,$$

if $u^{(r)}>0$ on $[b_2, \infty)$. On the other hand, if $u^{(j)}<0$ on $[a_1, \infty)$, then $v^{(j)}-B_1u^{(j)}<0$ on $[a_1, \infty)$ and again contradicts (6) or (7). Therefore,

(9) $$\lim_{x\to\infty}\frac{v^{(r)}(x)}{u^{(r)}(x)}=-\infty, \qquad 0\le r\le 2i+2k-1,$$

if $u^{(r)}<0$ on $[b_2, \infty)$. In view of (8) and (9), and Lemma 1, we have

$$\lim_{x\to\infty}\frac{v^{(r)}(x)}{u^{(r)}(x)}=\begin{cases}\infty, & 0\le r\le 2i-1,\\ (-1)^r\infty, & 2i\le r\le 2i+2k-1.\end{cases}$$

Turning to the case $2i+2k\le r\le n$, suppose that

(10) $$\lim_{x\to\infty}\frac{v^{(j)}(x)}{u^{(j)}(x)}<K,$$

for some j, $2i+2k\leq j\leq n$, and some constant K. Then

$$\frac{v^{(j)}(x)-Ku^{(j)}(x)}{u^{(j)}(x)}<0, \qquad x\in[\beta,\infty) \tag{11}$$

for some $\beta\geq b_2$. Because of (6) we may assume that $y(x)\equiv v(x)-Ku(x)>0$, $x\in[\beta,\infty)$. Since $y\in A_{i+k}$ by Remark 1, $\operatorname{sgn} y^{(j)}=\operatorname{sgn} u^{(j)}$ on $[\beta,\infty)$ by Lemma 1, which is incompatible with (11). Consequently, (10) cannot hold for any j, $2i+2k\leq j\leq n$, and

$$\lim_{x\to\infty}\frac{v^{(r)}(x)}{u^{(r)}(x)}=\infty, \qquad 2i+2k\leq r\leq n.$$

Proofs for (E_i) and (E_{iv}) are similar.

Remark 2. Lemma 2 for the case $r=0$ may be stated as follows: If u and v are eventually positive solutions of (E), $u\in A_i$, and

$$\limsup_{x\to\infty}\frac{v(x)}{u(x)}\neq 0 \qquad \left[\limsup_{x\to\infty}\frac{v(x)}{u(x)}\neq\infty\right],$$

then $v\in A_j$ for some $j\geq i$ $[j\leq i]$.

Lemma 3. *If the class A_l contains two solutions v_1 and v_2 of which every nontrivial linear combination again belongs to A_l, then A_l contains two solutions y_1 and y_2, each a linear combination of v_1 and v_2, such that*

$$\lim_{x\to\infty}\frac{y_2^{(r)}(x)}{y_1^{(r)}(x)}=\infty, \qquad r=0,1,\cdots,n. \tag{12}$$

Conversely, if $y_1, y_2\in A_l$ such that

$$\lim_{x\to\infty}\frac{y_2(x)}{y_1(x)}=\infty, \tag{13}$$

then every nontrivial linear combination of y_1 and y_2 belongs to A_l and (12) *holds.*

Proof. We may assume that $v_2>v_1>0$ on $[b,\infty)$ for some $b\geq a$. Since v_2-Kv_1 belongs to A_l for any constant K, $\lim_{x\to\infty}\{v_2(x)/v_1(x)\}$ exists. If the limit is infinite, put $y_i=v_i$, $i=1,2$. If the limit is finite and equal to C, then

$$\lim_{x\to\infty}\frac{v_2(x)-Cv_1(x)}{v_1(x)}=0;$$

put $y_1=v_2-Cv_1$ or $y_1=-(v_2-Cv_1)$ according as v_2-Cv_1 is eventually positive or negative, and $y_2=v_1$. In either case, we then have $\lim_{x\to\infty}\{y_2(x)/y_1(x)\}=\infty$, where both y_1 and y_2 are eventually positive. Evidently, $Y_K \equiv y_2 - Ky_1$ belongs to A_l for any constant K and is eventually positive. According to Lemma 1, $Y_K^{(r)}$, $r=0, 1, \cdots, n$, have constant signs on $[b_2, \infty)$ for some $b_2 \geq a$. Therefore, $\lim_{x\to\infty} \{y_2^{(r)}(x)/y_1^{(r)}(x)\}$, $r=0, 1, \cdots, n$, exist. Suppose that

$$\lim_{x\to\infty} \frac{y_2^{(j)}(x)}{y_1^{(j)}(x)} < M, \tag{14}$$

for some constant M and some j, $0 \leq j \leq n$. Then

$$\frac{y_2^{(j)}(x) - My_1^{(j)}(x)}{y_1^{(j)}(x)} < 0, \tag{15}$$

on $[d, \infty)$ for some d. Here, y_1 and $Y_M = y_2 - My_1$ belong to A_l and both are eventually positive; thus $y_1^{(j)}$ and $Y_M^{(j)}$ must have the same sign on $[d_1, \infty)$ for some d_1 by Lemma 1. But this conclusion is contrary to (15). Hence, (14) cannot hold for any j, $0 \leq j \leq n$, and any constant M, proving (12).

Conversely, assume that (13) holds, $y_1 > 0$, and $y_2 > 0$ on $[b_1, \infty)$ for some b_1. Every nontrivial linear combination of y_1 and y_2 is nonoscillatory and belongs to A_i, for some $i \leq l$, by Remark 1. It suffices to show that it cannot belong to A_i, $i < l$. If $C_1 y_1 + C_2 y_2 \equiv Y \in A_i$, $i < l$, for some non-zero constants C_1 and C_2, then

$$C_1 + C_2 \lim_{x\to\infty} \frac{y_2(x)}{y_1(x)} = \lim_{x\to\infty} \frac{Y(x)}{y_1(x)},$$

where the left-hand side is infinite, while the right-hand side is zero by Lemma 2. Thus every nontrivial linear combination of y_1 and y_2 belongs to A_l; and (12) now follows from (13), as shown in the first part of the proof.

Lemma 4. *Suppose that* (E) *has a nonoscillatory solution* $y \in A_j$ *which is eventually positive and a solution* w. *If*

$$\limsup_{x\to\infty} \frac{w(x)}{y(x)} = 0, \tag{16}$$

and w *is oscillatory, then* $y - Kw \in A_j$ *and it is eventually positive for any nonnegative constant* K. *Similarly, if*

$$\liminf_{x\to\infty} \frac{w(x)}{y(x)} = 0, \tag{17}$$

and w *is oscillatory, then* $y - Kw \in A_j$ *and it is eventually positive for any nonpositive constant* K.

If w *is eventually positive* [*negative*] *and* (16) [(17)] *holds, then* $y - Kw \in A_j$ *and it is eventually positive for any constant* K.

Proof. If (16) [(17)] holds, $u \equiv y - Kw$ is nonoscillatory for all nonnegative [nonpositive] constant K, regardless of whether or not w is oscillatory. To see this, assume the contrary; then there exists a sequence $\{\xi_i\}$ of real numbers, $\xi_i \to \infty$ as $i \to \infty$, such that $u(\xi_i)=0$, i.e., $w(\xi_i)/y(\xi_i)=K^{-1}>0$ [<0] for $i>N$ for some N (Of course, the assertion is trivial if $K=0$). But this contradicts (16) [(17)]. Hence, u must be nonoscillatory and eventually positive by (16) [(17)]. If w is eventually positive [negative], then $y-Kw$ is eventually positive for any negative [positive] constant K. Thus, $u \in A_i$ for some i by Lemma 1.

From the definition of u, we have

$$\frac{u(x)}{y(x)} = 1 - K\frac{w(x)}{y(x)}$$

and by Lemma 2,

$$\lim_{x\to\infty} \frac{u(x)}{y(x)} = \begin{Bmatrix} 0, & \text{if } i<j \\ \infty, & \text{if } i>j \end{Bmatrix} = 1 - K \lim_{x\to\infty} \frac{w(x)}{y(x)},$$

which is compatible with neither (16) nor (17). Consequently, $i=j$ and $y-Kw \in A_j$.

§ 3. Small and large solutions.

Hastings and Lazer [6] observed for (5) that every oscillatory solution $w=w(t) \to 0$ as $t \to \infty$ if $p \in C'[a, \infty)$, $p'(t) \leq 0$ and $\lim_{t\to\infty} p(t) = -\infty$. This result raises the following question: How fast does $w(t) \to 0$ as $t \to \infty$? In this connection we shall generalize the notion of principal solutions introduced by Leighton and Morse [13], to equations of higher order which may or may not be oscillatory. In some sense, a principal solution is "smaller" than all other linearly independent solutions [4, 12]. It is this property of a principal solution that we seek to preserve to the extent possible in our generalization.

Definition. A nonoscillatory solution Y_S of (E) is called a *small solution* of (E) if

$$\limsup_{x\to\infty} \frac{w(x)}{Y_S(x)} \neq 0,$$

for every solution w of (E). Introducing a companion concept, we shall say that a nonoscillatory solution Y_L of (E) is a *large solution* of (E) if

$$\limsup_{x\to\infty} \frac{w(x)}{Y_L(x)} \neq \infty,$$

for every solution w of (E).

The terms "small" and "large" are to be interpreted in the following way: For an arbitrary solution w, there exists positive constant K such that $|w(x)|$ is not bounded above by $K|Y_S(x)|$ for sufficiently large x, and $|w(x)|\leq K'|Y_L(x)|$, $x \in [a, \infty)$, for some positive constant K'. In Theorems 1 and 2 we shall discuss the existence of small and large solutions of (E).

It is well-known that A_0 is nonempty for (E_{ii}) and (E_{iii}) [5]; let $y \in A_0$ and assume $y>0$ without loss of generality. If A_0 is nonempty for (E_i) and (E_{iv}), it contains two eventually positive solutions y_1 and y_2 such that $\lim_{x\to\infty}\{y_2(x)/y_1(x)\}=\infty$ by [10, Theorem 5] and Lemma 3. Define

$$Y_S=\begin{cases} y \text{ for } (E_{ii}) \text{ and } (E_{iii}), \\ y_1 \text{ for } (E_i) \text{ and } (E_{iv}) \text{ if } A_0 \text{ is nonempty.}\end{cases}$$

Theorem 1. *If w is an oscillatory solution of* (E), *then*

$$\limsup_{x\to\infty} \frac{w(x)}{Y_S(x)} \neq 0, \tag{18}$$

and

$$\liminf_{x\to\infty} \frac{w(x)}{Y_S(x)} \neq 0. \tag{19}$$

If w is an eventually positive [*negative*] *solution,* (18) [(19)] *holds.*

Proof. Consider the cases (E_{ii}) and (E_{iii}). If (18) [(19)] does not hold, there exists a positive [negative] constant K and a point $\alpha\geq a$ such that $u\equiv Y_S-Kw\geq 0$ on $[\alpha, \infty)$ and $u(\alpha)=0$. According to Lemma 4, u belongs to A_0; but this is incompatible with Lemma 1 because $u(\alpha)=0$.

For (E_i) and (E_{iv}), assume that $Y_S>0$ on $[b, \infty)$ for some $b\geq a$, w is oscillatory and (18) [(19)] does not hold. Let $w(c)=0$ for some $c>b$. Then there exists a positive [negative] constant K_1 such that $u_1\equiv Y_S-K_1w\geq 0$ on $[c, \infty)$ and $u_1(\beta)=u_1'(\beta)=0$ for some $\beta\geq c$. But this is contrary to Lemma 1 since $u_1 \in A_0$ by Lemma 4. If w is eventually positive [negative] and (18) [(19)] does not hold, then

$$\lim_{x\to\infty} \frac{w(x)}{Y_S(x)}=0;$$

putting $y_0=w[-w]$, we see that

$$\lim_{x\to\infty} \frac{y_j(x)}{y_i(x)}=\infty, \qquad 0\leq i<j\leq 2.$$

By Lemma 3 in [9], there exists a solution $v=\sum_{k=0}^{2} \alpha_k y_k$ such that $v\geq 0$ on $[\xi, \infty)$

and $v(\zeta)=v'(\zeta)=0$ for some point $\zeta \in (\xi, \infty)$. This again contradicts Lemma 1 because $v \in A_0$ by Lemma 4, and completes the proof.

It was proved in [9] that no class A_l can contain three nonoscillatory solutions y_1, y_2, and y_3 such that

$$\lim_{x\to\infty} \frac{y_j(x)}{y_i(x)}=\infty, \qquad 1\leq i<j\leq 3.$$

Improving this result, we shall show that if A_l contains two solutions y_1 and y_2 such that

$$\lim_{x\to\infty} \frac{y_2(x)}{y_1(x)}=\infty, \tag{20}$$

and $y_2>y_1>0$ on $[b, \infty)$, then (E) cannot have a solution w satisfying

$$\limsup_{x\to\infty} \frac{w(x)}{y_1(x)}=\infty \quad \text{and} \quad \limsup_{x\to\infty} \frac{w(x)}{y_2(x)}=0. \tag{21}$$

Similarly, if Y_L is an eventually positive solution of (E_{iii}) $[(E_{iv})]$ belonging to the nonempty class $A_{n/2}[A_{(n-1)/2}]$ [8], (E_{iii}) $[(E_{iv})]$ cannot have a solution w such that

$$\limsup_{x\to\infty} \frac{w(x)}{Y_L(x)}=\infty. \tag{22}$$

Remark 3. Conditions of the type (21) and (22) may be written in equivalent forms using the limit inferior; this is easily seen when w is replaced by $-w$ and the relation $\limsup_{x\to\infty} g(x)=-\liminf_{x\to\infty} [-g(x)]$ is recalled.

Theorem 2. *Suppose that the class A_l of* (E) *contains two solutions y_1 and y_2 for which* (20) *holds and $y_2>y_1>0$ on $[b, \infty)$ for some $b\geq a$. Then for every solution w of* (E),

$$\limsup_{x\to\infty} \frac{w(x)}{y_1(x)}\neq\infty \quad or \quad \limsup_{x\to\infty} \frac{w(x)}{y_2(x)}\neq 0.$$

If Y_L is an eventually positive solution of (E_{iii}) $[(E_{iv})]$ *belonging to* $A_{n/2}[A_{(n-1)/2}]$, *then*

$$\limsup_{x\to\infty} \frac{w(x)}{Y_L(x)}\neq\infty.$$

For the proof of Theorem 2, we require the following lemmas.

Lemma 5. *Suppose that* (E) *has a nonoscillatory solution y which is eventually positive and an oscillatory solution w. If*

$$\limsup_{x\to\infty} \frac{w(x)}{y(x)} = \infty, \tag{23}$$

then $By+Cw$ *is an oscillatory solution for any constants* B *and* $C\neq 0$, $BC\leq 0$. *If, on the other hand,*

$$\liminf_{x\to\infty} \frac{w(x)}{y(x)} = -\infty, \tag{24}$$

then $By+Cw$ *is an oscillatory solution for any constants* B *and* $C\neq 0$, $BC\geq 0$.

Proof. Choose an arbitrary nonnegative [nonpositive] constant K. Due to (23) [(24)] there exists a sequence $\{\xi_i\}$ with $\xi_i\to\infty$ as $i\to\infty$ such that $w(\xi_i)/y(\xi_i)=K$, $i=1, 2, \cdots$. If we make the particular choice $K=-B/C$, then $By(\xi_i)+Cw(\xi_i)=0$, $i=1, 2, \cdots$, and the solution $By+Cw$ is oscillatory with $BC\leq 0$ [≥ 0].

Lemma 6. *Suppose that* (E) *has a solution* w *and two nonoscillatory solutions* y_1 *and* y_2 *for which* (20) *and* (21) *hold and* $y_2>y_1>0$ *on* $[b, \infty)$ *for some* $b\geq a$. *If* η *is an arbitrary point on* $[b, \infty)$, *there exists a solution* $v\equiv\alpha y_1+\beta y_2+\gamma w$, $\alpha, \beta>0, \gamma<0$, *such that* $v\geq 0$ *on* $[b, \infty)$ *and* $v(\zeta)=v'(\zeta)=0$ *for some* $\zeta\in[\eta, \infty)$.

Proof. Choose a constant $K>0$ such that $u\equiv w-Ky_1<0$ on $[b, \eta]$. In view of (21), w/y_1 cannot be bounded above by K on $[b, \infty)$; $u(\xi)>0$ for some $\xi>\eta$. Let σ be the first zero of u on (η, ∞). Then $u<0$ on $[b, \sigma)$ and $y_2-K_1u>0$ on $[b, \sigma)$ for any nonnegative constant K_1. On $[b, \infty)$, $u=w-Ky_1\leq w$ and

$$l_s=\limsup_{x\to\infty} \frac{u(x)}{y_2(x)}\leq \limsup_{x\to\infty} \frac{w(x)}{y_2(x)}=0;$$

thus,

$$\limsup_{x\to\infty} \frac{u(x)}{y_2(x)}=0$$

(because $l_s\geq 0$ if u is oscillatory, and u is eventually positive and $l_s\geq 0$ if u is nonoscillatory). Therefore, y_2-K_1u is eventually positive for any constant $K_1\geq 0$ by Lemma 4. If we choose $K_1>y_2(\xi)/u(\xi)>0$, then $y_2(\xi)-K_1u(\xi)<0$. Consequently, there exists a constant $K_2>0$ such that $v\equiv y_2-K_2u\geq 0$ on $[b, \infty)$ and $v(\zeta)=0$ for some point $\zeta\in[\eta, \infty)$ (Cf. [9, Proof of Lemma 3]). Since $v\geq 0$ on $[b, \infty)$, $v(\zeta)=0$ implies $v'(\zeta)=0$. Putting $\alpha=KK_2$, $\beta=1$, and $\gamma=-K_2$, we get $v=\alpha y_1+\beta y_2+\gamma w$ with $\alpha, \beta>0$ and $\gamma<0$.

This lemma generalizes Lemma 3 in [9].

Proof of Theorem 2. Assume to the contrary that (21) holds. Let $\{\eta_i\}$ be a sequence of numbers with $\eta_i \geq b$ and $\eta_i \to \infty$ as $i \to \infty$. For each i there exists a solution

$$v_i \equiv \alpha_i y_1 + \beta_i y_2 + \gamma_i w, \quad \alpha_i, \beta_i > 0, \quad \gamma_i < 0, \quad \alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1,$$

such that $v_i \geq 0$ on $[b, \infty)$ and $v_i(\zeta_i) = v_i'(\zeta_i) = 0$ for some $\zeta_i > \eta_i$ by Lemma 6. According to (20) and Lemma 3, for each $i, f_i \equiv \alpha_i y_1 + \beta_i y_2 \in A_l$ and $f_i > 0$ on $[b, \infty)$ since $y_2 > y_1 > 0$ on $[b, \infty)$ and $\alpha_i, \beta_i > 0$. Due to (21),

$$0 \leq \limsup_{x \to \infty} \frac{w(x)}{f_i(x)} \leq \limsup_{x \to \infty} \frac{w(x)}{\beta_i y_2(x)} = 0;$$

and $v_i \in A_l$ for every i by Lemma 4. Let

$$\alpha = \lim_{i \to \infty} \alpha_i, \quad \beta = \lim_{i \to \infty} \beta_i, \quad \gamma = \lim_{i \to \infty} \gamma_i$$

(take subsequences if necessary), and $Y \equiv \alpha y_1 + \beta y_2 + \gamma w$, $\alpha, \beta \geq 0$, $\gamma \leq 0$. Since $Y(x) = \lim_{i \to \infty} v_i(x)$ and $v_i \geq 0$ on $[b, \infty)$ for every i, $Y \geq 0$ on $[b, \infty)$, i.e., Y is a nonoscillatory solution. If $\beta \neq 0$, then $\beta\gamma \leq 0$, $\beta y_2 + \gamma w \in A_l$ and it is eventually positive by Lemma 4; thus $Y \in A_l$, for $y_1 > 0$ on $[b, \infty)$ and $\alpha \geq 0$ (Cf. Remark 1). Next consider the case $\beta = 0$: If w is oscillatory, Y would be oscillatory by Lemma 5 unless $\gamma = 0$; we therefore have $\gamma = 0$ and $Y = y_1 \in A_l$. If w is nonoscillatory, Y would be eventually negative by (21) unless $\gamma = 0$; thus, $\gamma = 0$ and again $Y = y_1 \in A_l$. Consequently, $Y \in A_l$ in any case.

From Lemma 1 we see that

$$Y > 0, \quad Y' > 0, \cdots, Y^{(\upsilon)} > 0, \tag{25}$$

on $[b_2, \infty)$ for some $b_2 \geq b$, where

$$\upsilon = \begin{cases} 2l & \text{for } (E_{\text{i}}) \text{ and } (E_{\text{iv}}), \\ 2l - 1 & \text{for } (E_{\text{ii}}) \text{ and } (E_{\text{iii}}). \end{cases}$$

The remainder of the proof is patterned after the proof used in [9, Theorem]. Since $\lim_{i \to \infty} v_i^{(j)}(b_2) = Y^{(j)}(b_2)$, there exists a number N such that $i > N$ implies

$$v_i^{(j)}(b_2) > \frac{Y^{(j)}(b_2)}{2} > 0, \quad j = 0, 1, \cdots, \upsilon, \tag{26}$$

and $\eta_i > b_2$. Furthermore, $v_i^{(\upsilon+1)} > 0$ on $[b, \infty)$; this follows from Lemma 1, for $v_i \in A_l$ and $v_i \geq 0$ on $[b, \infty)$. Hence,

$$v_i^{(\upsilon)}(b_2) \leq v_i^{(\upsilon)}(\tau), \qquad \tau \in [b_2, \infty). \tag{27}$$

From (26) with $j = \upsilon$ and (27), we get

$$v_i^{(\upsilon)}(\tau) > \frac{Y^{(\upsilon)}(b_2)}{2}, \qquad \tau \in [b_2, \infty). \tag{28}$$

Integrating (28) from b_2 to $x \in [b_2, \infty)$ and substituting in the resulting expression (26) with $j = \upsilon - 1$, we obtain

$$v_i^{(\upsilon-1)}(x) > \frac{Y^{(\upsilon)}(b_2)}{2}(x - b_2) + \frac{Y^{(\upsilon-1)}(b_2)}{2}.$$

Repeating a similar procedure $\upsilon - 1$ times, we arrive at

$$v_i(x) > \frac{Y^{(\upsilon)}(b_2)}{2(\upsilon!)}(x-b_2)^{\upsilon} + \frac{Y^{(\upsilon-1)}(b_2)}{2[(\upsilon-1)!]}(x-b_2)^{\upsilon-1} + \cdots + \frac{Y(b_2)}{2}, \qquad x \in [b_2, \infty). \tag{29}$$

But this inequality cannot hold throughout the interval $[b_2, \infty)$; for $x = \zeta_i > \eta_i > b_2$ (when $i > N$), the left-hand side $v_i(\zeta_i) = 0$, while the right-hand side is positive by (25). Thus (21) cannot hold, and this proves the first part of the theorem.

For the second part, $A_{n/2}[A_{(n-1)/2}]$ is nonempty for (E_{iii}) $[(E_{iv})]$ [8]. Assume that (22) holds and $Y_L > 0$ on $[b, \infty)$ for some $b \geq a$. Then there exist a sequence $\{\xi_i\}$, $\xi_i \to \infty$ as $i \to \infty$, and a solution

$$u_i \equiv B_i Y_L + C_i w, \quad B_i > 0, \quad C_i < 0, \quad B_i^2 + C_i^2 = 1,$$

such that $u_i > 0$ on $[b, \xi_i)$ and $u_i(\xi_i) = 0$, for each i. Let

$$B = \lim_{i\to\infty} B_i, \qquad C = \lim_{i\to\infty} C_i.$$

Then $B \geq 0$, $C \leq 0$, $B^2 + C^2 = 1$, and $V \equiv BY_L + Cw \geq 0$ on $[b, \infty)$. If w is oscillatory, V is oscillatory by Lemma 5 unless $C = 0$; we must have $C = 0$ and $V \in A_{n/2}[A_{(n-1)/2}]$. If w is nonoscillatory, then $V \geq 0$ on $[b, \infty)$ and (22) requires $C = 0$. Thus, $V \in A_{n/2}[A_{(n-1)/2}]$ in any case. By Lemma 1,

$$V > 0, \quad V' > 0, \cdots, V^{(n-1)} > 0, \tag{30}$$

on $[c, \infty)$ for some $c \geq b$. Choose a number N such that $i > N$ implies

$$u_i^{(j)}(c) > \frac{V^{(j)}(c)}{2} > 0, \qquad j = 0, 1, \cdots, n-1, \tag{31}$$

and $\xi_i > c$. On $[c, \xi_i]$, $u_i^{(n)} = -pu_i \geq 0$, i.e., $u_i^{(n-1)}(c) \leq u_i^{(n-1)}(\tau)$, $\tau \in [c, \xi_i]$. When this inequality is substituted in (31) with $j = n-1$, there results

$$u_i^{(n-1)}(\tau) > \frac{V^{(n-1)}(c)}{2}, \qquad \tau \in [c, \xi_i]. \tag{32}$$

Integrate (32) from c to $x \in [c, \xi_i]$ and substitute therein (31) with $j=n-2$, then

$$u_i^{(n-2)}(x) > \frac{V^{(n-1)}(c)}{2}(x-c) + \frac{V^{(n-2)}(c)}{2}, \qquad x \in [c, \xi_i].$$

Repeating a similar procedure $n-2$ times, we get

$$u_i(x) > \frac{V^{(n-1)}(c)}{2[(n-1)!]}(x-c)^{n-1} + \frac{V^{(n-2)}(c)}{2[(n-2)!]}(x-c)^{n-2} + \cdots + \frac{V(c)}{2}, \qquad x \in [c, \xi_i].$$

This inequality, however, cannot hold at $x=\xi_i$ because $u_i(\xi_i)=0$, while the right-hand side is positive by (30); and the proof is complete.

§ 4. Fundamental systems.

Suppose that Equation (E) is oscillatory. Then there exists an empty class A_i for some i. This is because if A_j is nonempty for all j, (E) has a fundamental system consisting of n nonoscillatory solutions of which every nontrivial linear combination is nonoscillatory [9, 10], that is, (E) is nonoscillatory. Let $A_{j_0}, A_{j_1}, \cdots, A_{j_{s-1}}, j_0 < j_1 < \cdots < j_{s-1}$, be nonempty classes and let $A_{j_s}, \cdots, A_{j_m}, j_s < \cdots < j_m$, be empty classes, where $m=[(n-1)/2]$ for (E_i) and (E_{iv}), and $m=[n/2]$ for (E_{ii}) and (E_{iii}). In view of [10, Theorem 5], we may choose a set N of eventually positive solutions consisting of

$$\begin{array}{ll}
y_{2i+1}, y_{2i+2} \in A_{j_i}, \quad i=0, 1, \cdots, s-1, & \text{for } (E_i), \\
y_1 \in A_0, \quad y_{2i}, y_{2i+1} \in A_{j_i}, \quad i=1, 2, \cdots, s-1, & \text{for } (E_{ii}), \\
y_1 \in A_0, \quad y_{2i}, y_{2i+1} \in A_{j_i}, \quad i=1, 2, \cdots, s-2, \quad \text{and} \quad y_{2s-2} \in A_{n/2}, & \text{for } (E_{iii}), \\
y_{2i+1}, y_{2i+2} \in A_{j_i}, \quad i=0, 1, \cdots, s-2, \quad \text{and} \quad y_{2s-1} \in A_{(n-1)/2} & \text{for } (E_{iv}),
\end{array}$$

for which (12) of Lemma 3 holds. Extend N to a fundamental system F by adjoining solutions $w_q, w_{q+1}, \cdots, w_n$;

$$F=\{y_1, y_2, \cdots, y_{q-1}, w_q, \cdots, w_n\},$$

where

$$q=\begin{cases} 2s+1 & \text{for } (E_i), \\ 2s & \text{for } (E_{ii}) \text{ and } (E_{iv}), \\ 2s-1 & \text{for } (E_{iii}). \end{cases}$$

We may assume that $w_q, \cdots, w_n$ are oscillatory solutions (Cf. [10]): For definiteness, consider (E_i) If w_l is nonoscillatory for some l, $q \le l \le n$, then $w_l \in A_{j_k}$ for some k, $0 \le k \le s-1$, and $\tilde{w}_l \equiv w_l - \sum_{i=1}^{2k+2} c_i y_i$ must be oscillatory for some constants $c_1, c_2, \cdots, c_{2k+2}$. We may replace $w_l \in F$ by $\tilde{w}_l$ to obtain a new fundamental system. Evidently, this procedure can be repeated until $w_q, \cdots, w_n$ are all oscillatory.

Lemma 7. *Suppose that A_{j_l} is nonempty, A_{j_k} is empty and $j_l<j_k$. If $y_r \in A_{j_l} \cap F$, then there exists an oscillatory solution $w_\rho \in F$ such that*

$$\liminf_{x\to\infty} \frac{w_\rho(x)}{y_r(x)} \neq \limsup_{x\to\infty} \frac{w_\rho(x)}{y_r(x)}. \tag{33}$$

Proof. This lemma will be proved by showing that unless (33) holds we can construct a solution ϕ_r of (E) which violates Lemma 1.

Suppose that (33) does not hold; then

$$\lim_{x\to\infty} \frac{w_i(x)}{y_r(x)}=0, \qquad i=q, q+1, \cdots, n, \tag{34}$$

since w_i is oscillatory while y_r is eventually positive. For any given $\alpha \in [a, \infty)$, we assert that there exists a nontrivial solution of the form

$$u(x) \equiv \sum_{i=1}^{\upsilon} c_i y_i(x) + \sum_{i=q}^{n} c_i w_i(x), \qquad c_i \text{ constant}, \tag{35}$$

such that

$$u(\alpha)=u'(\alpha)=\cdots=u^{(\mu)}(\alpha)=0, \tag{36}$$

where

$$\upsilon=\begin{cases} 2l & \text{for } (E_{\text{i}}) \text{ and } (E_{\text{iv}}), \\ 2l-1 & \text{for } (E_{\text{ii}}) \text{ and } (E_{\text{iii}}), \end{cases}$$

and

$$\mu=\begin{cases} 2j_l & \text{for } (E_{\text{i}}) \text{ and } (E_{\text{iv}}), \\ 2j_l-1 & \text{for } (E_{\text{ii}}) \text{ and } (E_{\text{iii}}). \end{cases}$$

Note that $\sum_{i=1}^{\upsilon} c_i y_i(x)$ is a linear combination of the solutions in $F\cap(A_{j_0}\cup\cdots\cup A_{j_{l-1}})$. The assertion will follow if we show that there are at least $\mu+2$ solutions in the linear combination (35). Evidently, there are $\upsilon+n-(q-1)$ solutions in it. Among the j_l+1 classes $A_0, A_1, \cdots, A_{j_l}$, there are $l+1$ nonempty classes $A_{j_0}, A_{j_1}, \cdots, A_{j_l}$ and at most $k-s$ empty classes $A_{j_s}, \cdots, A_{j_{k-1}}$ since $j_l<j_k$ and A_{j_k} is empty. Hence, $j_l+1\leq l+1+k-s$, and

$$\mu+2=\left.\begin{cases} 2j_l+2\leq 2l-2s+2k+2\leq 2l-2s+n & \text{for } (E_{\text{i}}) \\ 2j_l+1\leq 2l-2s+2k+1\leq 2l-2s+n & \text{for } (E_{\text{ii}}) \\ 2j_l+1\leq 2l-2s+2k+1\leq 2l-2s+n+1 & \text{for } (E_{\text{iii}}) \\ 2j_l+2\leq 2l-2s+2k+2\leq 2l-2s+n+1 & \text{for } (E_{\text{iv}}) \end{cases}\right\}=\upsilon+n-q+1,$$

i.e., $\mu+2\leq\upsilon+n-q+1$ for (E), proving the existence of u satisfying (35) and (36).

The term $\sum_{i=1}^{\upsilon} c_i y_i$ in (35) is a nonoscillatory solution belonging to A_σ for some $\sigma<j_l$ by Remark 1. Therefore,

$$\lim_{x\to\infty}\frac{u(x)}{y_r(x)}=0, \tag{37}$$

by Lemma 2 and (34). In fact, it will be shown that

$$\lim_{x\to\infty}\frac{u^{(i)}(x)}{y_r^{(i)}(x)}=0, \qquad i=0, 1, \cdots, n. \tag{38}$$

If u is oscillatory, the limits in (38) must exist; for if the limit does not exist for some j, $0\leq j\leq n$, there exists a number K such that $y_r^{(j)}-Ku^{(j)}$ has infinitely many zeros on $[a, \infty)$. But this is impossible because $y_r-Ku\in A_{j_l}$ by (37) and Lemma 4, and $y_r^{(j)}-Ku^{(j)}$ is eventually of constant sign by Lemma 1. Hence the limits exist, and they must be zero. On the other hand, if u is nonoscillatory, $u\in A_\tau$ for some τ, $0\leq\tau\leq j_l$, by (37) and Lemma 2. Therefore, (38) follows from Lemma 2 if $\tau<j_l$ and from Lemma 3 if $\tau=j_l$.

We are ready to construct the function ϕ_r. Noting that $y_r^{(i)}>0$, $i=0, 1, \cdots, \mu$, on $[\alpha, \infty)$ for some α since $y_r\in A_{j_l}\cap F$, we deduce from (38) that $y_r^{(i)}-Cu^{(i)}$, $i=0, 1, \cdots, \mu$, are eventually positive for any constant C. Furthermore, $y_r^{(i)}(\alpha)-Cu^{(i)}(\alpha)=y_r^{(i)}(\alpha)>0$, $i=0, 1, \cdots, \mu$, due to (36). Therefore, we may choose a constant C_1 such that $\phi_r\equiv y_r-C_1u$ has the following properties:

$$\phi_r^{(i)}(x)\geq 0, \quad x\in[\alpha, \infty), \quad i=0, 1, \cdots, \mu, \quad \phi_r^{(\kappa)}(\zeta)=0,$$

for some κ, $0\leq\kappa\leq\mu$, and some $\zeta\in(\alpha, \infty)$.

If $\kappa<\mu$, $\phi_r^{(\kappa)}(\zeta)=0$ implies $\phi_r^{(\kappa+1)}(\zeta)=0$ since $\phi_r^{(\kappa)}\geq 0$ on $[\alpha, \infty)$; continuing this argument we obtain successively $\phi_r^{(\kappa+2)}(\zeta)=\cdots=\phi_r^{(\mu+1)}(\zeta)=0$. Consequently, $\phi_r\geq 0$ on $[\alpha, \infty)$ and $\phi_r^{(\mu+1)}(\zeta)=0$. But this is incompatible with Lemma 1 because $\phi_r\in A_{j_l}$ as can be easily seen: If u is oscillatory, then $\phi_r\in A_{j_l}$ by (37) and Lemma 4. If $u\in A_{j_l}$, then $\phi_r\in A_{j_l}$ by (37) and Lemma 3. Finally, if $u\in A_\tau$ for some $\tau<j_l$, then $\phi_r\in A_{j_l}$ by Remark 1. This completes the proof.

Under the conditions of Lemma 7, choose a constant K such that

$$\liminf_{x\to\infty}\frac{w_\rho(x)}{y_r(x)}<K<\limsup_{x\to\infty}\frac{w_\rho(x)}{y_r(x)},$$

then $w_\rho-Ky_r$ is an oscillatory solution.

Suppose that a class A_λ of (E) is empty. If the family $\{A_0, A_1, \cdots, A_\lambda\}$ contains ι empty classes and $\lambda+1-\iota$ nonempty classes, there are at least 2ι oscillatory solutions w_k in the fundamental system F and $2(\lambda+1-\iota)[2(\lambda+1-\iota)-1]$ oscillatory solutions of the form $w_i - K_i y_i$, where K_i is a constant and $w_i, y_i \in F$, for (E_i) and (E_{iv}) [(E_{ii}) and (E_{iii})].

We thus have the following result.

Theorem 3. *If A_λ is empty,* (E_i) *and* (E_{iv}) *have at least $2\lambda+2$ linearly independent oscillatory solutions, and* (E_{ii}) *and* (E_{iii}) *have at least $2\lambda+1$ linearly independent oscillatory solutions.*

Equation (E) is $k-(n-k)$ disfocal if and only if its adjoint equation (E^+) is $(n-k)-k$ disfocal [18]. Moreover, $A_{[k/2]}$ is nonempty if and only if (E) is eventually $k-(n-k)$ disfocal on $[a, \infty)$, provided $(-1)^{n-k}p(x)<0$ [2, 10]. Therefore, $A_{[k/2]}$ of (E) is empty if and only if $A_{[(n-k)/2]}$ of (E^+) is empty, provided $(-1)^{n-k}p(x) <0$, $x \in [a, \infty)$; more specifically, A_j is empty for (E) if and only if $A_{\gamma-j}$ is empty for (E^+), where γ is equal to $(n-2)/2$ for (E_i^+), $(n-1)/2$ for (E_{ii}^+) and (E_{iv}^+), and $n/2$ for (E_{iii}^+).

We can now easily prove the following statements for the self-adjoint equations (E_i) and (E_{iii}).

Theorem 4. *If*

$$y^{(2m)}+py=0, \quad p>0, \quad m\geq 2, \tag{39}$$

is oscillatory on $[a, \infty)$, it has at least $m+2$ or $m+1$ linearly independent oscillatory solutions according as m is even or odd.

Theorem 5. *If*

$$y^{(2m)}+py=0, \quad p<0, \quad m\geq 2,$$

is oscillatory on $[a, \infty)$, it has at least $m+1$ or $m+2$ linearly independent oscillatory solutions according as m is even or odd.

Since (39) which is (E_i) is oscillatory, A_j is empty for some j, $0\leq j\leq m-1$, and therefore A_{m-1-j} is also empty. This means that there is an empty class A_k with $k \geq m/2$ [$(m-1)/2$] if m is even [odd]. For example, take $k=\max(j, m-1-j)$. By Theorem 3, (E_i) has at least $2k+2$ linearly independent oscillatory solutions, where $2k+2\geq m+2$ if m is even and $2k+2\geq m+1$ if m is odd. This proves Theorem 4.

Theorem 5 may be proved in a similar manner.

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