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Boundary Value Problems for Systems of Second Order Ordinary Differential Equations

By

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Dedicated to Professor Taro Ura on his sixtieth birthday

§ 1. Introduction

This paper is concerned with the existence of solutions of the boundary value problem

$$x''=f(t, x, x'), \tag{1}$$

$$x(0)=a, \qquad x(1)=b, \tag{2}$$

where x is an n-vector and the prime denotes differentiation with respect to t.

Scorza-Dragoni [17] first proved that the equation (1) has a solution satisfying (2) for arbitrary a and b whenever $f: I\times R^n\times R^n\to R^n$ is bounded and continuous, where and through this paper $I=[0, 1]$. Hartman [5] (or refer to [6]) obtained an existence result for the problem (1)–(2) by imposing growth conditions on f which yield a priori bounds of $x'(t)$ in terms of a bound of $x(t)$. Such an idea had been found out by Nagumo [13] in 1937 for scalar second order equations. Hartman's result has been developed by Bernfeld, Ladde and Lakshmikantham [1], Lasota and Yorke [10], Schmitt and Thompson [16] and so on.

In the case where $n=1$, Nagumo [14] has obtained a beautiful existence theorem for the problem (1)–(2). We have extended his result in [8, 9] by using topological properties of solution curves. In this paper, we consider a further extension of the result to vector equations. Our main theorem is Theorem 3 in Section 4, and the proof is based on the degree theory for a certain class of set-valued mappings which will be called *regular* mappings.

§ 2. Regular mapping and the degree

For a set $A\subset R^m$, we use the following notations; $\bar{A}$ is the closure, ∂A is the boundary and co A is the convex hull. Furthermore, we set

$$\text{Comp}\,(R^m)=\{A\colon A \text{ is a nonempty compact set in } R^m\},$$
$$\text{Conv}\,(R^m)=\{A\in \text{Comp}\,(R^m)\colon A \text{ is convex}\}.$$

By Caratheodory's lemma (refer to [4, p. 28]), co A is written as

$$\mathrm{co}\, A=\left\{\sum_{i=0}^{m} r_i p_i : \sum_{i=0}^{m} r_i=1,\ r_i\geqq 0,\ p_i \in A,\ i=0, 1, \cdots, m\right\}.$$

Therefore, if A is compact, then so is co A. Let M be a metric space, and let $C(M, R^m)$ be the set of all continuous mappings from M into R^m. For a given mapping Φ: $M\to\mathrm{Comp}\,(R^m)$, Φ^*: $M\to\mathrm{Conv}\,(R^m)$ denotes the mapping defined by $\Phi^*(x)=\mathrm{co}\,\Phi(x)$ for $x \in M$, while $A(\Phi)$ denotes the set of all sequences $\{\Phi_k\}$ in $C(M, R^m)$ satisfying the following condition:

(R) If $\{\Phi_{k_j}\}$ is a subsequence of $\{\Phi_k\}$ and if $\{x_j\}$ is a sequence in M converging to an x in M, then the sequence $\{\Phi_{k_j}(x_j)\}$ contains a subsequence which converges to some point in $\Phi(x)$.

The element $\{\Phi_k\}$ of $A(\Phi)$ will be called *an approximate sequence* of Φ. A mapping Φ: $M\to\mathrm{Comp}\,(R^m)$ is called *upper semicontinuous* if, for every $x \in M$ and an open set V containing $\Phi(x)$, there exists a neighborhood W of x such that $\Phi(W)\subset V$, where $\Phi(W)=\cup\{\Phi(y): y \in W\}$. It is clear that if Φ is upper semicontinuous, then so is Φ^*.

Cellina [2] has proved the following lemma (see also [3, Lemma 1]).

Lemma 1. *If* Φ: $M\to\mathrm{Conv}\,(R^m)$ *is upper semicontinuous and if M is compact, then* $A(\Phi)$ *is nonempty.*

Let D be a bounded and open set in R^m, and let Φ: $\bar{D}\to\mathrm{Conv}\,(R^m)$ be upper semicontinuous. Applying Lemma 1, Cellina and Lasota [3] defined the degree of Φ by

$$d(\Phi, D, p)=\lim_{k\to\infty} d(\Phi_k, D, p) \tag{3}$$

for a point $p \in R^m\backslash\Phi(\partial D)$ and a $\{\Phi_k\} \in A(\Phi)$, where $d(\Phi_k, D, p)$ is the degree of the continuous mapping Φ_k, refer to [11] or [15]. This statement involves that

(i) the limit in the right hand side of (3) exists for every $\{\Phi_k\} \in A(\Phi)$ and it is independent of the choice of $\{\Phi_k\}$,

(ii) $d(\Phi, D, p)\neq 0$ implies $p \in \Phi(D)$,

(iii) if $\Phi(\cdot, \cdot)$: $I\times\bar{D}\to\mathrm{Conv}\,(R^m)$ is upper semicontinuous and if $p \in R^m\backslash\Phi(I, \partial D)$, then $d(\Phi(0, \cdot), D, p)=d(\Phi(1, \cdot), D, p)$,

(iv) for Φ satisfying $\Phi(x)=\{x\}$ on $\bar{D}$, $d(\Phi, D, p)=1$ if and only if $p \in D$.

Hukuhara [7] and Ma [12] have also introduced the same degree as in the above by different approaches from that in [3].

A mapping Φ: $M\to\mathrm{Comp}\,(R^m)$ is said to be *regular* if $A(\Phi)$ is nonempty. Lemma 1 shows that an upper semicontinuous mapping from M into Conv (R^m) is regular when M is compact.

Lemma 2. *Consider a mapping* $\Phi: M \to \mathrm{Comp}(R^m)$. *If there is a* $\phi \in C(M, R^m)$ *such that* $\phi(x) \in \Phi(x)$ *for all* $x \in M$, *then* Φ *is regular. Conversely, if* Φ *is regular and single-valued,* $\Phi(x)=\{\phi(x)\}$, *then* $\phi \in C(M, R^m)$.

Proof. The first assertion of the lemma is clear since the sequence $\{\Phi_k\}$ in $C(M, R^m)$ defined by $\Phi_k=\phi$ satisfies (R). We prove the last assertion. Suppose that ϕ is not continuous. Then there exist $\varepsilon>0$, $x \in M$ and a sequence $\{x_j\}$ in M converging to x such that $|\phi(x)-\phi(x_j)| \geqq \varepsilon$ for all j, where $|\cdot|$ is any norm in R^m. Let $\{\Phi_k\} \in A(\Phi)$ be fixed. By (R), the sequence $\{\Phi_k\}$ converges to ϕ at every point in M, and hence there exists a subsequence $\{\Phi_{k_j}\}$ of $\{\Phi_k\}$ satisfying $|\Phi_{k_j}(x_j)-\phi(x_j)|<1/j$ for all j. Therefore we have $\varepsilon \leqq |\phi(x)-\phi(x_j)|<|\phi(x)-\Phi_{k_j}(x_j)|+1/j$. On the other hand, by using (R), we have $|\phi(x)-\Phi_{k_j}(x_j)| \to 0$ as $j \to \infty$ by taking a subsequence if necessary, a contradiction. q.e.d.

The following lemma is trivial.

Lemma 3. (a) *For mappings* Φ, $\Psi: M \to \mathrm{Comp}(R^m)$, *we have* $A(\Phi) \subset A(\Psi)$ *if* $\Phi(x) \subset \Psi(x)$ *for* $x \in M$. *Especially, if* $\Phi: M \to \mathrm{Comp}(R^m)$ *is regular, then* $\Phi^*: M \to \mathrm{Conv}(R^m)$ *is regular.*

(b) *If* $\Phi(\cdot, \cdot): I \times M \to \mathrm{Comp}(R^m)$ *is regular, then* $\Phi(t, \cdot): M \to \mathrm{Comp}(R^m)$ *is regular for every* $t \in I$.

(c) *Let* M_1 *and* M_2 *be metric spaces. Suppose that* $g_1: M_1 \to M_2$ *and* $g_2: R^m \to R^n$ *are continuous and that* $\Phi: M_2 \to \mathrm{Comp}(R^m)$ *is regular. Then the composite mapping* $g_2 \circ \Phi \circ g_1: M_1 \to \mathrm{Comp}(R^n)$ *is regular.*

Examining the arguments in Cellina and Lasota [3], we can generalize the concept of the degree for a regular mapping by (3) if $p \in R^m \setminus \Phi^*(\partial D)$, and we can obtain the following theorem.

Theorem 1. (a) $d(\Phi, D, p) \neq 0$ *implies* $p \in \Phi(D)$ *whenever* $\Phi: \bar{D} \to \mathrm{Comp}(R^m)$ *is regular.*

(b) *If* $\Phi(\cdot, \cdot): I \times \bar{D} \to \mathrm{Comp}(R^m)$ *is regular and if* $p \in R^m \setminus \Phi^*(I, \partial D)$, *then* $d(\Phi(0, \cdot), D, p)=d(\Phi(1, \cdot), D, p)$.

Remark 1. Clearly, we have

$$d(\Phi, D, p)=d(\Phi^*, D, p) \tag{4}$$

for a regular mapping Φ. Even if Φ is not regular, Φ^* may be regular, which suffices to define $d(\Phi, D, p)$ by (4). In this case, we cannot conclude that $p \in \Phi(D)$ under $d(\Phi, D, p) \neq 0$, though $d(\Phi, D, p) \neq 0$ implies $p \in \Phi^*(D)$. This is seen in the following example.

Example. Let $D=(-1, 1)$, and let $\Phi: \bar{D}\to\mathrm{Comp}(R)$ be the mapping defined by $\Phi(x)=\{x/|x|\}$ for $x\neq 0$ and $\Phi(0)=\{-1, 1\}$. Then Φ is upper semicontinuous, and hence Φ^* is regular. Since $\Phi^*(\partial D)$ does not contain 0, $d(\Phi^*, D, 0)$ is defined. Clearly, the sequence $\{\Phi_k\}$ in $C(\bar{D}, R)$ defined by

$$\Phi_k(x)=\begin{cases} 1, & 1/k\leqq x\leqq 1,\\ kx, & -1/k<x<1/k,\\ -1 & -1\leqq x\leqq -1/k \end{cases}$$

is an approximate sequence of Φ^*. Therefore we have $d(\Phi^*, D, 0)=1$ and $0\in\Phi^*(D)$, while $0\notin\Phi(D)$.

§ 3. Solution mapping

Consider a differential equation

$$\text{(E)}\qquad x'=h(t, x),$$

where $h: I\times R^m\to R^m$ is continuous, and assume that

(C) every solution of (E) is continuable over I.

A subset S of $I\times R^m$ will be called *a positively invariant set* of (E) if every solution curve of (E) starting from a point in S remains in S on its right maximal interval of existence, that is, if every solution x of (E) satisfies $(t, x(t))\in S$ for $\tau\leqq t\leqq 1$ whenever $(\tau, x(\tau))\in S$ for some $\tau\in I$. Clearly, the intersection and the union of positively invariant sets are positively invariant. Similarly, the concept of the *negatively invariant set* of (E) is defined.

For $(\tau, \xi)\in I\times R^m$, we put

$$(5)\qquad \Phi(\tau, \xi)=\{x(1): x \text{ is a solution of (E) satisfying } x(\tau)=\xi\}.$$

Then, by well-known Kamke's theorem (see [6, Theorem 3.2, p.p. 14–15]) and the assumption (C), $\Phi(\tau, \xi)$ is a compact set in R^m, and we have the following theorem.

Theorem 2. *The solution mapping* $\Phi: I\times R^m\to\mathrm{Comp}(R^m)$ *defined by* (5) *is regular, and it satisfies that* $\Phi(1, \xi)=\{\xi\}$ *for* $\xi\in R^m$.

Proof. It is well-known that the mapping h in the equation (E) admits a sequence $\{h_k\}$ in $C(I\times R^m, R^m)$ with the following properties:

(A1) $\{h_k\}$ converges to h uniformly on every compact set in $I\times R^m$.

(A2) Every solution of

$$(E_k)\qquad x'=h_k(t, x)$$

is uniquely determined by initial data and it is continuable over I for $k=1, 2, \cdots$.

Let $\Phi_k: I\times R^m\to R^m$ be the mapping defined by $\Phi_k(\tau, \xi)=x_k(1)$ for each k, where x_k is the solution of (E_k) satisfying $x_k(\tau)=\xi$. Then Φ_k are continuous. It follows from (A1) and Kamke's theorem that the sequence $\{\Phi_k\}$ satisfies (R) if $M=I\times R^m$ in (R). Therefore Φ is regular and $\{\Phi_k\}\in A(\Phi)$.

The last assertion of the theorem is clear. q.e.d.

§ 4. Boundary value problems

In this section, we give an existence theorem for the problem (1)–(2). The equation (1) is equivalent to the system

$$x'=y, \qquad y'=f(t, x, y). \tag{6}$$

Let $I=[0, 1]$ and $J=\{1, 2, \cdots, n\}$. For two vectors x and y in R^n, we write $x\leqq y$ when $x_i\leqq y_i$ holds for each $i\in J$, where and hereafter the suffix i denotes the i-th component of a vector, and hence i runs over the set J. The scalar product of x and y will be denoted by $\langle x, y\rangle$, that is, $\langle x, y\rangle=\sum_{i\in J} x_i y_i$.

For given twice continuously differentiable functions $\alpha, \beta: I\to R^n$ satisfying $\alpha(t)\leqq\beta(t)$ on I, set

$$\omega=\{(t, x)\in I\times R^n: \alpha(t)\leqq x\leqq\beta(t)\}.$$

Let Ω be the compact set defined by

$$\Omega=\{(t, x, y)\in\omega\times R^n: \phi(t, x)\leqq y\leqq\psi(t, x)\},$$

where ϕ and ψ are continuously differentiable functions from ω into R^n satisfying $\phi(t, x)\leqq\psi(t, x)$ on ω. We assume that the function f in the equation (6) is defined and continuous on Ω.

Theorem 3. *Suppose that the following inequalities hold on ω or on Ω for each $i\in J$;*

$$\alpha_i'(t)\geqq\phi_i(t, x) \qquad \text{if } x_i=\alpha_i(t), \tag{7}$$

$$\beta_i'(t)\leqq\psi_i(t, x) \qquad \text{if } x_i=\beta_i(t), \tag{8}$$

$$\alpha_i''(t)\geqq f_i(t, x, y) \quad \text{if } x_i=\alpha_i(t), \quad y_i=\alpha_i'(t), \tag{9}$$

$$\beta_i''(t)\leqq f_i(t, x, y) \quad \text{if } x_i=\beta_i(t), \quad y_i=\beta_i'(t), \tag{10}$$

$$f_i(t, x, y)\geqq\frac{\partial}{\partial t}\phi_i(t, x)+\left\langle\frac{\partial}{\partial x}\phi_i(t, x), y\right\rangle \qquad \text{if } y_i=\phi_i(t, x), \tag{11}$$

$$f_i(t, x, y)\leqq\frac{\partial}{\partial t}\psi_i(t, x)+\left\langle\frac{\partial}{\partial x}\psi_i(t, x), y\right\rangle \qquad \text{if } y_i=\psi_i(t, x). \tag{12}$$

Then, for a given $b \in R^n$ *with* $\alpha(1) \leqq b \leqq \beta(1)$, *the equation* (6) *has at least one solution* (x, y) *defined on* I *which satisfies* $x(1)=b$ *and*

$$(t, x(t), y(t)) \in \Omega \qquad \text{for } t \in I. \tag{13}$$

In particular, if $\alpha(0)=a=\beta(0)$ *holds, then the solution satisfies* (2).

Remark 2. The condition (9) will be meaningless when the set $\{(t, x, y) \in \Omega: x_i=\alpha_i(t), y_i=\alpha_i'(t)\}$ is empty. This is similar for the condition (10).

Proof. The proof is fairly complicated and lengthy, so we proceed in the seven steps.

Step 1. We shall construct a bounded and continuous extension $F: I \times R^n \times R^n \to R^n$ of f so that the following inequalities hold for each $i \in J$;

$$\alpha_i''(t) > F_i(t, x, y) \quad \text{if } x_i < \alpha_i(t), \quad y_i=\alpha_i'(t), \tag{14}$$

$$\beta_i''(t) < F_i(t, x, y) \quad \text{if } x_i > \beta_i(t), \quad y_i=\beta_i'(t), \tag{15}$$

$$F_i(t, x, y) \geqq f_i(t, x, \hat{y}) \quad \text{for } (t, x, y) \in \omega \times R^n, \quad y_i \leqq \phi_i(t, x), \tag{16}$$

$$F_i(t, x, y) \leqq f_i(t, x, \hat{y}) \quad \text{for } (t, x, y) \in \omega \times R^n, \quad y_i \geqq \psi_i(t, x), \tag{17}$$

where $\hat{y}$ is the vector with the j-th component, $j \in J$, defined by

$$\hat{y}_j = \begin{cases} \phi_j(t, x), & y_j < \phi_j(t, x), \\ y_j, & \phi_j(t, x) \leqq y_j \leqq \psi_j(t, x), \\ \psi_j(t, x), & y_j > \psi_j(t, x) \end{cases} \tag{18}$$

which depends on t, x and y.

First of all, for each $i \in J$, we shall define F_i on the domain

$$V_i=\{(t, x, y) \in \omega \times R^n: y_i \in R, \quad \phi_j(t, x) \leqq y_j \leqq \psi_j(t, x) \quad \text{for } j \in J \setminus \{i\}\}$$

so as to satisfy

$$\alpha_i''(t) \geqq F_i(t, x, y) \qquad \text{on } A_i, \tag{19}$$

$$\beta_i''(t) \leqq F_i(t, x, y) \qquad \text{on } B_i, \tag{20}$$

$$F_i(t, x, y) \geqq f_i(t, x, \hat{y}) \qquad \text{on } V_i^- \tag{21}$$

and

$$F_i(t, x, y) \leqq f_i(t, x, \hat{y}) \qquad \text{on } V_i^+, \tag{22}$$

where

$$A_i=\{(t, x, y) \in V_i\colon x_i=\alpha_i(t),\ y_i=\alpha_i'(t)\},$$
$$B_i=\{(t, x, y) \in V_i\colon x_i=\beta_i(t),\ y_i=\beta_i'(t)\},$$
$$V_i^-=\{(t, x, y) \in V_i\colon y_i<\phi_i(t, x)\},$$
$$V_i^+=\{(t, x, y) \in V_i\colon y_i>\psi_i(t, x)\}.$$

Here, notice that $V_i=V_i^-\cup\Omega\cup V_i^+$ and that the inequalities (19) and (20) are already satisfied on $A_i\cap\Omega$ and on $B_i\cap\Omega$, respectively, by (9) and (10). Therefore, since $A_i\subset\Omega\cup V_i^+$ and $B_i\subset\Omega\cup V_i^-$ by (7) and (8), it is not difficult to obtain a bounded and continuous extension F_i of f_i on the domain V_i which satisfies (19) through (22).

For an arbitrary $(t, x, y) \in \omega\times R^n$, define $F_i(t, x, y)$ by

$$F_i(t, x, y)=F_i(t, x, \hat{y}^*),$$

where $\hat{y}^*=(\hat{y}_1^*, \cdots, \hat{y}_n^*)$ with $\hat{y}_i^*=y_i$ and $\hat{y}_j^*=\hat{y}_j$ given by (18) for $j \in J\setminus\{i\}$ which depends on i, t, x and y. Here, we note that $(t, x, \hat{y}^*)$ belongs to V_i. Finally, for an arbitrary $(t, x, y) \in I\times R^n\times R^n$ and $i \in J$, we set

$$F_i(t, x, y)=\begin{cases} F_i(t, \bar{x}, y)+\dfrac{x_i-\beta_i(t)}{1+x_i-\beta_i(t)}, & x_i>\beta_i(t),\\ F_i(t, \bar{x}, y), & \alpha_i(t)\leqq x_i\leqq\beta_i(t),\\ F_i(t, \bar{x}, y)-\dfrac{\alpha_i(t)-x_i}{1+\alpha_i(t)-x_i}, & x_i<\alpha_i(t),\end{cases}$$

where $\bar{x}$ is the vector with the j-th component, $j \in J$, defined by

$$\bar{x}_j=\begin{cases}\beta_j(t), & x_j>\beta_j(t),\\ x_j, & \alpha_j(t)\leqq x_j\leqq\beta_j(t),\\ \alpha_j(t), & x_j<\alpha_j(t)\end{cases} \tag{23}$$

which depends on t and x. Here, we note that $(t, \bar{x}, y)$ belongs to $\omega\times R^n$. It is easy to see that $F(t, x, y)$ satisfies all required conditions.

Step 2. Instead of the equation (6), we consider the equation

$$x'=y, \qquad y'=F(t, x, y). \tag{24}$$

The boundedness of F assures that every solution of (24) is continuable over I. Suppose that (x, y) is a solution of (24) satisfying $x_i(t)<\alpha_i(t)$ and $y_i(t)=\alpha_i'(t)$ for some $i \in J$ and $t \in I$. Then, by (14), we have

$$y_i'(t)=F_i(t, x(t), y(t))<\alpha_i''(t).$$

Therefore we can easily observe that

$$N_i=\{(t, x, y) \in I\times R^n\times R^n: x_i<\alpha_i(t), y_i>\alpha_i'(t)\}$$

is a negatively invariant set of (24) for each $i \in J$, while

$$P_i=\{(t, x, y) \in I\times R^n\times R^n: x_i<\alpha_i(t), y_i\leqq\alpha_i'(t)\}$$

is a positively invariant set of (24). Similarly, by (15),

$$M_i=\{(t, x, y) \in I\times R^n\times R^n: x_i>\beta_i(t), y_i<\beta_i'(t)\}$$

and

$$Q_i=\{(t, x, y) \in I\times R^n\times R^n: x_i>\beta_i(t), y_i\geqq\beta_i'(t)\}$$

are, respectively, a negatively invariant set and a positively invariant set of (24).

Let X_i, $i \in J$, be the set defined by

$$X_i=\{(t, x, y) \in I\times R^n\times R^n: \alpha_i(t)\leqq x_i\leqq\beta_i(t)\}.$$

Then the outside of X_i is the disjoint union of P_i, Q_i, N_i and M_i. Since $M_i\cup N_i$ is negatively invariant, the set $Y_i=P_i\cup X_i\cup Q_i$ is positively invariant. Therefore the intersection $Y=\cap\{Y_i: i \in J\}$ is positively invariant.

Step 3. Let ε and δ be arbitrary fixed numbers such that $0<\varepsilon<\delta<1$. Then we shall obtain a continuous mapping ρ: $[\varepsilon, \delta]\times R^n\to R^n$ which satisfies

$$\phi(t, x)\leqq\rho(t, x)\leqq\psi(t, x) \quad \text{for } (t, x) \in \omega, \qquad \varepsilon\leqq t\leqq\delta \tag{25}$$

and

$$(t, x, \rho(t, x)) \in Y \qquad \text{for } (t, x) \in [\varepsilon, \delta]\times R^n. \tag{26}$$

Let C: $\{(t, x) \in \omega: \varepsilon\leqq t\leqq\delta\}\to R^n$ be the mapping such that the i-th component $C_i(t, x)$ is linear in x_i and satisfies $C_i(t, x)=\min\{\alpha_i'(t), \psi_i(t, x)\}$ at $x_i=\alpha_i(t)$, $=\max\{\beta_i'(t), \phi_i(t, x)\}$ at $x_i=\beta_i(t)$. Since $\alpha_i'(t)=\beta_i'(t)$ when $\alpha_i(t)=\beta_i(t)$ for some $t \in [\varepsilon, \delta]$, $C_i(t, x)$ turns out to be continuous by (7) and (8) if we set $C_i(t, x)=\alpha_i'(t)$ when $\alpha_i(t)=\beta_i(t)$. Define ρ_i: $[\varepsilon, \delta]\times R^n\to R$, $i \in J$, by

$$\rho_i(t, x)=\begin{cases}\phi_i(t, \bar{x}), & C_i(t, \bar{x})<\phi_i(t, \bar{x}),\\ C_i(t, \bar{x}), & \phi_i(t, \bar{x})\leqq C_i(t, \bar{x})\leqq\psi_i(t, \bar{x}),\\ \psi_i(t, \bar{x}), & \psi_i(t, \bar{x})<C_i(t, \bar{x}),\end{cases}$$

where $\bar{x}$ is the vector given by (23). Then we can easily see that ρ is continuous and satisfies (25). When $x_i<\alpha_i(t)$, we have $C_i(t, \bar{x})\leqq\psi_i(t, \bar{x})$ since $\bar{x}_i=\alpha_i(t)$, and hence $\rho_i(t, x)=\max\{\phi_i(t, \bar{x}), C_i(t, \bar{x})\}\leqq\alpha_i'(t)$ by (7). Consequently, we have $(t, x, \rho(t, x)) \in P_i$ if $x_i<\alpha_i(t)$. Similarly, we have $(t, x, \rho(t, x)) \in Q_i$ if $x_i>\beta_i(t)$. Thus, $(t, x.$

$\rho(t, x))$ belongs to Y_i for each $i \in J$, namely, (26) holds.

By Theorem 2, the mapping $\Phi: [\varepsilon, \delta] \times R^n \times R^n \to \text{Comp}(R^n \times R^n)$ defined by

$$\Phi(\tau, \xi, \eta) = \{(x(\delta), y(\delta)) : (x, y) \text{ is a solution of (24) through } (\tau, \xi, \eta)\}$$

is regular. We define two continuous mappings $\gamma: [\varepsilon, \delta] \times R^n \to [\varepsilon, \delta] \times R^n \times R^n$ and $\pi: R^n \times R^n \to R^n$ by $\gamma(t, x) = (t, x, \rho(t, x))$ and $\pi(x, y) = x$. By Lemma 3 (c), the composite mapping $\Psi = \pi \circ \Phi \circ \gamma: [\varepsilon, \delta] \times R^n \to \text{Comp}(R^n)$ is regular.

Step 4. Let r be a number satisfying $r > \max\{|\alpha_i(t)|, |\beta_i(t)| : \varepsilon \leqq t \leqq \delta, i \in J\}$, and let D be the set defined by

$$D = \{x \in R^n : |x_i| < r \text{ for all } i \in J\}.$$

Then D is a bounded open set in R^n. The restriction of Ψ to $[\varepsilon, \delta] \times \bar{D}$, denoted by Ψ again, is regular. By the assumption, there exists a continuous mapping $p: I \to R^n$ satisfying $p(1) = b$ and

$$\alpha(t) \leqq p(t) \leqq \beta(t) \qquad \text{on } I. \tag{27}$$

We want to prove

$$d(\Psi(\varepsilon, \cdot), D, p(\delta)) = 1. \tag{28}$$

The definition of r and (27) imply that $p(\delta) \in D$. Clearly, Ψ satisfies that $\Psi(\delta, x) = \{x\}$ for all x in $\bar{D}$, and hence we have $d(\Psi(\delta, \cdot), D, p(\delta)) = 1$.

Let (t, x) be an arbitrary point in $[\varepsilon, \delta] \times \partial D$. Then we have that $|x_i| = r$ for some $i \in J$. First, consider the case where $x_i = -r$. As was seen in Step 3, we have $\gamma(t, x) = (t, x, \rho(t, x)) \in P_i$ because $x_i = -r < \alpha_i(t)$. Since P_i is positively invariant, the set $\Psi(t, x)$ is contained in the convex set $\{x \in R^n : x_i < \alpha_i(\delta)\}$. Therefore we have $\Psi^*(t, x) \subset \{x \in R^n : x_i < \alpha_i(\delta)\}$. Similarly, in the case where $x_i = r$, we have $\Psi^*(t, x) \subset \{x \in R^n : x_i > \beta_i(\delta)\}$. It follows from (27) that $\Psi^*(t, x)$ does not contain $p(\delta)$ for all $(t, x) \in [\varepsilon, \delta] \times \partial D$, namely,

$$p(\delta) \in R^n \setminus \Psi^*([\varepsilon, \delta], \partial D).$$

By Theorem 1 (b), we obtain that $d(\Psi(\varepsilon, \cdot), D, p(\delta)) = d(\Psi(\delta, \cdot), D, p(\delta))$. Thus, (28) is proved.

Step 5. It follows from (28) and Theorem 1 (a) that there exists a $\xi \in D$ satisfying $p(\delta) \in \Psi(\varepsilon, \xi)$. In other words, the equation (24) has a solution (x, y) such that

$$x(\varepsilon) = \xi, \qquad y(\varepsilon) = \rho(\varepsilon, \xi) \tag{29}$$

and that $x(\delta) = p(\delta)$. We show that the solution satisfies

$$\alpha(t) \leqq x(t) \leqq \beta(t) \qquad \text{on } [\varepsilon, \delta]. \tag{30}$$

By (26) and (29), we have $(\varepsilon, x(\varepsilon), y(\varepsilon)) \in Y$. If the solution does not satisfy (30), then the solution curve enters $P_i \cup Q_i$ for some $i \in J$ because Y is positively invariant. Namely, there exists a $t \in [\varepsilon, \delta]$ such that $(t, x(t), y(t)) \in P_i \cup Q_i$. Since $P_i \cup Q_i$ is positively invariant, either $x_i(\delta) < \alpha_i(\delta)$ or $x_i(\delta) > \beta_i(\delta)$ holds. This contradicts $x(\delta) = p(\delta)$ and (27), and hence we have (30).

Step 6. We shall prove that the solution satisfies

$$(t, x(t), y(t)) \in \Omega \qquad \text{on } [\varepsilon, \delta]. \tag{31}$$

Suppose that (31) is false. Then there exists a subinterval $[\sigma, \tau]$ of $[\varepsilon, \delta]$ such that

$$(\sigma, x(\sigma), y(\sigma)) \in \Omega, \quad (t, x(t), y(t)) \notin \Omega \quad \text{for } \sigma < t \leqq \tau, \tag{32}$$

where we note that $(\varepsilon, x(\varepsilon), y(\varepsilon)) \in \Omega$ by (25), (29) and (30).

Let U: $\{(t, x, y) \in \omega \times R^n : \sigma \leqq t \leqq \tau\} \to R$ be the mapping defined by $U(t, x, y) = \sum_{i \in J} \operatorname{dist}(y_i, [\phi_i(t, x), \psi_i(t, x)])$, namely,

$$U(t, x, y) = \sum_{i \in G_0} [\phi_i(t, x) - y_i] + \sum_{i \in H_0} [y_i - \psi_i(t, x)], \tag{33}$$

where $G_0 = \{i \in J: y_i < \phi_i(t, x)\}$ and $H_0 = \{i \in J: y_i > \psi_i(t, x)\}$, which depend on t, x and y, and we understand that for empty G_0 or H_0 the corresponding sum makes zero. It is clear that $U(t, x, y) = 0$ if and only if $(t, x, y) \in \Omega$. Along the solution curve, put

$$u(t) = U(t, x(t), y(t)) \qquad \text{for } \sigma \leqq t \leqq \tau,$$

where we note (30). Since clearly U is Lipschitz continuous in (t, x, y), u is absolutely continuous. The relation (32) implies $u(\sigma) = 0$ and $u(t) > 0$ for $\sigma < t \leqq \tau$.

We show that u satisfies

$$\log u(\tau) - \log u(s) \leqq nK(\tau - s) \qquad \text{for } \sigma < s < \tau, \tag{34}$$

where $K = \max\{|(\partial/\partial x_j)\phi_i(t, x)|, |(\partial/\partial x_j)\psi_i(t, x)| : (t, x) \in \omega, i, j \in J\}$. We put

$$\begin{aligned} \Gamma(t, x, y) = & \sum_{i \in G_0} \Big\{ \sum_{j \in G_0} \Big(\frac{\partial}{\partial x_j} \phi_i(t, x) \Big) [\phi_j(t, x) - y_j] \\ & \qquad + \sum_{j \in H_0} \Big(\frac{\partial}{\partial x_j} \phi_i(t, x) \Big) [\psi_j(t, x) - y_j] \Big\} \\ & + \sum_{i \in H_0} \Big\{ \sum_{j \in G_0} \Big(\frac{\partial}{\partial x_j} \psi_i(t, x) \Big) [y_j - \phi_j(t, x)] \\ & \qquad + \sum_{j \in H_0} \Big(\frac{\partial}{\partial x_j} \psi_i(t, x) \Big) [y_j - \psi_j(t, x)] \Big\}, \end{aligned}$$

where G_0 and H_0 are those as in (33). Then Γ is continuous and satisfies $|\Gamma(t, x, y)| \leqq nKU(t, x, y)$, and hence

$$(35) \qquad |\Gamma(t, x(t), y(t))| \leqq nKu(t) \qquad \text{for } \sigma < t \leqq \tau.$$

Let s be fixed with $\sigma < s < \tau$. Then we have $u(t) > 0$ on $[s, \tau]$ and the function

$$(36) \qquad \mu(t) = \log u(t) + \int_s^t \Gamma(w, x(w), y(w))dw/u(w)$$

is absolutely continuous on $[s, \tau]$. Differentiating (36), we have

$$(37) \qquad u(t)\mu'(t) = u'(t) + \Gamma(t, x(t), y(t)) \qquad \text{a.e. on } [s, \tau].$$

We want to show that $\mu'(t) \leqq 0$ a.e. on $[s, \tau]$. Let $t \in [s, \tau]$ be fixed. Then there exist two subset G and H of J and a sequence $\{\nu_k\}$ of nonzero numbers such that

$$\lim_{k\to\infty} \nu_k = 0, \quad s \leqq t + \nu_k \leqq \tau \quad \text{for } k = 1, 2, \cdots$$

and that $u(\theta)$ is expressed in

$$(38) \qquad u(\theta) = \sum_{i\in G} [\phi_i(\theta, x(\theta)) - y_i(\theta)] + \sum_{i\in H} [y_i(\theta) - \psi_i(\theta, x(\theta))]$$

for $\theta = t, t+\nu_k, k = 1, 2, \cdots$. Clearly, $G_0 \subset G \subset \{i \in J: y_i(t) \leqq \phi_i(t, x(t))\}$ and $H_0 \subset H \subset \{i \in J: y_i(t) \geqq \psi_i(t, x(t))\}$. Here, we emphasize that G and H do not depend on θ. Therefore $\Gamma(t, x(t), y(t))$ is expressed in

$$(39) \qquad \begin{aligned} \Gamma(t, x(t), y(t)) = \sum_{i\in G} \Big\{ &\sum_{j\in G} \Big[\frac{\partial}{\partial x_j}\phi_i(t, x(t))\Big][\phi_j(t, x(t)) - y_j(t)] \\ &+ \sum_{j\in H} \Big[\frac{\partial}{\partial x_j}\phi_i(t, x(t))\Big][\psi_j(t, x(t)) - y_j(t)]\Big\} \\ + \sum_{i\in H} \Big\{ &\sum_{j\in G} \Big[\frac{\partial}{\partial x_j}\psi_i(t, x(t))\Big][y_j(t) - \phi_j(t, x(t))] \\ &+ \sum_{j\in H} \Big[\frac{\partial}{\partial x_j}\psi_i(t, x(t))\Big][y_j(t) - \psi_j(t, x(t))]\Big\}. \end{aligned}$$

When u is differentiable at t, the equality $u'(t) = \lim_{k\to\infty} [u(t+\nu_k) - u(t)]/\nu_k$ and (38) imply

$$(40) \qquad \begin{aligned} u'(t) = &\sum_{i\in G} \Big\{\frac{\partial}{\partial t}\phi_i(t, x(t)) + \Big\langle\frac{\partial}{\partial x}\phi_i(t, x(t)), y(t)\Big\rangle - F_i(t, x(t), y(t))\Big\} \\ &+ \sum_{i\in H} \Big\{F_i(t, x(t), y(t)) - \frac{\partial}{\partial t}\psi_i(t, x(t)) - \Big\langle\frac{\partial}{\partial x}\psi_i(t, x(t)), y(t)\Big\rangle\Big\}. \end{aligned}$$

Substitute (39) and (40) into (37). Then a direct calculation gives

$$
(41)\qquad \begin{aligned} u(t)\mu'(t) = & \sum_{i\in G}\left\{\frac{\partial}{\partial t}\phi_i(t, x(t)) + \left\langle\frac{\partial}{\partial x}\phi_i(t, x(t)), \hat{y}(t)\right\rangle - F_i(t, x(t), y(t))\right\} \\ & + \sum_{i\in H}\left\{F_i(t, x(t), y(t)) - \frac{\partial}{\partial t}\psi_i(t, x(t)) - \left\langle\frac{\partial}{\partial x}\psi_i(t, x(t)), \hat{y}(t)\right\rangle\right\}, \end{aligned}
$$

where $\hat{y}(t)$ is the vector given by (18) with $x=x(t)$. By (16) and (17), we have

$$
(42)\qquad F_i(t, x(t), y(t)) \geqq f_i(t, x(t), \hat{y}(t)), \qquad \text{for } i\in G
$$

and

$$
(43)\qquad F_i(t, x(t), y(t)) \leqq f_i(t, x(t), \hat{y}(t)), \qquad \text{for } i\in H.
$$

It follows from (41) through (43) that

$$
\begin{aligned} u(t)\mu'(t) \leqq & \sum_{i\in G}\left\{\frac{\partial}{\partial t}\phi_i(t, x(t)) + \left\langle\frac{\partial}{\partial x}\phi_i(t, x(t)), \hat{y}(t)\right\rangle - f_i(t, x(t), \hat{y}(t))\right\} \\ & + \sum_{i\in H}\left\{f_i(t, x(t), \hat{y}(t)) - \frac{\partial}{\partial t}\psi_i(t, x(t)) - \left\langle\frac{\partial}{\partial x}\psi_i(t, x(t)), \hat{y}(t)\right\rangle\right\}. \end{aligned}
$$

Therefore, by the assumptions (11) and (12), we obtain that $u(t)\mu'(t)\leqq 0$ because $\hat{y}_i(t) = \phi_i(t, x(t))$ for $i\in G$ and $\hat{y}_i(t)=\psi_i(t, x(t))$ for $i\in H$. This implies $\mu'(t)\leqq 0$ a.e. on $[s, \tau]$. Thus, we have $\mu(\tau)-\mu(s)\leqq 0$. On the other hand, it follows from (35) and (36) that

$$
\begin{aligned} \mu(\tau)-\mu(s) &= \log u(\tau) - \log u(s) + \int_s^\tau \Gamma(w, x(w), y(w))dw/u(w) \\ &\geqq \log u(\tau) - \log u(s) - nK\int_s^\tau dw, \end{aligned}
$$

and hence we obtain (34).

Making $s\to\sigma$ in (34), we arrive a contradiction since the left hand side of (34) tends to $+\infty$ (note $u(\sigma)=0$). Thus, we have (31).

Step 7. Let $\{\varepsilon_k\}$ and $\{\delta_k\}$ be two sequences in the open interval (0, 1) such that $\lim_{k\to\infty}\varepsilon_k=0$ and $\lim_{k\to\infty}\delta_k=1$. As was seen in the above argument, for each integer k, the equation (24) has a solution (x^k, y^k) which satisfies $x^k(\delta_k)=p(\delta_k)$ and

$$
(44)\qquad (t, x^k(t), y^k(t))\in\Omega \qquad \text{on } [\varepsilon_k, \delta_k].
$$

We may assume that (x^k, y^k) are defined on I because every solution of (24) is continuable over I. Since Ω is compact and F is bounded, the family $\{(x^k, y^k): k=1, 2, \cdots\}$ is uniformly bounded and equicontinuous on I. By taking a subsequence if necessary, we may assume that $\{(x^k, y^k)\}$ converges to a solution (x, y) of (24) uniformly on I. Since $x^k(\delta_k)=p(\delta_k)\to b$ as $k\to\infty$, we have $x(1)=b$. Furthermore, we can conclude that (13) holds by (44). At the same time, this shows that (x, y) is a solution of (6). This completes the proof.

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