Funkcialaj Ekvacioj, **24** (1981), 201-210

Stabilities for Equilibria of Competing-Species Reaction-Diffusion Equations with Homogeneous Dirichlet Condition

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§ 1. Introduction and preliminaries

This article can be considered as an extension of Leung's work in [7]. The reaction-diffusion equation

$$(1.1)\qquad \begin{aligned}\frac{\partial u_1}{\partial t}&=\sigma_1\Delta u_1+u_1[a+f_1(u_1, u_2)]\\ \frac{\partial u_2}{\partial t}&=\sigma_2\Delta u_2+u_2[b+f_2(u_1, u_2)]\end{aligned}$$

is studied, where $\Delta=\sum_{i=1}^n \partial^2/\partial x_i^2$, a, b, σ_1, σ_2 are positive constants. f_i: $R^2\rightarrow R$ have Hölder continuous partial derivatives up to second order in compact sets, $i=1, 2$. Further, we assume that

$$(1.2)\qquad \begin{aligned}&f_i(0, 0)=0, \quad i=1, 2;\\ &\frac{\partial f_i}{\partial u_j}<0, \text{ each } i, j=1 \text{ or } 2\end{aligned}$$

for (u_1, u_2) in the first open quadrant; and (for Theorem 2.1 only) there exists a positive constant C, such that

$$(1.3)\qquad b+f_2(0, C)<0.$$

The system (1.1) together with assumptions (1.2) is a model for biological competing species interactions, where $u_i(x, t)$, $i=1, 2$ represent the concentration of two species at position $x=(x_1, \cdots, x_n)$ and time $t\geq 0$. σ_1, σ_2 are diffusion rates; a, b are growth rates, and f_1, f_2 describe interactions as in [7].

Many studies related to (1.1) can be found in [1] to [3], [6] to [8], [12] and numerous others. Careful study of Dirichlet problems for (1.1) is made in [7]. However, the homogeneous condition for both species:

This research was partially supported by a grant from the National Science Foundation.

(1.4) $$u_i(x, t)\equiv 0,\ i=1, 2 \quad \text{for } t\geq 0,\ x \in \delta\mathscr{D},$$

where $\delta\mathscr{D}$ is the boundary of a bounded domain $\mathscr{D}$, had not been treated satisfactorily in [7]. (Except for Theorem 3.2, when both a and b are small relative to σ_1, σ_2 respectively). When one or both a and b are not small, the existence of nontrivial equilibrium states had been observed in Theorem 3.3 and remark (ii) in [7]. However, their stabilities had not been studied. The construction for the upper and lower solutions in [7] simply do not carry over easily to the case when both u_i vanish on the boundary and one or both a, b are not small. In order to construct the upper and lower solutions for stability analysis in these situations, more elaborate procedures had to be made in order to deal with condition (1.4). This article carefully studies this stability problem, which is of practical as well as theoretical interest.

Let $\lambda_1 > 0$ be the principal eigenvalue for the Dirichlet problem $\Delta u+\lambda u=0$ in $\mathscr{D}$, $u=0$ on $\delta\mathscr{D}$. Vaguely speaking, this article proves that when $a<\sigma_1\lambda_1$, $b>\sigma_2\lambda_1$, u_1 tends to zero and u_2 has a stable positive equilibrium in the interior. When $a>\sigma_1\lambda_1$, and $b>\sigma_2\lambda_1$, there exists a stable equilibrium with both u_1, u_2 positive in the interior, provided suitable conditions are satisfied. One can interpret that stable positive equilibrium bifurcates from 0 when the growth rates a, b increase, or diffusion rates σ_1, σ_2 decrease. Of course, the size of the domain, which determines λ_1, is also an important parameter in the bifurcation process.

We consider equation (1.1) for $x=(x_1, \cdots, x_n) \in \mathscr{D}$, where $\mathscr{D}$ is a bounded open connected subset of R^n, $n\geq 1$ with boundary $\delta\mathscr{D}$. Let $H^{2+l}(\bar{\mathscr{D}})$ denote the Banach space of all real-valued functions u continuous on $\bar{\mathscr{D}}$ with all first and second derivatives also continuous in $\bar{\mathscr{D}}$, with finite value for the norm

$$|u|_G^{(2+l)}=\sum_{0\leq|\alpha|\leq 2}\sup_{\mathscr{D}}|D^\alpha u|+\sum_{|\alpha|=2}\sup\frac{|[D^\alpha u](x)-[D^\alpha u](y)|}{|x-y|^l}.$$

We assume that $\delta\mathscr{D} \in H^{2+l}$. For any $T>0$, let $\mathscr{D}_T=\mathscr{D}\times(0, T)$. $H^{2+l,1+l/2}(\bar{\mathscr{D}}_T)$ denotes the Banach space of all real-valued functions u having all derivatives of the form $D^\alpha D_t^r u$ with $2r+|\alpha|\leq 2$ continuous on $\bar{\mathscr{D}}_T$ and having finite norm $|u|_{\mathscr{D}_T}^{2+l}$, as indicated in [7]. For more details of these symbols and norms, see [7] or [12]. Finally, let ν denote the outward unit normal at the boundary $\delta\mathscr{D}$ of $\mathscr{D}$.

§ 2. Stability for equilibria with homogeneous Dirichlet data

The main results of this article are Theorems 2.1 and 2.2. The following lemma had been proved in Section 2 of [7], and will be used later. We state it here for convenience:

Lemma 2.1. *Let $v_i(x, t)$, $w_i(x, t)$, $(x, t) \in \bar{\mathscr{D}}\times[t_0, \infty)$, $i=1, 2$ be functions in $H^{2+l,1+l/2}(\bar{\mathscr{D}}\times[t_0, T])$, each $T>t_0$, satisfying the inequalities:*

$$0 \le v_i \le w_i, \quad i=1,2,$$

$$\sigma_1 \Delta v_1 + v_1[a+f_1(v_1, w_2)] - \frac{\partial v_1}{\partial t} \ge 0$$

(2.1)
$$\sigma_1 \Delta w_1 + w_1[a+f_1(w_1, v_2)] - \frac{\partial w_1}{\partial t} \le 0$$

$$\sigma_2 \Delta v_2 + v_2[b+f_2(w_1, v_2)] - \frac{\partial v_2}{\partial t} \ge 0$$

$$\sigma_2 \Delta w_2 + w_2[b+f_2(v_1, w_2)] - \frac{\partial w_2}{\partial t} \le 0.$$

Let $(u_1(x,t), u_2(x,t))$ *with* $u_i \in H^{2+l,1+l/2}(\bar{\mathscr{D}} \times [t_0, T])$, *each* $T > t_0$, $i=1,2$, *be a solution of the reaction-diffusion equations* (1.1) *with initial-boundary conditions such that*

(2.2)
$$v_i(x, t_0) \le u_i(x, t_0) \le w_i(x, t_0), \quad x \in \mathscr{D},\ i=1,2,$$
$$v_i(x, t) \le u_i(x, t) \le w_i(x, t), \quad (x,t) \in \delta\mathscr{D} \times [t_0, \infty),\ i=1,2.$$

Then $(u_1(x,t), u_2(x,t))$ *will satisfy*

(2.3)
$$v_i(x,t) \le u_i(x,t) \le w_i(x,t), \quad (x,t) \in \bar{\mathscr{D}} \times [t_0, \infty).$$

We shall refer to v_1, v_2 as lower solutions and w_1, w_2 as upper solutions of the respective equations in (1.1) and initial boundary conditions associated with (2.2).

When $a < \sigma_1 \lambda_1$ and $b > \sigma_2 \lambda_1$, we now prove that under the homogeneous Dirichlet condition (1.4), there is a stable solution of (1.1) with $u_1 \equiv 0$ and $u_2(x) > 0$ for $x \in \mathscr{D}$.

Theorem 2.1. *Suppose* $a < \sigma_1 \lambda_1$, $b > \sigma_2 \lambda_1$, *and* $u_2^*(x) \in H^{2+l}(\bar{\mathscr{D}})$ *is a solution of*

$$\sigma_2 \Delta u + u[b + f_2(0, u)] = 0 \text{ in } \mathscr{D},\ u = 0 \text{ on } \delta\mathscr{D},$$

with $u_2^*(x) > 0$ *for* $x \in \mathscr{D}$. *Let* $(u_1(x,t), u_2(x,t))$ *with* $u_i \in H^{2+l,1+l/2}(\bar{\mathscr{D}} \times [0, T])$, *each* $T > 0$, $i=1,2$, *be a solution of the reaction-diffusion equations* (1.1) *with initial-boundary conditions*

$$u_i(x, 0) = \theta_i(x) \ge 0, \quad x \in \bar{\mathscr{D}},$$
$$u_i(x, t) = 0, \quad x \in \delta\mathscr{D},\ t \ge 0,$$

$i=1,2$, *where* $\theta_i \in H^{2+l}(\bar{\mathscr{D}})$, $i=1,2$ *satisfy the compatibility conditions of order* 1 *at* $t=0$ *as described in* [5], p. 319. *Then* $(u_1(x,t), u_2(x,t)) \to (0, u_2^*(x))$, *as* $t \to \infty$, *uniformly for* $x \in \bar{\mathscr{D}}$, *provided that* θ_1, θ_2 *and all their first partial derivatives are close enough to* 0, u_2^* *respectively and their corresponding first partial derivatives.*

Proof. The fact that $u_2^*(x)$ exists can be readily seen as follows. Let $\omega(x)$ be the principal eigenfunction for the eigenvalue problem $\Delta u+\lambda u=0$ in $\mathscr{D}$, $u=0$ on $\delta\mathscr{D}$, with $\lambda=\lambda_1$ as the principal eigenvalue (thus $\omega(x)>0$ in $\mathscr{D}$). For $\delta>0$ sufficiently small, we have $\sigma_2\Delta(\delta\omega)+(\delta\omega)[b+f_2(0,\delta\omega)]=\delta\omega[-\sigma_2\lambda_1+b+f_2(0,\delta\omega)]>0$. While for $K>0$ sufficiently large (1.3) implies that $\sigma_2\Delta K+K[b+f_2(0,K)]<0$. Thus by [11] or [2], $u_2^*(x)$ exists in $H^{2+l}(\bar{\mathscr{D}})$ with $\delta\omega(x)<u_2^*(x)<K$, $x\in\bar{\mathscr{D}}$. We now proceed to apply Lemma 2.1 by constructing appropriate v_i, w_i, $i=1,2$. Let $v_1(x,t)\equiv 0$ for $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$. If $\theta_1(x)\equiv 0$, define $w_1(x,t)\equiv 0$; otherwise, define $w_1(x,t)$ as the solution of the initial boundary value problem: $\sigma_1\Delta w_1+w_1[a+f_1(w_1,0)]-\partial w_1/\partial t=0$ for $(x,t)\in\mathscr{D}\times(0,\infty)$, $w_1(x,0)=\theta_1(x)$ for $x\in\mathscr{D}$, $w_1(x,t)=0$ for $(x,t)\in\delta\mathscr{D}\times[0,\infty)$. Exactly as in Theorem 3.1 in [7], we have $0\le w_1(x,t)\le K_1e^{-\alpha_1 t}$ for $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$, and some positive constants K_1, α_1.

We let $v_2(x,t)=[1-k(x)e^{-lt}]u_2^*(x)$, $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$, where $k(x)=-\varepsilon u_2^*(x)+C$, $0<C<1$, and l, ε are positive small constants to be chosen. We have

$$
\begin{aligned}
&\sigma_2\Delta v_2+v_2[b+f_2(w_1,v_2)]-\frac{\partial v_2}{\partial t}\\
&\quad=v_2[f_2(w_1,v_2)-f_2(0,v_2)+f_2(0,v_2)-f_2(0,u_2^*)]\\
&\qquad-e^{-lt}\Big[\sigma_2(\Delta k)u_2^*+2\sum_{i=1}^{n}k_{x_i}u_{2x_i}^*\sigma_2+klu_2^*\Big]\\
&\quad\ge u_2^*\Big[-qKe^{-\alpha_1 t}+(1-C)\min_{(1-C)\le s\le 1}\Big|\frac{\partial f_2}{\partial u_2}(0,su_2^*)\Big|k(x)e^{-lt}u_2^*\\
&\qquad-e^{-lt}(\sigma_2\Delta k+kl)\Big]-2\sigma_2e^{-lt}\sum_{i=1}^{n}k_{x_i}u_{2x_i}^*.
\end{aligned}
\tag{2.4}
$$

Here $-q=\min_{x\in\bar{\mathscr{D}}}\min_{\substack{0\le s_1\le K\\ 1-C\le s_2\le 1}}(\partial f_2/\partial u_1)(s_1,s_2u_2^*(x))<0$, and ε is small enough such that $k(x)>0$ in $\bar{\mathscr{D}}$. Choose $l=\varepsilon\sigma_2$, thus $|u_2^*e^{-lt}(\sigma_2\Delta k+kl)|=|u_2^*e^{-lt}(-\sigma_2\varepsilon\Delta u_2^*+\varepsilon\sigma_2(-\varepsilon u_2^*+C))|\le\varepsilon\sigma_2u_2^*e^{-lt}R$, for some constant $R>0$ independent of ε, for all $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$. In a neighborhood $\mathscr{F}$ of $\delta\mathscr{D}$ in $\bar{\mathscr{D}}$, we have $-2\sigma_2e^{-lt}\sum_{i=1}^{n}k_{x_i}u_{2x_i}^*-\varepsilon\sigma_2u_2^*e^{-lt}R=2\sigma_2e^{-lt}\varepsilon\sum_{i=1}^{n}u_{2x_i}^{*2}-\varepsilon\sigma_2u_2^*e^{-lt}R>\sigma_2\varepsilon e^{-lt}P$ for some positive constant P independent of ε (because $u_2^*(x)>\delta\omega(x)$, thus $\sum_{i=1}^{n}u_{2x_i}^{*2}\ne 0$ on $\delta\mathscr{D}$). Further, the first term $-u_2^*qKe^{-\alpha_1 t}$ in the last line of (2.4) will in absolute value $<(1/2)\sigma_2\varepsilon e^{-lt}P$, provided that ε is small enough so that $l=\varepsilon\sigma_2<\alpha_1$, and t is large enough; and the second term in the same line is always ≥ 0 for $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$. Consequently, $\sigma_2\Delta v_2+v_2[b+f_2(w_1,v_2)]-\partial v_2/\partial t\ge 0$ for $x\in\mathscr{F}$, t large enough. In the complement of $\mathscr{F}$ in $\mathscr{D}$, the second term in the last line of (2.4) is bounded below by Qe^{-lt} for some positive constant Q which can remain unchanged if $\varepsilon>0$ is reduced. The remaining terms will have absolute value $<(Q/2)e^{-lt}$ for small enough $\varepsilon>0$ and t large enough, as before. Therefore the expression in (2.4) is ≥ 0, for $x\in\mathscr{D}\backslash\mathscr{F}$, t large enough; and $v_2(x,t)$ is a lower solution for $t\ge\bar{T}$, for some large $\bar{T}$.

Next, we let $w_2(x,t)=[1+\hat{k}(x)e^{-\hat{l}t}]u_2^*(x)$, $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$, where $\hat{k}(x)=\hat{C}-$

$\hat{\varepsilon}u_2^*(x)$, and $\hat{C}$, $\hat{l}$, $\hat{\varepsilon}$ are positive constants to be chosen ($\hat{\varepsilon}$ at least small enough so that $\hat{k}(x)>0$ in $\bar{\mathscr{D}}$). We have

$$\begin{aligned}(2.5)\qquad &\sigma_2\Delta w_2+w_2[b+f_2(0,w_2)]-\frac{\partial w_2}{\partial t}\\ &=w_2[f_2(0,w_2)-f_2(0,u_2^*)]+e^{-\hat{l}t}\Big[\sigma_2(\Delta\hat{k})u_2^*+2\sigma_2\sum_{i=1}^n \hat{k}_{x_i}u_{2x_i}^*+\hat{k}\hat{l}u_2^*\Big]\\ &\le u_2^*\Big[\min_{0\le s\le 1+\hat{C}}\Big|\frac{\partial f_2}{\partial u_2}(0,s)\Big|(-\hat{k}e^{-\hat{l}t}u_2^*)+e^{-\hat{l}t}(\sigma_2\Delta\hat{k}+\hat{k}\hat{l})\Big]\\ &\qquad\qquad +2\sigma_2e^{-\hat{l}t}\sum_{i=1}^n \hat{k}_{x_i}u_{2x_i}^*.\end{aligned}$$

Choose $\hat{l}=\hat{\varepsilon}\sigma_2$, thus $|u_2^*e^{-\hat{l}t}(\sigma_2\Delta\hat{k}+\hat{k}\hat{l})|\le\hat{\varepsilon}\sigma_2u_2^*e^{-\hat{l}t}\hat{R}$, for some $\hat{R}>0$ independent of $\hat{\varepsilon}$, for all $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$. In a neighborhood $\mathscr{F}$ of $\partial\mathscr{D}$ in $\bar{\mathscr{D}}$, we have

$$2\sigma_2e^{-\hat{l}t}\sum_{i=1}^n \hat{k}_{x_i}u_{2x_i}^*+\hat{\varepsilon}\sigma_2u_2^*e^{-\hat{l}t}\hat{R}=-2\sigma_2e^{-\hat{l}t}\hat{\varepsilon}\sum_{i=1}^n u_{2x_i}^{*2}+\hat{\varepsilon}\sigma_2u_2^*e^{-\hat{l}t}\hat{R}<-\sigma_2\hat{\varepsilon}e^{-\hat{l}t}\hat{P}<0,$$

for some positive constant $\hat{P}$ independent of $\hat{\varepsilon}$. Consequently $\sigma_2\Delta w_2+w_2[b+f_2(0,w_2)]-\partial w_2/\partial t<0$ for $x\in\mathscr{F}$, $t\ge 0$. In the complement of $\mathscr{F}$ in $\bar{\mathscr{D}}$, the first term in the last line of (2.5) is bounded above by $-\hat{Q}e^{-\hat{l}t}$ for some positive constant $\hat{Q}$, and the remaining terms will have absolute value $<(\hat{Q}/2)e^{-\hat{l}t}$ for small enough $\hat{\varepsilon}$. The expression in (2.5) is therefore <0 for $x\in\bar{\mathscr{D}}\backslash\mathscr{F}$, $t\ge 0$. $w_2(x,t)$ is an upper solution.

Since $u_2^*(x)>\delta\omega(x)$, $\delta>0$, and $u_2^*\in H^{2+l}(\bar{\mathscr{D}})$, the outward unit normal derivative of u_2^* is bounded above by a negative constant. Thus, by choosing $\theta_1(x)$, $\theta_2(x)$ and their first partial derivative to be sufficiently close to that of 0 and $u_2^*(x)$ and their first partial derivatives respectively, we have $v_i(x,\bar{T})\le u_i(x,\bar{T})\le w_i(x,\bar{T})$ for $x\in\bar{\mathscr{D}}$, $i=1,2$. By Lemma 2.1, we conclude that $v_i(x,t)\le u_i(x,t)\le w_i(x,t)$, for $(x,t)\in\bar{\mathscr{D}}\times[\bar{T},\infty)$, $i=1,2$. By the choice of v_i, w_i, we see that $(u_1(x,t),u_2(x,t))\to(0,u_2^*(x))$ as $t\to\infty$, uniformly for $x\in\bar{\mathscr{D}}$. This completes the proof.

The more interesting case occurs when both $a>\sigma_1\lambda_1$ and $b>\sigma_2\lambda_1$. One looks for the possibility of an equilibrium state for the homogeneous Dirichlet problem with *both* species u_i having positive concentrations in the interior. One also looks for conditions with which these positive solutions become asymptotically stable as $t\to\infty$.

Theorem 2.2. *Suppose* $a>\sigma_1\lambda_1$, $b>\sigma_2\lambda_1$, *let* $(\bar{u}_1(x),\bar{u}_2(x))$, *with* $\bar{u}_i(x)\in H^{2+l}(\bar{\mathscr{D}})$, $i=1,2$ *be a solution of the homogeneous Dirichlet problem* (1.1), (1.2) *and* (1.4). *Suppose that* $\bar{u}_i(x)>0$ *in* $\mathscr{D}$, $\partial\bar{u}_i/\partial\nu<0$ *on* $\partial\mathscr{D}$, $i=1,2$, *and*

$$(2.6)\qquad \sup_{x\in\mathscr{D}}\left|\frac{\bar{u}_i(x)}{\bar{u}_j(x)}\cdot\frac{(\partial f_j/\partial u_i)(\bar{u}_1(x),\bar{u}_2(x))}{(\partial f_j/\partial u_j)(\bar{u}_1(x),\bar{u}_2(x))}\right|<\inf_{x\in\mathscr{D}}\left|\frac{\bar{u}_i(x)}{\bar{u}_j(x)}\cdot\frac{(\partial f_i/\partial u_i)(\bar{u}_1(x),\bar{u}_2(x))}{(\partial f_i/\partial u_j)(\bar{u}_1(x),\bar{u}_2(x))}\right|<\infty$$

for each $1\le i,j\le 2$, $i\ne j$, *then* $(\bar{u}_1(x),\bar{u}_2(x))$ *is asymptotically stable.* (Here, asymptotic stability is interpreted to mean that for any solution $(u_1(x,t), u_2(x,t))$ with $u_i \in H^{2+l,1+l/2}(\bar{\mathscr{D}}\times[0,T])$, each $T>0$, $i=1,2$, of the reaction-diffusion equations (1.1) with boundary conditions $u_i(x,t)=0$, and initial conditions $u_i(x,0)$ whose values and first partial derivatives are close enough to that of $\bar{u}_i(x)$ respectively for all $x \in \bar{\mathscr{D}}$, $i=1,2$, one has $u_i(x,t)\to\bar{u}_i(x)$ uniformly as $t\to+\infty$, $i=1,2$.) (Remark: The existence of a solution $(\bar{u}_1(x), \bar{u}_2(x))$ with all the properties described will be shown later in an example.)

Proof: Assumption (2.6) implies that there are ρ_1, ρ_2 close enough to 1 with $\rho_1<1<\rho_2$ such that for each $x \in \mathscr{D}$,

$$0<\frac{\bar{u}_i(x)}{\bar{u}_j(x)}\frac{\max_{\rho_1\le s,\tau\le\rho_2}|(\partial f_j/\partial u_i)(s\bar{u}_1(x),\tau\bar{u}_2(x))|}{\min_{\rho_1\le s\le 1}|(\partial f_j/\partial u_j)(s\bar{u}_1(x),\bar{u}_2(x))|}$$
$$<\inf_{x\in\mathscr{D}}\frac{\bar{u}_i(x)}{\bar{u}_j(x)}\left\{\frac{\min_{\rho_1\le s,\tau\le\rho_2}|(\partial f_i/\partial u_i)(s\bar{u}_1(x),\tau\bar{u}_2(x))|}{\max_{\rho_1\le s\le 1}|(\partial f_i/\partial u_j)(s\bar{u}_1(x),\bar{u}_2(x))|}\right\}-\varepsilon_1<\infty \tag{2.7}$$

for each $1\le i,j\le 2$, $i\ne j$, where ε_1 is a small positive number. We will construct appropriate lower and upper solutions v_i, w_i, and apply Lemma 2.1. Let

$$G(x)=\bar{u}_2(x)\min_{\rho_1\le s\le\tau\le\rho_2}\left|\frac{\partial f_2}{\partial u_2}(s\bar{u}_1(x),\tau\bar{u}_2(x))\right|\cdot\left(\bar{u}_1(x)\max_{\rho_1\le s\le 1}\left|\frac{\partial f_2}{\partial u_1}(s\bar{u}_1(x),\bar{u}_2(x))\right|\right)^{-1},$$

for $x \in \mathscr{D}$; and let α be a number, $1<\alpha<\rho_2$ such that $(1-\rho_1)>(\alpha-1)\inf_{x\in\mathscr{D}} G(x)$. Define $w_2(x,t)=p(x,t)\bar{u}_2(x)$, $p(x,t)=1+(\alpha-1-\varepsilon_4\bar{u}_2(x))e^{-mt}$, where ε_4 and m are positive constants to be determined later (one condition on ε_4 is $\varepsilon_4\max_{x\in\bar{\mathscr{D}}}\bar{u}_2(x)<\alpha-1$). On the other hand, define $v_1(x,t)=q(x,t)\bar{u}_1(x)$, $q(x,t)=1-(1-\beta(x))e^{-mt}$, where $\beta(x)=1-(\alpha-1)\inf_{x\in\mathscr{D}}G(x)+\varepsilon_2(\alpha-1)+\varepsilon_3(\alpha-1)\bar{u}_1(x)$, ε_2 and ε_3 are small positive constants satisfying $\varepsilon_2+\varepsilon_3\max_{x\in\bar{\mathscr{D}}}\bar{u}_1(x)<\varepsilon_1<\inf_{x\in\mathscr{D}}G(x)$. (Observe that $\rho_1<\beta(x)<1$). We have

$$\begin{aligned}
&\sigma_2\Delta w_2[b+f_2(v_1,w_2)]-\frac{\partial w_2}{\partial t}\\
&\quad=p(x,t)\bar{u}_2[f_2(v_1,w_2)-f_2(v_1,\bar{u}_2)+f_2(v_1,\bar{u}_2)-f_2(\bar{u}_1,\bar{u}_2)]\\
&\qquad+e^{-mt}\left[m(\alpha-1-\varepsilon_4\bar{u}_2(x))\bar{u}_2-\bar{u}_2\sigma_2\varepsilon_4\Delta\bar{u}_2-2\sigma_2\varepsilon_4\sum_{i=1}^{n}\bar{u}_{2x_i}^2\right]\\
&\quad\le p(x,t)\bar{u}_2\left[\max_{1\le\tau\le\rho_2}\left\{\frac{\partial f_2}{\partial u_2}(v_1,\tau\bar{u}_2)\right\}\{(\alpha-1)\bar{u}_2e^{-mt}-\varepsilon_4\bar{u}_2^2e^{-mt}\}\right.\\
&\qquad\left.-\min_{\rho_1\le s\le 1}\left\{\frac{\partial f_2}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\right\}\cdot\{(1-\hat{\beta})\bar{u}_1e^{-mt}-\varepsilon_3(\alpha-1)\bar{u}_1^2e^{-mt}\}\right]+e^{-mt}[\cdots]
\end{aligned} \tag{2.8}$$

where $[\cdots]$ represents the terms inside the brackets immediately before the inequality sign $\le$, and $\hat{\beta}=1-(\alpha-1)\inf_{x\in\mathscr{D}}G(x)+\varepsilon_2(\alpha-1)$. Set $\varepsilon_4=m=\varepsilon_3$; thus

$$\Big| p(x,t)\bar{u}_2\Big[\max_{1\le\tau\le\rho_2}\Big\{\frac{\partial f_2}{\partial u_2}(v_1,\tau\bar{u}_2)\Big\}(-\varepsilon_4\bar{u}_2^2e^{-mt})$$
$$+\min_{\rho_1\le s\le 1}\Big\{\frac{\partial f_2}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\Big\}\varepsilon_3(\alpha-1)\bar{u}_1^2e^{-mt}\Big]$$
$$+e^{-mt}[m(\alpha-1-\varepsilon_4\bar{u}_2(x))\bar{u}_2-\bar{u}_2\sigma_2\varepsilon_4\varDelta\bar{u}_2]\Big|\le\varepsilon_4e^{-mt}\bar{u}_2(x)K_1$$

for all $x\in\mathscr{D}$, where K_1 is some positive constant. In a neighborhood Ω of $\partial\mathscr{D}$ in $\bar{\mathscr{D}}$, we have $-2\sigma_2\varepsilon_4\sum_{i=1}^n\bar{u}_{2x_i}^2e^{-mt}+\varepsilon_4e^{-mt}\bar{u}_2(x)K_1<0$, for all $t\ge 0$, since $\bar{u}_2=0$ on $\partial\mathscr{D}$. Further,

$$\max_{1\le\tau\le\rho_2}\Big\{\frac{\partial f_2}{\partial u_2}(v_1(x,t),\tau\bar{u}_2(x))\Big\}(\alpha-1)\bar{u}_2(x)$$
$$-\min_{\rho_1\le s\le 1}\Big\{\frac{\partial f_2}{\partial u_1}(s\bar{u}_1(x),\bar{u}_2(x))\Big\}(1-\hat{\beta})\bar{u}_1(x)$$
$$\le\max_{\rho_1\le s\le\tau\le\rho_2}\Big\{\frac{\partial f_2}{\partial u_2}(s\bar{u}_1(x),\tau\bar{u}_2(x))\Big\}(\alpha-1)\bar{u}_2(x)$$
$$+\max_{\rho_1\le s\le 1}\Big|\frac{\partial f_2}{\partial u_1}(s\bar{u}_1(x),\bar{u}_2(x))\Big|((\alpha-1)G(x)-\varepsilon_2(\alpha-1))\bar{u}_1(x)$$
$$=-\min_{\rho_1\le s\le\tau\le\rho_2}\Big|\frac{\partial f_2}{\partial u_2}(s\bar{u}_1(x),\tau\bar{u}_2(x))\Big|(\alpha-1)\bar{u}_2(x)$$
$$+\bar{u}_2(x)\min_{\rho_1\le s\le\tau\le\rho_2}\Big|\frac{\partial f_2}{\partial u_2}(s\bar{u}_1(x),\tau\bar{u}_2(x))\Big|(\alpha-1)$$
$$-\varepsilon_2(\alpha-1)\bar{u}_1(x)\max_{\rho_1\le s\le 1}\Big|\frac{\partial f_2}{\partial u_1}(s\bar{u}_1(x),\bar{u}_2(x))\Big|<0,$$

for all $x\in\mathscr{D}$, $t\ge 0$. Consequently, we have $\sigma_2\varDelta w_2+w_2[b+f_2(v_1,w_2)]-\partial w_2/\partial t<0$, for $x\in\Omega$, $t\ge 0$. For $x\in\mathscr{D}\backslash\Omega$, two terms in (2.8) satisfy the inequality:

$$p(x,t)\bar{u}_2\Big[\max_{1\le\tau\le\rho_2}\Big\{\frac{\partial f_2}{\partial u_2}(v_1,\tau\bar{u}_2)\Big\}(\alpha-1)\bar{u}_2e^{-mt}$$
$$-\min_{\rho_1\le s\le 1}\Big\{\frac{\partial f_2}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\Big\}(1-\hat{\beta})\bar{u}_1(x)e^{-mt}<-\varepsilon_2K_2e^{-mt},$$

for some $K_2>0$, all $t\ge 0$; and for such (x,t), the sum of all the other remaining terms after the inequality sign $\le$ in (2.8) can be reduced to less than $(1/2)\varepsilon_2K_2e^{-mt}$ in absolute value, by choosing $\varepsilon_4=m=\varepsilon_3$ sufficiently small. We therefore have $\sigma_2\varDelta w_2+w_2[b+f_2(v_1,w_2)]-\partial w_2/\partial t<0$, for $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$, and $w_2(x,t)$ is an upper solution.

For v_1, we have the inequality:

$$\sigma_1 \Delta v_1 + v_1[a+f_1(v_1, w_2)] - \frac{\partial v_1}{\partial t}$$

$$\begin{aligned} &= q(x,t)\bar{u}_1[f_1(v_1,w_2)-f_1(v_1,\bar{u}_2)+f_1(v_1,\bar{u}_2)-f_1(\bar{u}_1,\bar{u}_2)] \\ &\quad + e^{-mt}\Big[-m(1-\beta(x))\bar{u}_1+\bar{u}_1\sigma_1\varepsilon_3(\alpha-1)\Delta\bar{u}_1+2\sigma_1\varepsilon_3(\alpha-1)\sum_{i=1}^n \bar{u}_{1x_i}^2\Big] \\ &\geq q(x,t)\bar{u}_1\Big[\min_{1\leq\tau\leq\rho_2}\Big\{\frac{\partial f_1}{\partial u_2}(v_1,\tau\bar{u}_2)\Big\}\{(\alpha-1)\bar{u}_2e^{-mt}-\varepsilon_4\bar{u}_2^2e^{-mt}\} \\ &\quad - \max_{\rho_1\leq s\leq 1}\Big\{\frac{\partial f_1}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\Big\}\cdot\{(1-\hat{\beta})\bar{u}_1e^{-mt}-\varepsilon_3(\alpha-1)\bar{u}_1^2e^{-mt}\}\Big]+e^{-mt}[\cdots] \end{aligned} \tag{2.9}$$

where $[\cdots]$ represents the terms inside the brackets immediately before the inequality sign $\geq$. Due to the choice $\varepsilon_4 = m = \varepsilon_3$ made previously, one has the inequality

$$\begin{aligned} &\Big| q(x,t)\bar{u}_1\Big[\min_{1\leq\tau\leq\rho_2}\Big\{\frac{\partial f_1}{\partial u_2}(v_1,\tau\bar{u}_2)\Big\}(-\varepsilon_4\bar{u}_2^2e^{-mt}) \\ &\quad - \max_{\rho_1\leq s\leq 1}\Big\{\frac{\partial f_1}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\Big\}(-\varepsilon_3(\alpha-1)\bar{u}_1^2e^{-mt})\Big] \\ &\quad + e^{-mt}[-m(1-\beta(x))\bar{u}_1+\bar{u}_1\sigma_1\varepsilon_3(\alpha-1)\Delta\bar{u}_1]\Big| \leq \varepsilon_4 e^{-mt}\bar{u}_1(x)K_3 \end{aligned}$$

for all $x \in \bar{\mathscr{D}}$, where K_3 is some positive constant. In a neighborhood $\tilde{\Omega}$ of $\partial\mathscr{D}$ in $\bar{\mathscr{D}}$, we have $2\sigma_1\varepsilon_3(\alpha-1)\sum_{i=1}^n \bar{u}_{1x_i}^2 e^{-mt} - \varepsilon_4 e^{-mt}\bar{u}_1(x)K_3 > 0$, for all $t \geq 0$, since $\bar{u}_1 = 0$ on $\partial\mathscr{D}$. Further,

$$\begin{aligned} &\min_{1\leq\tau\leq\rho_2}\Big\{\frac{\partial f_1}{\partial u_2}(v_1,\tau\bar{u}_2)\Big\}(\alpha-1)\bar{u}_2 - \max_{\rho_1\leq s\leq 1}\Big\{\frac{\partial f_1}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\Big\}(1-\hat{\beta})\bar{u}_1 \\ &\geq -\max_{\rho_1\leq s,\tau\leq\rho_2}\Big|\frac{\partial f_1}{\partial u_2}(s\bar{u}_1,\tau\bar{u}_2)\Big|(\alpha-1)\bar{u}_2 + \min_{\rho_1\leq s\leq 1}\Big|\frac{\partial f_1}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\Big|\cdot(1-\hat{\beta})\bar{u}_1 \\ &\geq -(\alpha-1)\bar{u}_1\min_{\rho_1\leq s\leq 1}\Big|\frac{\partial f_1}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\Big|\Big(\inf_{x\in\mathscr{D}} G(x)-\varepsilon_1\Big) \\ &\quad + \min_{\rho_1\leq s\leq 1}\Big|\frac{\partial f_1}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\Big|\cdot(1-\hat{\beta})\bar{u}_1 \\ &= -\bar{u}_1(x)\min_{\rho_1\leq s\leq 1}\Big|\frac{\partial f_1}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\Big|\cdot(\varepsilon_2-\varepsilon_1)(\alpha-1) > 0, \end{aligned}$$

for all $x \in \mathscr{D}$, $t \geq 0$. The second $\geq$ sign in the last sentence is due to hypothesis (2.6). Consequently, we have $\sigma_1\Delta v_1 + v_1[a+f_1(v_1,w_2)] - \partial v_1/\partial t > 0$, for $x \in \tilde{\Omega}$, $t \geq 0$. For $x \in \mathscr{D}\backslash\tilde{\Omega}$, two terms in (2.9) satisfy the inequality:

$$q(x,t)\bar{u}_1\Big[\min_{1\leq\tau\leq\rho_2}\Big\{\frac{\partial f_1}{\partial u_2}(v_1,\tau\bar{u}_2)\Big\}\cdot(\alpha-1)\bar{u}_2e^{-mt}$$

$$-\max_{\rho_1\leq s\leq 1}\left\{\frac{\partial f_1}{\partial u_1}(s\bar{u}_1,\bar{u}_2)\right\}(1-\beta)\bar{u}_1e^{-mt}\Big]>(\varepsilon_1-\varepsilon_2)K_4e^{-mt}$$

for some $K_4>0$, all $t\geq 0$; and for such (x, t), the sum of all the other remaining terms after the inequality sign $\geq$ in (2.9) can be reduced to less than $(1/2)(\varepsilon_1-\varepsilon_2)K_4e^{-mt}$ in absolute value, by reducing the size of $\varepsilon_4=m=\varepsilon_3$. We therefore have $\sigma_1\Delta v_1+v_1[a+f_1(v_1, w_2)]-\partial v_1/\partial t>0$, for $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$, and $v_1(x,t)$ is a lower solution.

Since all the first partial derivatives of f_1 and f_2 have the same sign, we can interchange the role of $\bar{u}_1, f_1$ with $\bar{u}_2, f_2$ respectively and construct lower and upper solutions v_2, w_1 in exactly the same manner as before. Here v_2, w_1 are of the form $v_2=\tilde{q}(x,t)\bar{u}_2(x)$, $w_1=\tilde{p}(x,t)\bar{u}_1(x)$ with $\tilde{p}(x,t)$, $\tilde{q}(x,t)$ analogous to $p(x,t)$, $q(x,t)$ respectively. ($\tilde{p}(x,t)\to 1^+$, $\tilde{q}(x,t)\to 1^-$, as $t\to\infty$, all $x\in\bar{\mathscr{D}}$).

Finally, we have $v_i(x,t)\to\bar{u}_i(x)$ from below, and $w_i(x,t)\to\bar{u}_i(x)$ from above, as $t\to\infty$, uniformly for $x\in\bar{\mathscr{D}}$, $i=1,2$. When the initial conditions $u_i(x,0)$ and their partial derivatives are close to that of $\bar{u}_i(x)$ in the sense described in the theorem, we have $v_i(x,0)\leq u_i(x,0)\leq w_i(x,0)$, $x\in\mathscr{D}$. (Note that we have $\partial\bar{u}_i/\partial\nu<0$ on $\delta\mathscr{D}$). Applying Lemma 2.1, we clearly have $(\bar{u}_1(x), \bar{u}_2(x))$ as an asymptotically stable solution.

As an example for a solution of (1.1) satisfying all the properties described in Theorem 2.2, we consider

$$\text{(2.10)}\qquad \begin{aligned}&450u_1''(x)+u_1[500-494u_1-u_2]=0\\&450u_2''(x)+u_2[500-u_1-494u_2]=0\end{aligned}\qquad \text{for } x\in(0,\pi),\quad '=\frac{d}{dx},$$

with boundary condition $u_i(0)=u_i(\pi)=0$ for $i=1,2$. The function $\phi_1(x)=.05\sin x$ satisfies $450\phi_1''+\phi_1[500\text{–}494\phi_1-\tilde{u}_2]=\phi_1[50\text{–}24.7\sin x-\tilde{u}_2]\geq 0$, for each $0\leq\tilde{u}_2\leq 2$, $x\in[0,\pi]$. On the other hand $\phi_2(x)\equiv 2$ satisfies $450\phi_2''+\phi_2[500\text{–}494\phi_2-\tilde{u}_2]=2[-488-\tilde{u}_2]<0$, for each $0\leq\tilde{u}_2\leq 2$. Similarly, letting $\psi_1(x)=.05\sin x$ and $\psi_2(x)\equiv 2$ and substituting into the second equation for u_2, we conclude by means of [11] that there is a solution $(\bar{u}_1(x), \bar{u}_2(x))$ to the homogeneous Dirichlet problem for (2.10) in $H^{2+l}([0,\pi])$, with $.05\sin x\leq\bar{u}_i(x)\leq 2$, $x\in[0,\pi]$, $i=1,2$. Clearly, the conditions $\partial\bar{u}_i/\partial\nu<0$ at $x=0,\pi$ and $\bar{u}_i(x)>0$ in $(0,\pi)$ are satisfied. To check the inequality in (2.6), we have to estimate the size of the first derivatives. First, (2.10) implies that $|\bar{u}_2''(x)|\leq 2/450[500+2+494(2)]<7$, hence

$$|\bar{u}_2'(x)|\leq 2\max_{0\leq x\leq\pi}|\bar{u}_2(x)|\cdot\frac{1}{\pi}+\frac{1}{2}\max_{0\leq x\leq\pi}|\bar{u}_2''(x)|\,\pi\leq 2(2)\frac{1}{\pi}+\frac{1}{2}(7)\pi<13$$

for $x\in[0,1]$ (see [4], p. 139 for estimate). By symmetry of (2.10), we also have $|\bar{u}_1'(x)|<13$ for $x\in[0,1]$. Since $|\bar{u}_2|\leq 2$, we have

$$\inf_{x\in(0,1)}\left\{\frac{\bar{u}_1(x)}{\bar{u}_2(x)}\right\}\geq\min_{0\leq x\leq 2/13}\left\{\frac{.05\sin x}{13x}\right\}=\frac{.05\sin(2/13)}{2}>.00375;$$

on the other hand

$$\sup_{x\in(0,1)}\left\{\frac{\bar{u}_1(x)}{\bar{u}_2(x)}\right\}\leq\max_{0\leq x\leq 2/13}\left\{\frac{13x}{.05\sin x}\right\}=\frac{2}{.05\sin(2/13)}<267.$$

For (2.10), we identify $f_1(u_1, u_2)=500\text{–}494u_1-u_2$ and $f_2(u_1, u_2)=500-u_1-494u_2$. We have $\partial f_2/\partial u_1=-1$, $\partial f_2/\partial u_2=-494$, $\partial f_1/\partial u_1=-494$, $\partial f_1/\partial u_2=-1$; and clearly the inequality (2.6) is satisfied for $i=1, j=2$ for this example $(\bar{u}_1(x), \bar{u}_2(x))$. By symmetry, (2.6) is also true for $i=2, j=1$. Consequently, we can apply Theorem 2.2 to assert the asymptotic stability of this solution.

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(Ricevita la 29-an de septembro, 1980)