Weakly Equicontinuous Flows

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§ 1. Introduction.

The concept of equicontinuity is central in Topological Dynamics. Yet it can be defined only in flows whose phase spaces are either uniform spaces or metric spaces. In 1970, O. Hajek [7] suggested that one can characterize equicontinuity by using the first limit prolongation relation $J$ which was introduced by T. Ura [10]. He conjectured that for a flow $(X, T)$ the property $J(A) \subset A$, where $A$ is the diagonal of $X \times X$, may be a candidate for a topological version of equicontinuity. In section 2 we will show that the property $J(A) \subset A$ is a necessary but not a sufficient condition for equicontinuity (Example 2.7). A flow that has the property $J(A) \subset A$ and possesses an additional property, called $M$, is called a weakly equicontinuous flow. It is shown that the concepts of equicontinuity and weak equicontinuity are equivalent provided that all the orbit closures in the phase space $X$ are compact (Theorem 2.9). The concept of weak equicontinuity of a flow $(X, T)$ is shown to be somewhere between equicontinuity of $(X, T)$ and that of characteristic 0 of $(X \times X, T)$ (Theorem 2.10).

Hajek [7] also mentioned that one of the main obstacles in lifting results from Dynamical Systems theory to Topological Dynamics is time orientation. For in the former the phase group is the additive group of reals which is endowed with a natural order relation. While in the later this orientation of time is not available and thus we have no analogue to concepts such as wandering points, dispersive points, positive (negative) prolongation relations $D^+(D^-)$, positive (negative) limit prolongation $J^+(J^-)$, etc.. In [3] we used Hahn's idea [6], of ordering a group using its semigroups, to lift most of the above quoted properties to Topological Dynamics theory. In section 3, we pursue this approach to define the properties of wandering, $P$-wandering, regionally recurrence, $P$-regionally recurrence and dispersiveness for a flow whose phase group $T$ is an abelian topological group. At the end of the section we give a deep decomposition theorem for weakly equicontinuous flows.

Throughout this paper $(X, T)$ will denote a flow (or a transformation group), where $X$ is assumed to be locally compact and Hausdorff and $T$ is a topological group [5]. For properties that are undefined here we refer the reader to [3] and [7].
§ 2. Weakly equicontinuous flows.

Following [7] we say that \( t_i \to \infty \) for a net \( \{t_i\} \) in \( T \) if the net \( \{t_i\} \) is ultimately outside each compact subset of \( T \). For a flow \((X, T)\) we shall define the limit set, the first prolongation set and the first prolongation limit set of \( x \) in \( X \), respectively, by

\[
L(x) = \{ y \mid x_t \to y \text{ for some } t \to \infty \}, \quad D(x) = \{ y \mid x, t_i \to y \text{ for some } x_i \to x, \{t_i\} \text{ in } T \}
\]

and

\[
J(x) = \{ y \mid x, t_i \to y \text{ for some } x_i \to x, t_i \to \infty \}.
\]

A flow \((X, T)\) is said to be of characteristic 0 if \( D(x) = xT \) for all \( x \in X \). A subset \( M \) of \( X \) is said to be Liapunov stable if for every neighborhood \( U \) of \( M \) there exists a neighborhood \( V \) of \( M \) such that \( VT \subset U \). A flow \((X, T)\) is said to be Liapunov stable if \( xT \) is Liapunov stable for every \( x \in X \).

**Definition 2.1.** A point \( x \in X \) is said to have property \( M \) if whenever there are nets \( x_i \to x \), \( y_i \to y \) and \( \{t_i\} \) in \( T \) such that the net \( \{x, t_i\} \) is convergent, then the net \( \{y, t_i\} \) is also convergent.

**Definition 2.2.** A point \( x \in X \) is said to be \( T \)-weakly equicontinuous if \( J(x, x) \subset A \) and \( x \) has property \( M \).

Notice that the property \( J(x, x) \subset A \) is equivalent to the property \( D(x, x) \subset A \).

If a flow \((X, T)\) satisfies one of the point properties defined above at each of its points, then the flow is said to have that property.

**Lemma 2.3.** Let \( x \) be \( T \)-weakly equicontinuous. Then the following hold. (1) If there are nets \( \{x_i\} \) in \( X \) and \( \{t_i\} \) in \( T \) such that \( x_i \to x \) and \( x, t_i \to y \), then \( x t_i \to y \).

(2) If for a net \( \{t_i\} \) in \( T \) and \( x, y \in X \), \( x_t \to y \), then \( y t_i \to x \).

**Proof.** The proof is straightforward and is thus omitted.

**Theorem 2.4.** Let \((X, T)\) be a flow, where \( X \) is equipped with a uniformity \( \mathcal{U} \). If \((X, T)\) is pointwise equicontinuous, then it is weakly equicontinuous.

**Proof.** Let \( x \in X \). Assume that there exist points \( z, y \in X \) such that \( (z, y) \in J(x, x) \). Then there are nets \( \{z_i\} \), \( \{y_i\} \) in \( X \) and \( t_i \to \infty \) in \( T \) such that \( \lim z_i = \lim y_i = x, z_t \to z \) and \( y_t \to y \). Let \( \alpha \) be an index in \( \mathcal{U} \) and let \( \beta \in \mathcal{U} \) be a symmetric index such that \( \beta^3 \subset \alpha \). Then \( (y, t_i, z) = (z_t, z, z, t_i)(z, t_i, z) \in \beta^3 \subset \alpha \) for all \( i \geq i_0 \). Thus the net \( y, t_i \to z \) and consequently \( z = y \). This proves that \( J(x, x) \subset A \) and that \( x \) has property \( M \). Thus the proof is now complete.

**Definition 2.5.** A point \( x \) is \( T \)-regular if for any open set \( U \) in \( X \) whenever \( xS \subset V \) for a subset \( S \) of \( T \), then there exists an open neighborhood \( V \) of \( x \) such that \( VS \subset U \) [8].

Let \( X^* = X \cup \{\infty\} \) be the one point compactification of \( X \). Then the flow \((X, T)\) can be extended to \((X^*, T)\) in a natural way by letting \( t = \infty \) for all \( t \in T \). If \( J_* \)
and $\Delta_\alpha$ denote, respectively, the prolongation limit relation in $X^*$ and the diagonal in $X^* \times X^*$, then $x \in X$ is $T$-weakly equicontinuous iff $J_\alpha(x, x) \subseteq \Delta_\alpha$.

**Theorem 2.6.** Let $(X, T)$ be a flow. If a point $x \in X$ is $T$-regular, then it is $T$-weakly equicontinuous.

**Proof.** Let $x \in X$. Assume there are points $z, y \in X$ such that $(z, y) \in J(x, x)$. Then, using the terminology of 2.4, $z, t \rightarrow z$ and $y, t \rightarrow y$. Claim that $xt \rightarrow z$. Assume the contrary, that is, $xt \rightarrow b \in X^*$, $b \neq z$. Let $U$ be an open neighborhood of $b$ such that $z \notin U$. If $b = \infty$, then $W = U - \{\infty\}$ is an open set in $X$, otherwise put $W = U$. Let $S = \{t_j \in \{t_i\} | xt_j \in W\}$. Then we may assume that $xS \subseteq W$. Since $x$ is $T$-regular, there exists an open neighborhood $V$ of $x$ such that $xS \subseteq W$. Hence the net $\{xt_i\}$ is eventually in $W$ and thus we have a contradiction. Similarly, one can show that $xt_i \rightarrow y$ and hence $y = z$. The proof is now complete.

The following example will show that the converse of Theorem 2.4 and 2.6 is false.

**Example 2.7.** Consider the continuous flow $(X, T)$, where $T$ is the additive group of reals and $X$ is a subset of the plane defined as $X = \{(x, y) | x > 0, n = 1, 2, \cdots \} \cup \{(x, 0) | x > 0\}$, which arises from the differential system $x_1' = -x_1, x_2' = -x_2$. The flow is weakly equicontinuous, not regular, not equicontinuous and not distal. The flow is wandering, dispersive, not recurrent and of characteristic 0. Furthermore, $(X^\alpha, T)$ is of characteristic 0 for all cardinals $\alpha > 0$.

**Theorem 2.8.** A flow $(X, T)$ is weakly equicontinuous if $(X^\alpha, T)$ is weakly equicontinuous for all cardinals $\alpha > 0$.

**Proof.** The proof is simple and is thus omitted.

**Theorem 2.9.** Let $(X, T)$ be a flow and let $xT$ be compact for all $x \in X$. Then the following statements are pairwise equivalent

(a) $(X, T)$ is weakly equicontinuous

(b) $(X \times X, T)$ is Liapunov stable

(c) $(X, T)$ is regular

(d) $(X, T)$ is pointwise equicontinuous.

**Proof.** (a)→(b). Let $(x, y) \in X \times X$. Assume that $(x, y)T$ is not Liapunov stable. Then there exist an open neighborhood $U$ of $(x, y)T$, a net $\{V_\alpha\}$ of open neighborhoods of $(x, y)T$ with $V_\alpha \subseteq U$ for all $\alpha$, and a net $\{t_\alpha\}$ in $T$ such that $V_\alpha t_\alpha \not\subseteq U$ for all $\alpha$. Hence there exists a net $\{b_\alpha, c_\alpha\} \rightarrow (b, c) \in (x, y)T$ such that $(b_\alpha, c_\alpha) t_\alpha \not\in U$ for all $\alpha$. Since $(b, c)t_\alpha \rightarrow (d, e) \in (x, y)T \subseteq U$ and $(b, c)$ is $T$-weakly equicontinuous (Theorem 2.8), it follows that $(b_\alpha, c_\alpha) t_\alpha \rightarrow (d, e) \in U$. Hence we have a contradiction.

(b)→(c). This follows from [4, 4.4] and [5, 4.24].
(c)→(d). This follows from [4, 4.5].
(d)→(a). This is just theorem 2.4.

**Theorem 2.10.** If a flow \((X, T)\) is weakly equicontinuous, then \((X \times X, T)\) is of characteristic 0.

**Proof.** This follows easily from Theorem 2.8.

**Corollary 2.11.** If a flow \((X, T)\) is weakly equicontinuous, then \(\bar{x}T\) is minimal for all \(x \in X\).

**Proof.** This follows from Theorem 2.10 and [4; 1.5, 2.7].

The following example will show that the converse of the above theorem is false in general.

**Example 2.12.** Consider the continuous flow \((X, T)\) defined by the system

\[
\begin{align*}
r' &= -r^2 \sin \theta \\
\theta' &= 1
\end{align*}
\]

for \(r > 0\).

The flow \((X, T)\) is not weakly equicontinuous but \((X \times X, T)\) is of characteristic 0.

§ 3. A decomposition Theorem.

By an ordering of \(T\) we mean a relation \(R \subseteq T \times T\) such that \(R\) is a semigroup under the composition \((s_1, s_2)(t_1, t_2) = (s_1t_1, s_2t_2)\) and if \(t \in T\) and \((r_1, r_2) \in R\), then \((tr_1, tr_2) \in R\). Instead of \((t_1, t_2) \in R\) we will write \(t_1 < t_2(R)\). Such an ordering \(R\) on \(T\) can be shown to be transitive [6].

Let \(\mathcal{R}\) be the family of all orderings of \(T\) and let \(\mathcal{P}\) be the family of all semigroups in \(T\). Define the following maps

\[
\begin{align*}
f &: \mathcal{P} \to \mathcal{R}; f(P) &= \{(s, t) \in T \times T | ts^{-1} \in P\} \\
g &: \mathcal{R} \to \mathcal{P}; g(R) &= \{t \in T | (e, t) \in R\},
\end{align*}
\]

where \(e\) denotes the identity of \(T\). It can be easily shown that the maps \(f\) and \(g\) are mutually inverse and each one of them is both injective and surjective.

We say that \(T\) is directed if each two element in \(T\) has an upper bound and is strongly directed if each compact set in \(T\) has an upper bound. It was shown in [6] that \(T\) is strongly directed by a relation \(R \in \mathcal{R}\) iff \(g(R) = P\) is a replete semigroup in \(T\). Let \(\mathcal{P}\) be the subfamily of all replete semigroups in \(T\) and let \(\mathcal{R}\) be the corresponding subfamily of orderings of \(T\). From now on we will deal only with the sets \(\mathcal{P}\) and \(\mathcal{R}\).

In [3] we used the above idea of ordering \(T\) to develop the notions of \(P\)-exten-
Weakly Equicontinuous Flows

...syndetic, $P$-limit relation $L^p$, $P$-prolongation relation $D^p$, $P$-limit prolongation relation $J^p$, $P$-recurrent, $P$-almost periodic, $P$-characteristic 0 and $P$-dispersive.

In this section we will follow the definitions and the terminology of [3] and [7]. The phase group $T$ is assumed, throughout this section, to be abelian. The set $P$ denotes a certain replete semigroup in $T$ and $R=f(P)$ is the corresponding ordering of $T$.

**Definition 3.1.** A point $x \in X$ is said to be $P$-dispersive if for any point $y \in X$ there are open neighborhoods $U$ of $x$ and $V$ of $y$ and $s \in T$ such that $Ut \cap V=\emptyset$ for all $s < t(R)$. A point $x \in X$ is dispersive if it is $P$-dispersive for all $P \in \mathcal{P}$.

**Definition 3.2.** A point $x \in X$ is said to be $P$-wandering if there exist $s \in T$ and an open neighborhood $U$ of $x$ such that $UT \cap U=\emptyset$ for all $s < t(R)$. A point $x \in X$ is wandering if it is $P$-wandering for all $P \in \mathcal{P}$. A point $x \in X$ is $P$-nonwandering if it is not $Q$-wandering for all $Q \subset P$, $Q \in \mathcal{P}$ and it is non-wandering if it is $P$-nonwandering for all $P \in \mathcal{P}$.

**Definition 3.3.** A point $x \in X$ is $P$-regionally recurrent if for every open neighborhood $U$ of $x$ there exists a $P$-extensive set ([1] and [3]) $S$ in $T$ such that $U \cap Us \neq \emptyset$ for all $s \in S$.

**Theorem 3.4.** [3] A point $x \in X$ is $p$-dispersive iff $J^p(x) = \emptyset$.

**Theorem 3.5.** Let $(X, T)$ be a flow. Then the following statements are pairwise equivalent for a point $x \in X$.

(a) $x$ is $P$-regionally recurrent
(b) $x$ is $P$-nonwandering
(c) $x \in J^q(x)$ for all $Q \subset P$, $Q \in \mathcal{P}$.

**Proof.** (a)$\Rightarrow$(b). This follows from [3; 2.2, 2.3].

(b)$\Rightarrow$(c). Assume that $x$ is $P$-nonwandering. Let $\{ U_\alpha | \alpha \in I \}$ be a neighborhood filter of $x$ ordered by inclusion. Then for each $U_\alpha$, $t \in T$ and $Q \subset P$, $Q \in \mathcal{P}$ there is $s_\alpha \in T$ such that $t < s_\alpha(R)$, where $R=f(Q)$ and $U_\alpha \cap Us_\alpha \neq \emptyset$. Thus there exists a net $\{ x_\alpha | \alpha \in I \}$ converging to $x$ and such that the net $\{ x_\alpha t | (\alpha, t) \in I \times T \}$, where $T$ is directed by $R$, is frequently in every $U_\alpha$. It follows from [3; 2.8] that $x \in J^q(x)$.

(c)$\Rightarrow$(a). The proof can be accomplished by simply reversing the steps in the proof of (b)$\Rightarrow$(c).

**Lemma 3.6.** For any $y \in L^p(x)$ we have $J^p(x) \subset J^p(y)$.

**Proof.** The proof is simple and is thus omitted.

**Theorem 3.7.** Let $(X, T)$ be a weakly equicontinuous flow. Then $(X, T)$ is $P$-
dispersive iff it is $P$-wandering.

**Proof.** Assume that $(X, T)$ is $P$-dispersive. Then according to Theorem 3.4 we have $J^p(x) = \phi$ for every $x \in X$. Thus from Theorem 3.5 it follows that $(X, T)$ is $P$-wandering. Conversely, assume that $(X, T)$ is $P$-wandering. Suppose that there exists $x \in X$ such that $J^p(x) \neq \phi$. Since $\overline{xT} = D(x)$ is minimal (Corollary 2.11) and $J^p(x)$ is a closed and invariant subset of $D(x)$, $J^p(x) = \overline{xT}$. Consequently $x \in J^p(x)$ and hence we have a contradiction. The proof is now complete.

**Definition 3.8.** A point $x \in X$ is said to be of $P$-characteristic 0 if $xP = D^p(x)$ [3].

**Lemma 3.9** [3]. If a point $x \in X$ is of $P$-characteristic 0, then $L^p(x) = J^p(x)$.

**Theorem 3.10.** Let $(X, T)$ be a weakly equicontinuous flow. Then $(X \times X, T)$ is of $P$-characteristic 0 for all $P \in \mathcal{P}$.

**Proof.** The proof is similar to the proof of Theorem 2.10.

**Corollary 3.11.** Let $(X, T)$ be a weakly equicontinuous flow. Then for each $x \in X$ and $P \in \mathcal{P}$, $J^p(x) = L^p(x)$.

**Proof.** This follows from Theorem 3.10 and Lemma 3.9.

**Theorem 3.12.** Let $(X, T)$ be a weakly equicontinuous flow. Then $(X, T)$ is nonwandering iff it is pointwise almost periodic.

**Proof.** Assume that $(X, T)$ is nonwandering. Then from Theorem 3.5 we have $x \in J^p(x)$ for all $x \in X$ and $P \in \mathcal{P}$. Hence $(X, T)$ is pointwise recurrent [3; 3.7] or [1; 2.2]. This implies that $(X, T)$ is pointwise almost periodic [5; 7.05]. Conversely, assume that $(X, T)$ is pointwise almost periodic. Then $x \in L^p(x) = J^p(x)$ for all $x \in X$ and $P \in \mathcal{P}$ [3; 3.7]. Hence, $(X, T)$ is nonwandering.

Consider the following subsets of $X$. $X_1 = \{x \in X | J^p(x) \neq \phi \text{ for all } P \in \mathcal{P}\}$, $X_2 = \{x \in X | J^p(x) = \phi \text{ for some } P \in \mathcal{P}\}$, $X_3 = \text{The set of } T\text{-equicontinuous points in } X$ [5], $X_4 = X - X_3$: the set of $T\text{-nonequicontinuous points in } X$ [5].

**Theorem 3.13.** For a weakly equicontinuous flow $(X, T)$ the following statements are valid.

(a) $X = X_1 \cup X_3$ is a decomposition of $X$ into two disjoint open and closed invariant sets.

(b) The subflow $(X_1, T)$ is pointwise almost periodic and each point in $X_2$ is $P$-dispersive for some $P \in \mathcal{P}$.

(c) $X_1 \subset X_4$ and $X_3$ is invariant.

(d) If $X_3$ is connected, then either $X_4 = X_1$ or $X_4 = \phi$. 
We will give a proof of Theorem 3.5. Hence we follow [3].

Let $(X, T)$ be a weakly equicontinuous flow, where $X$ is assumed connected. If there exists a point in $X$ which is $P$-almost periodic for some $P \in \mathcal{P}$, then $(X, T)$ is pointwise equicontinuous.

Proof. Apply Theorem 3.13 and [1, 2.5].

References

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