An Abstract Threshold Theorem for One Parameter Families of Positive Noncompact Operators

By

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In a recent paper [2], K. Cooke and J. Kaplan study the delay integral equation

(1.1) \[ x(t) = \int_{t-\tau}^{t} a(s)x(s)(1-x(s))ds \]

where \( a(t) \) is a positive \( \omega \)-periodic, continuous function. Positive \( \omega \)-periodic solutions of (1.1) are sought. This equation (more general nonlinearities are allowed) is derived from a model for the spread of an epidemic. More recent papers of R. Nussbaum [10] and the author [12] sharpen the results in [2] so that the behavior of solutions of (1.1) is well-understood. For (1.1) there is a critical value of the parameter \( \tau \), say \( \tau_0 \), such that if \( \tau < \tau_0 \) all nonnegative solutions of (1.1) tend to zero exponentially fast as \( t \to \infty \). For each \( \tau > \tau_0 \) there exists a unique \( \omega \)-periodic, positive solution of (1.1). These solutions form a continuous curve parametrized by \( \tau \) for \( \tau > \tau_0 \) and \( x \) decreases pointwise and uniformly to zero as \( \tau \downarrow \tau_0 \), i.e., these solutions bifurcate from the zero solution at \( \tau = \tau_0 \).

Similar results have been obtained more recently by S. Busenberg and K. L. Cooke [1] for the delay differential equation

(1.2) \[ x'(t) = b(t)x(t-T)(1-x(t)) - cx(t) \]

The equation (1.2), in which \( b(t) \) is a \( \omega \)-periodic, models the spread of an infection in a population by a vector agent. Positive \( \omega \)-periodic solutions of (1.2) are shown to exist if \( c \) is smaller than a certain critical value; again, bifurcation occurs at the critical value of \( c \) from the identically zero solution. It should be noted that both (1.1) and (1.2) possess the trivial solution \( x \equiv 0 \).

The analysis in [1, 10, 12] amounts to the study of fixed points of a one parameter family of completely continuous operators, \( T_\tau : \mathcal{K} \to \mathcal{K} \), where \( \mathcal{K} \) is the cone of nonnegative \( \omega \)-periodic functions in the Banach space of \( \omega \)-periodic functions. In fact, for (1.1), the operator is just the right hand side. In [10] and [12] a bifurcation theory is developed for a one parameter family of (nonlinear) completely continuous operators \( \{T_\tau\}_{\tau>0} \) mapping a cone in a Banach space into itself (positive operator).
The complete continuity of the operators $T_{\tau}$ plays an essential role in the theory—both in the spectral analysis of certain linearized operators (the Krein-Rutman theorem) and in the existence of fixed points (Leray-Schauder degree).

The aim of this work is to investigate the extent to which the complete continuity hypotheses can be avoided in the bifurcation problem

$$(1.3) \quad T_{\tau}x = x, \quad T_{0}0 = 0$$

where $\{T_{\tau}\}_{\tau \in \tau}$, $\tau$ an interval of reals, is a one parameter family of positive operators. The essential requirements we will demand of the family $\{T_{\tau}\}$ are that each $T_{\tau}$ be a monotone and concave operator on certain conical intervals in the cone. These properties are already present in the operators associated with the problems (1.1) and (1.2). Our main result, Theorem 3.1, is a "threshold" theorem for the existence of positive solutions of (1.3). Positive solutions of (1.3) are shown to exist if and only if the parameter $\tau$ exceeds a certain threshold, $\tau_{0}$. These solutions form a continuous curve $x$ for $\tau > \tau_{0}$. For each $\tau$, $x$, is shown to be the unique solution of (1.3). An attractive feature of the theorem is that the positive fixed points can be obtained by successive approximations. Various sufficient conditions are given for bifurcation to occur at $\tau_{0}$ although these are not very satisfying. An open problem remains as to whether the assumptions of theorem 3.1 are sufficient to imply the bifurcation of positive solutions of (1.3) at $\tau_{0}$, i.e., $x_{\tau} \to 0$ as $\tau \to \tau_{0}$.

In section 2 we collect some definitions and preliminary results. In section 3 we prove our main result, theorem 3.1. In section 4, theorem 3.1 is applied to obtain the existence of positive almost periodic solutions of (1.1) when $a(t)$ is positive and almost periodic.

§ 2. Preliminary results and notation

In this section we collect some definitions and results which will be useful in the following sections. Recall that a cone $K$ in a Banach space $X$ is a nonempty, closed set with the properties that (i) if $x, y \in K$ then $x + y \in K$, (ii) if $x \in K$ and $\alpha > 0$ then $\alpha x \in K$, (iii) $x \in K$ and $-x \in K$ imply $x = 0$. If $K$ is a cone and $x - y \in K$ we write $x \geq y$; if $K$ has nonempty interior we write $x > y$ when $x - y \in \text{int } K$. A cone $K$ is called solid if it has nonempty interior (int $K$) and is reproducing if $K - K = X$. A cone $K$ is normal if every order interval $\langle u, v \rangle = \{x: u \leq x \leq v\}$ is bounded. Let $A$ be a linear operator leaving invariant a cone $K$ (a positive linear operator). Let $r(A)$ denote the spectral radius of $A$. Following S. Karlin [4], we define

$$\lambda(A) = \sup \{\lambda \geq 0: Ax \geq \lambda x \text{ for some } x \in K \setminus \{0\}\}.$$

It follows from [theorem 5.4, ch. 2, 6] that $\lambda(A) \leq r(A)$ when $K$ is reproducing. Equality holds if $A$ is compact [7]. The well-known Krein-Rutman theorem [7] asserts
that if $A$ is compact, $K - K = X$ and $r(A) > 0$ then $r(A)$ is an eigenvalue of $A$ corresponding to an eigenvector (simple) in $K$. Since we wish to avoid compactness assumptions we will be forced to work with $\tilde{\lambda}(A)$ rather than $r(A)$. A positive linear operator $A$ is said to be strongly positive on a solid cone $K$ if for each $x \in K - \{0\}$ there exists a positive integer $n(x) = n$ such that $A^n x > 0$. We will require the following simple result.

**Lemma 2.1.** If $A$ is strongly positive and $\tilde{\lambda}(A) > 0$ then given $\varepsilon > 0$, there exists $u \in \text{int } K$ such that

$$Au \geq (\tilde{\lambda}(A) - \varepsilon)u.$$  

**Proof.** By definition of $\tilde{\lambda}(A)$ there exists $x \in K \setminus \{0\}$ such that $Ax \geq (\tilde{\lambda}(A) - \varepsilon)x$. Let $u = A^n x$ where $n = n(x)$ in the definition of strongly positive.

Suppose now that $\{A_t\}_{t > 0}$ is a one parameter family of strongly positive, strongly continuous, positive linear operators with the property that the map $\tau \mapsto A_\tau x$ is non-decreasing, i.e., if $\tau_1 < \tau_2$ then $A_{\tau_1} x \leq A_{\tau_2} x$ for every $x \in K$. For such families we have the following:

**Lemma 2.2.** If $\tilde{\lambda}(A_\tau) > 0$ for all $\tau > 0$ then the map $\tau \mapsto \tilde{\lambda}(A_\tau)$ is non-decreasing and lower semicontinuous.

**Proof.** Let $0 < \tau_1 < \tau_2$ and $\varepsilon > 0$ be arbitrary. Then there exists $x \in K \setminus \{0\}$ such that $A_{\tau_1} x \geq (\tilde{\lambda}(A_{\tau_1}) - \varepsilon)x$. Hence, $A_{\tau_2} x \geq A_{\tau_1} x \geq (\tilde{\lambda}(A_{\tau_1}) - \varepsilon)x$, so $\tilde{\lambda}(A_{\tau_2}) \geq \tilde{\lambda}(A_{\tau_1}) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary this proves the first assertion. Let $\tilde{\lambda}(A_{\tau_0}) > r > 0$. Then there exists $\delta > 0$ and $u \in \text{int } K$ such that $A_{\tau_0} u \geq (r + \delta)u$ by lemma 2.1. Now

$$A_{\tau} u = A_{\tau_0} u + (A_{\tau} - A_{\tau_0})u$$

$$\geq (r + \delta)u + (A_{\tau} - A_{\tau_0})u$$

$$\geq (r + \delta/2)u + [\delta/2u + (A_{\tau} - A_{\tau_0})u]$$

$$\geq (r + \delta/2)u$$

if $|\tau - \tau_0|$ is sufficiently small since $\delta/2u \in \text{int } K$ and $(A_{\tau} - A_{\tau_0})u \rightarrow 0$ as $\tau \rightarrow \tau_0$. Thus $\tilde{\lambda}(A_{\tau}) > r$ if $|\tau - \tau_0|$ is sufficiently small. This proves the lower semicontinuity.

In most cases of interest to us, the map $\tau \mapsto A_\tau$ is continuous in the uniform topology. In this case, if $A_\tau$ is compact for each $\tau$, then $\tau \mapsto \tilde{\lambda}(A_\tau) = r(A_\tau)$ is continuous [11].

If $K$ is a solid, normal cone it is possible ([5]) to remetrize $\text{int } K$ so that it becomes a complete metric space using the metric

$$\rho(x, y) = \min \{\alpha : x \leq e^\alpha y \text{ and } y \leq e^\alpha x\}.$$  

Furthermore if $u, v \in \text{int } K$ and $u \leq v$ then $\rho$ is equivalent to the norm induced metric
on the order interval \( \langle u, v \rangle \). With these preliminaries out of the way we can turn to families of nonlinear operators. The family \( \{T_{\tau}\}_{\tau \in I} \), of nonlinear operators \( T_{\tau} : K \rightarrow K \) is said to be uniformly concave on \( \langle u, v \rangle \) (\( u, v \in \text{int} \, K \)) for \( \tau \in I \) if (i) each \( T_{\tau} \) is monotone on \( \langle 0, v \rangle \) i.e., \( 0 \leq x \leq y \leq v \) implies \( T_{\tau}x \leq T_{\tau}y \), (ii) \( T_{\tau}u > 0 \) for each \( \tau \in I \) and for any interval \([a, b] \subset (0, 1)\) there exists \( \eta = \eta(a, b) > 0 \) such that

\[
T_{\tau}(\mu x) \geq (1 + \eta)\mu T_{\tau}x, \quad a \leq \mu \leq b, \quad x \in \langle u, v \rangle, \quad \tau \in I.
\]

We can now state a theorem which is a trivial modification of a result due to Krasnosel’skii [5]. It is assumed that \( K \) is solid and normal.

**Theorem 2.3.** Let the family \( \{T_{\tau}\}_{\tau \in I} \) leave the interval \( \langle u, v \rangle \) (\( u, v \in \text{int} \, K \)) invariant and be uniformly concave on \( \langle u, v \rangle \) for \( \tau \in I \). Then \( T_{\tau} \) has a unique fixed point \( x_{\tau} \) in \( \langle u, v \rangle \) for each \( \tau \in I \) which is the limit of the successive approximations \( x_{n+1} = T_{\tau}x_{n} \), for any \( x_{0} \in \langle u, v \rangle \). Moreover the map \( \tau \mapsto x_{\tau} \) is a continuous map from \( I \) into \( K \).

This theorem is based on the observation that in the metric \( \rho \) on \( \langle u, v \rangle \):

\[
\rho(T_{\tau}x, T_{\tau}y) \leq \rho(x, y) - \ln[1 + \eta(e^{-\rho(u, v)}, e^{-\rho(x, y)})]
\]

for \( x, y \in \langle u, v \rangle \), \( \tau \in I \). Thus \( T_{\tau} \) is a uniform-in-\( \tau \) “generalized” contraction on the complete metric space \( \langle u, v \rangle \) and a simple extension of the contraction mapping theorem [5] gives the theorem.

**§ 3. An abstract threshold theorem**

In this section we formulate and prove an abstract threshold theorem. It will be assumed without further mention that the cone \( K \) is normal, reproducing, and solid. Let \( \{T_{\tau}\}_{\tau \geq 0} \) be a family of cone preserving (nonlinear) operators on \( K \) satisfying the following conditions:

1. \( T_{\tau}0 = 0 \) for all \( \tau > 0 \).
2. There exists \( \tau_{1} > 0 \) and \( v \in \text{int} \, K \) such that \( T_{\tau}(\langle 0, v \rangle) \subseteq \langle 0, v \rangle \) for \( \tau \leq \tau_{1} \) and \( T_{\tau} \) is monotone on \( \langle 0, v \rangle \).
3. The map \( \tau \mapsto T_{\tau}x \) is increasing for each \( x \in \langle 0, v \rangle \) and is continuous in \( \tau \) uniformly for \( x \in \langle 0, v \rangle \).
4. \( T'_{\tau}(0) \), the Frechet derivative of \( T_{\tau} \) at \( x = 0 \), exists and is strongly positive. The family \( \{T'_{\tau}(0)\}_{\tau \geq 0} \) is strongly continuous in \( \tau \). \( \lambda(T'_{\tau}(0)) > 1 \).
5. \( T'_{\tau}(0)y > T_{\tau}y \) for \( y \in \langle 0, v \rangle \cap \text{int} \, K \), \( \tau \leq \tau_{1} \).
6. \( \{T_{\tau}\} \) is uniformly concave for \( \tau \leq \tau_{1} \) on \( \langle 0, v \rangle \) in the following sense:

For all \( u \in \langle 0, v \rangle \cap \text{int} \, K \) and all \( [a, b] \subset (0, 1) \) there exists \( \eta = \eta(a, b, u) > 0 \) such that \( T_{\tau}(\delta x) \geq \delta(1 + \eta)T_{\tau}x, \quad a \leq \delta \leq b, \quad x \in \langle u, v \rangle, \quad \tau \leq \tau_{1} \). Moreover, \( \eta(a, b, \cdot) \) is non-decreasing in \( u \).
Theorem 3.1. Let (1)–(6) hold. Then there exists $\tau_0 \in [0, \tau_1)$ such that \{ $\tau$: $\lambda(T'_0(0)) > 1$ = $(\tau_0, \infty) \} \cap [0, \tau_1)$ such that

(a) For each $\tau \in (\tau_0, \tau_1]$, $T_\tau$ has a unique nonzero fixed point $x_\tau$ in $\langle 0, v \rangle$ which lies in $\text{int} \ K$. Also $x_\tau$ is the limit of the successive approximations $x_n = T_\tau x_{n-1}$, for any $x_0 \in \langle 0, v \rangle - \{0\}$.

(b) The map from $(\tau_0, \tau_1]$ into $\text{int} \ K \cap \langle 0, v \rangle$ given by $\tau \to x_\tau$ is a continuous, nondecreasing map in the sense that $\tau_2 < \tau_3$ implies $x_{\tau_2} \leq x_{\tau_3}$.

(c) If $\tau_0 > 0$ then $T_\tau$ has no fixed points in $\langle 0, v \rangle$ for $\tau \leq \tau_0$ other than the zero fixed point. Moreover $\inf_{\tau > \tau_0} x_\tau = 0$.

Remark 1. By $\inf_{\tau > \tau_0} x_\tau = 0$ we mean that no element $u$ of $K$ other than zero satisfies $0 \leq u \leq x_\tau$, for all $\tau > \tau_0$. In order to see the difference between $\inf_{\tau > \tau_0} x_\tau = 0$ and $x_\tau \to 0$ as $\tau \to \tau_0$, let $K$ be the cone of non-negative functions in $C[0, 1]$ and $x_\tau(t) = t^{1/(\tau-1)}$, $\tau > 1$. Then $\inf_{\tau > 1} x_\tau = 0$ but $\lim_{\tau \to \tau_0} x_\tau$ does not exist in $C[0, 1]$.

Remark 2. If, in addition to (1)–(6), we have that $T(\tau, x) \equiv T_\tau x$ is $C^*$ in $(\tau, x)$ in a neighborhood containing $(0, \tau_1) \times \langle 0, v \rangle \cap \text{int} \ K$ and if $T_\tau x > T'_\tau(x)x$ for each $x \in \text{int} \ K \cap \langle 0, u \rangle$, $\tau \leq \tau_1$, then $\tau \to x_\tau$ is $C^*$.

It should be clear that theorem 3.1 is a sample theorem. The particular interval $(0, \infty)$ parametrizing the family of operators $T_\tau$ is purely arbitrary. The family $\{ T_\tau \}$ could be decreasing on an interval instead of increasing, in which case $x_\tau$ would be decreasing in $\tau$.

It should be emphasized that (5) and (6) as well as the assumption in Remark 1, $T_\tau x > T'_\tau(x)x$, are essentially concavity assumptions on the nonlinear operator $T_\tau$. This is made clear in the figure below in which $T_\tau: R^+ \to R^+ \text{ defined by } T_\tau x = \tau x(1 - x)_+.$

![Figure 3.1: $\tau_1 = 2$](image)

The essential information on the linearized operator $T'_\tau(0)$ we use in theorem 3.1 to obtain the threshold $\tau_0$ is that $1 - \frac{\lambda}{\lambda}(T'_\tau(0))$ eventually becomes positive. We do not require that it change sign.
It is natural to ask under what conditions bifurcation occurs at \( \tau_0 \). Of course, if \( T_\tau x \) is a smooth function of both variables, it is necessary that \( I - T'_\tau(0) \) be noninvertible and in particular \( r(T'_\tau(0)) \geq 1 \). In the following lemma we give some sufficient conditions for bifurcation to occur at \( \tau_0 \).

**Lemma 3.2.** If any one of the following is satisfied then \( x, y > 0 \) as \( \tau \searrow \tau_0 \).

(i) There exists \( u \in K - \{ 0 \} \) and \( n \geq 1 \) such that

\[
u \leq T^n_\tau x_n \quad \text{for all } \tau > \tau_0,\]

(ii) \( K \) is regular, i.e., if \( \{x_n\}_{n=1}^\infty \) satisfies

\[
0 \leq x_n \leq \cdots \leq x_{n-1} \leq \cdots \leq x_2 \leq x_1
\]

then \( \lim_{n \to \infty} x_n \) exists. Of course, it is sufficient if \( \lim_{\tau \to \tau_0} x_n \) exists in \( K \).

(iii) \( X \subset X_1, K \subset K_1 \) where \( K_1 \) is a cone in the Banach space \( X_1 \) and \( \|x\| \leq M \|\cdot\|_{x_1} \) for some \( M > 0 \). \( T_\tau \) can be extended to \( K_1 \) so that (1)–(6) holds and bifurcation occurs at \( \tau_0 \) in \( K_1 \).

**Proof of theorem 3.1.** We first note that from (3) it follows that the map \( \tau \to T'_\tau(0)x \) is nondecreasing for each \( x \in K \). Hence lemma 2.2 applies to the family \( \{T'_\tau(0)\}_{\tau \geq 0} \) yielding that \( \{\tau : \lambda(T'_\tau(0)) > 1\} \) is an open interval, \((\tau_0, \infty)\), for some \( \tau_0 \in [0, \tau_1) \).

If \( \tau_0 < \tau^* \leq \tau_1 \) there exists by lemma 2.1 \( y > 0 \) such that \( T'_\tau(0)y \geq (1 + \delta)y \) for some \( \delta > 0 \). Then, for \( 0 < \epsilon < 1 \),

\[
T_\tau(\epsilon y) = T_\tau(\epsilon y) + (T_\tau(\epsilon y) - T_\tau(\epsilon y))
\]

\[
= T_\tau(0)(\epsilon y) + R(\epsilon, \epsilon y) + (T_\tau(\epsilon y) - T_\tau(\epsilon y))
\]

\[
\geq \epsilon y + \epsilon \left( \frac{\delta}{2} y + \epsilon^{-1} R(\epsilon, \epsilon y) \right) + \epsilon \left( \frac{\delta}{2} y + T_\tau(\epsilon y) - T_\tau(\epsilon y) \right)
\]

where \( R(\epsilon, y) = T_\tau y - T_\tau(0)y \). Since \( y > 0 \) we can choose \( \epsilon \) so small that the second term on the right is in \( K \). Fix such an \( \epsilon \) and then require \( |\tau - \tau^*| \) to be so small that the third term lies in \( K \). Hence, if \( \tau^* \leq \tau_1 \), then \( T_\tau(\langle \epsilon y, v \rangle) \subset \langle \epsilon y, v \rangle \) for \( \tau \) near \( \tau^* \).

It follows from (6) that \( \{T_\tau\}_{\tau \nearrow \tau^*} \) is uniformly concave on \( \langle \epsilon y, v \rangle \) in the sense of theorem 2.3. Hence, by that theorem, \( T_\tau \) has a unique fixed point \( x_\tau \) in \( \langle \epsilon y, v \rangle \) for each sufficiently near \( \tau^* \), the fixed point \( x_\tau \) is the limit of the successive approximations \( x_n = T_\tau x_{n-1}, n = 1, 2, \ldots \) for any \( x_0 \in \langle \epsilon y, v \rangle \). Moreover the map \( \tau \to x_\tau \) for \( \tau \) near \( \tau^* \) is continuous. Since \( \tau^* \in (\tau_0, \tau_1] \) is arbitrary we have established the existence of a fixed point for each \( T_\tau \), \( \tau \in (\tau_0, \tau_1] \).

We next show that if \( T_\tau \) has a nonzero fixed point then it lies in \( \text{int} K \). Let \( T_\tau x = x, x \in \langle 0, v \rangle - \{0\} \). Then for \( 0 < \epsilon < 1 \),
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\[ x = T^n(x) \geq T^p_\tau(\varepsilon x) \]
\[ = T'_\tau(0)^n(\varepsilon x) + o(\varepsilon) \]
\[ = \varepsilon(T'_\tau(0)^nx + O(\varepsilon)). \]

The right hand side is in int \( K \) if \( \varepsilon \) is sufficiently small and \( n = n(x) \) in the definition of strong positivity for \( T'_\tau(0) \).

In order to establish that \( T_\tau \) has a unique fixed point for each \( \tau \in (\tau_0, \tau_1) \) suppose the contrary, i.e., assume \( x, y \in (0, \nu) - \{0\} \) are fixed points of \( T_\tau \). If \( x \neq y \) then we may assume that \( y - x \notin K \) (if \( y - x \in K \) then \( x - y \notin K \), so relabel). \( K \) is closed so there exists \( \lambda, 0 < \lambda < 1 \), such that \( y - \lambda x \notin K \). On the other hand, by the previous paragraph \( y \in int K \) so \( y - \nu x \in K \) for small positive \( \nu \). Since \( K \) is closed it is clear that there exists a maximal \( \mu > 0 \) such that \( y - \mu x \in K \) and that \( 0 < \mu < \lambda < 1 \). Now, by the monotonicity and concavity of \( T_\tau \):

\[ y = T_\tau y \geq T_\tau(\mu x) \geq (1 + \eta)\mu T_\tau x = (1 + \eta)\mu x. \]

Since \( \eta > 0 \), this contradicts the maximality of \( \mu \). Hence we must conclude that \( x = y \).

The uniqueness together with existence and local continuity of \( \tau \mapsto x_\tau \) proves the assertion that the map \( \tau \mapsto x_\tau \) is continuous on \( (\tau_0, \tau_1) \). The map \( \tau \mapsto x_\tau \), nondecreasing since if

\[ \tau_0 < \tau_2 < \tau_3 < \tau_1 \]

then

\[ T_{\tau_0} x_{\tau_2} \geq T_{\tau_0} x_{\tau_3} = x_{\tau_3}. \]

Hence \( T_{\tau_0} \langle x_{\tau_2}, v \rangle \subset \langle x_{\tau_3}, v \rangle \) and by theorem 2.3 and uniqueness of \( x_{\tau_3}, x_{\tau_2} \in \langle x_{\tau_3}, v \rangle \).

We now show that each \( x_\tau \) can be obtained by successive approximations \( x_\tau = T_\tau x_{\tau-1} \) starting from any \( x_0 \in (0, \nu), x_0 \neq 0 \). By an argument similar to that used to show that fixed points lie in int \( K \) it may be shown that \( T_{\tau_0}^n x_0 > 0 \) for some positive integer \( n \). Now \( T_{\tau_0}^n(0) y \geq (1 + \delta) y \) for some \( \delta > 0, y > 0 \) so

\[ T_{\tau_0}(\varepsilon y) = T_{\tau_0}^n(0)(\varepsilon y) + R(\tau, \varepsilon y) \]
\[ = \varepsilon (1 + \delta) y + R(\tau, \varepsilon y) \geq \varepsilon y \]

if \( \varepsilon \) is sufficiently small by an argument similar to one above. We can choose \( \varepsilon \) so small that \( \varepsilon y \leq T_{\tau_0}^n x_0 \). Thus theorem 2.3 completes the proof that \( T_{\tau_0}^n x_0 \rightarrow x_\tau \), as \( p \rightarrow \infty \).

Now suppose \( \tau_0 > 0 \). If \( \tau \leq \tau_0 \) and \( T_\tau x = x \) with \( x \in (0, \nu), x \neq 0 \) then \( x \in int K \) and \( T_{\tau_0} x \geq T_\tau x = x \). Thus \( T_{\tau_0} \langle x, v \rangle \subset \langle x, v \rangle \) and so \( T_{\tau_0} \) has a nonzero fixed point \( x \in (0, \nu) \cap int K \). But then, by (5), \( T_{\tau_0}(0)x = T_{\tau_0} x + \omega \), where \( \omega \in int K \). Since \( \omega \geq \delta x \) for some \( \delta > 0 \) we have \( T_{\tau_0}(0)x \geq (1 + \delta) x \) which contradicts \( \lambda(T_{\tau_0}(0)) \leq 1 \). This proves
that $T_r$ can have no nonzero fixed points in $\langle 0, u \rangle$ for $\tau \leq \tau_0$.

Still assuming $\tau_0 > 0$, suppose $\lim_{\tau \to \tau_0} \text{dist}(x_\tau, \partial K) = r > 0$. Then $x_\tau + B(0, r/2) \subseteq \text{int} K$ for all $\tau \in (\tau_0, \tau_1]$ where $B(0, r/2) = \{x \in X : \|x\| \leq r/2\}$. Choose $u \in B(0, r/2) \cap \text{int} K$. Then $x_\tau - u \in K$ for $\tau \in (\tau_0, \tau]$. Hence, if $0 < \varepsilon < 1$,

$$T_{\tau_0}(\varepsilon x_\tau) = T_{\tau_0}(\varepsilon x_\tau) + T_{\tau_0}(e \varepsilon x_\tau) - T_{\tau_0}(e \varepsilon x_\tau)$$

$$\geq \varepsilon(1 + \eta(e, \varepsilon, x_\tau)) T_{\tau_0}(x_\tau) + T_{\tau_0}(e \varepsilon x_\tau) - T_{\tau_0}(e \varepsilon x_\tau)$$

$$\geq \varepsilon(1 + \eta(e, \varepsilon, u)) x_\tau + T_{\tau_0}(e \varepsilon x_\tau) - T_{\tau_0}(e \varepsilon x_\tau)$$

$$\geq \varepsilon x_\tau + \eta e(e, \varepsilon, u) u + T_{\tau_0}(e \varepsilon x_\tau) - T_{\tau_0}(e \varepsilon x_\tau)$$

where we have used the fact that $\eta(e, \varepsilon, \cdot)$ is nondecreasing in the last variable and that $x_\tau \geq u, \tau \in (\tau_0, \tau_1]$. With $\varepsilon$ fixed, choose $\tau$ sufficiently near to $\tau_0$ so that $\varepsilon \eta u + T_{\tau_0}(e \varepsilon x_\tau) - T_{\tau_0}(e \varepsilon x_\tau) \geq 0$. This can be done by (3). Then $T_{\tau_0}(e \varepsilon x_\tau) \geq e \varepsilon x_\tau$ and this implies, by theorem 2.3, that $T_{\tau_0}$ has a nonzero fixed point which contradicts an earlier argument. Hence $\lim_{\tau \to \tau_0} \text{dist}(x_\tau, \partial K) = 0$. But this implies $\inf \tau > \tau_0 x_\tau = 0$ for all $\tau > \tau_0, u \in K - \{0\}$, then

$$x_\tau = T_{\tau_0}^p x_\tau \geq T_{\tau_0}^p u.$$ 

Since $p$ can be chosen so $T_{\tau_0}^p u > 0$ (independent of $\tau$) this contradicts $\lim_{\tau \to \tau_0} \text{dist}(x_\tau, \partial K) = 0$.

**Proof of remark 2.** Since $x_\tau - T_{\tau_0}'(x_\tau) x_\tau = T_{\tau_0}(x_\tau) - T_{\tau_0}'(x_\tau) x_\tau \in \text{int} K$ we conclude from [6, theorem 5.5, ch. 2] that $r(T_{\tau_0}'(x_\tau)) < 1$. The implicit function theorem then guarantees that $\tau \to x_\tau$ is $C^k$. Note that in this case we can dispense with the uniformity of $\eta$ in $\tau$ for $\tau \leq \tau_1$ assumed in (6).

**Proof of Lemma 3.2.**

(i) If $T_{\tau_0}^p x_\tau \geq u$ for all $\tau > \tau_0$ for some $n \geq 0$ (independent of $\tau$) and some $u \in K - \{0\}$ then

$$x_\tau = T_{\tau_0}^p x_\tau \geq T_{\tau_0}^p x_\tau \geq T_{\tau_0}^p u.$$ 

Since $p$ can be chosen so $T_{\tau_0}^p u > 0$, the above inequality is easily seen to contradict $\lim_{\tau \to \tau_0} \text{dist}(x_\tau, \partial K) = 0$. In fact $x_\tau + B(0, (1/2) \text{dist}(T_{\tau_0}^p u, \partial K)) \subseteq \text{int} K$ for all $\tau > \tau_0$.

(ii) If $\lim_{\tau \to \tau_0} x_\tau = x$ exists then $T_{\tau_0} x = x$. But $T_{\tau_0}$ has only the zero fixed point.

(iii) Obvious.

§ 4. **Almost periodic solutions of (1.1)**

In this section we apply the threshold theorem 3.1 to obtain almost periodic solutions of (1.1). Before beginning, we need to recall some properties of almost periodic functions. The hull of an almost periodic function $f(t)$ is the closure, is the
uniform topology, of the translates \( \{ f(t+\alpha) \}_{\alpha \in \mathcal{R}} \) of \( f \). It is well known \cite{3} that the hull of \( f \) is compact in the uniform topology, indeed, this can be taken as a definition for a continuous bounded function on \( \mathcal{R} \) to be almost periodic. If \( f(t) \) is almost periodic then \( f \) has a Fourier series, \( f(t) \sim \sum_{n=1}^{\infty} a_n e^{i t n} \), where \( \{ \lambda_n \}_{n=1}^{\infty} \) is the countable set of \( \lambda \)'s for which \( \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(s) e^{-i s t} \mathop{ds} \neq 0 \). The \( a_n \)'s are determined by the formula

\[
a_n = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(s) e^{-i s t} \mathop{ds}.
\]

The module of \( f \), \( \text{mod}(f) \), is the smallest subgroup of \( (\mathcal{R}, +) \) containing the set \( \{ \lambda_n \}_{n=1}^{\infty} \). The space of real almost periodic function is a Banach space when provided with the norm, \( \| f \| = \sup_{t \in \mathcal{R}} |f(t)| \). We write \( AP(\mathcal{R}) \) for the Banach space of all real almost periodic functions. Let \( K = \{ x \in AP(\mathcal{R}) \colon \text{x(t)} \geq 0 \text{ for all } t \in \mathcal{R} \} \). \( K \) is easily seen to be a cone with interior: \( \text{int} K = \{ x \in K \colon \text{inf \x > 0} \} \). \( K \) is normal since every order interval \( \langle u, v \rangle \) is clearly bounded. \( K \) is also reproducing since if \( x \in AP(\mathcal{R}) \) then \( x = x^+ - x^- \) where \( x^+(t) = \max \{ x(t), 0 \} \) and \( x^-(t) = \max \{ -x(t), 0 \} \) are in \( K \).

We consider the equation

\[
(4.1), \quad x(t) = \int_{t - \tau}^{t} a(s)x(s)(1-x(s))ds
\]

where \( a(t) \) is almost periodic and \( a \in \text{int } K \). The problem of finding positive almost periodic solutions of (4.1) reduces to finding fixed points for the operator

\[
[T, x](t) = \int_{t - \tau}^{t} a(s)x(s)(1-x(s))ds,
\]

where \( \{ \quad \} \) means the positive part of \( \{ \quad \} \). To see that the right hand side is an almost periodic function it suffices to show that \( B(t) = \int_{t - \tau}^{t} b(s)ds \) is almost periodic whenever \( b(t) \) is almost periodic. Let \( \{ B(t+\alpha_n) \} \) be a sequence of translates of \( B(t) \) and \( \{ b(t+\alpha_n) \} \) the corresponding set for \( b(t) \). Since \( b(t) \in AP(\mathcal{R}) \) there is a subsequence \( b(t+\alpha_{n_k}) \rightarrow c(t) \) uniformly on \( \mathcal{R} \). Clearly

\[
B(t+\alpha_{n_k}) = \int_{t - \tau}^{t} b(s+\alpha_{n_k})ds \rightarrow \int_{t - \tau}^{t} c(s)ds
\]

uniformly on \( \mathcal{R} \). This shows that \( \{ B(t+\alpha) \}_{\alpha \in \mathcal{R}} \) is precompact in the uniform topology and hence \( B \) is almost periodic. The Frechet derivative of \( T, x = 0 \) is given by

\[
(T'(0)h)(t) = \int_{t - \tau}^{t} a(s)h(s)ds, \quad h \in AP(\mathcal{R}).
\]

If \( T \) were a \( k \)-set-contraction on \( K \) then \( T'_0 \) would be a \( k \)-set-contraction \cite{8}. This is easily seen not to be the case. In fact if \( a(t) \equiv 1 \), the linear operator \( S \), defined by
If $\exists$ $\alpha > 0$ in $\set C$, $z(0) = 0$ then $(S_n^x)(t) \equiv 0$ on $[t_0 - \tau, t_0]$ implying that $(S_n^{n-1}x)(t) \equiv 0$ on $[t_0 - 2\tau, t_0]$, $\ldots$, implying that $x(t) \equiv 0$ on $[t_0 - n\tau, t_0]$ which is a contradiction. Thus $(S_n^x)(t) > 0$ for all $t \in \set R$. Let $y(t) = (S_n^x)(t)$ and suppose $\inf_{t \in \set A} y(t) = 0$. Then there exists a sequence $(\alpha_n)_{n=1}^\infty \subseteq \set R$ such that $y(\alpha_n) \to 0$ as $n \to \infty$. Since $(y(t + \alpha_n))_{n=1}^\infty$ and $(x(t + \alpha_n))_{n=1}^\infty$ are precompact, we may as well assume $y(t + \alpha_n) \to z(t)$ uniformly on $\set R$ and that $x(t + \alpha_n) \to w(t)$ uniformly on $\set R$ where $z \in \Hull (y)$, $w \in \Hull (x)$. It is easy to see that $z = S_n^x w$. But $z(0) = 0$ so $w(t) \equiv 0$ on $[-n\tau, 0]$ for every $w \in \Hull (x)$, and $[a, b] \subseteq \set Z$, $b - a \leq M$. This contradiction proves that $S_n^x x \in \Int K$.

The continuity of $(T'_{\tau_1}(0))_{\tau_1 > 0}$ in the uniform operator topology follows from the inequality

$$(T'_{\tau_1}(0)x(t) - T'_{\tau_2}(0)x(t)) = \int_{t-\tau_1}^{t-\tau_2} a(s)x(s)ds \leq ||a|| ||(\tau_1 - \tau_2)|| \|x\|$$

if $\tau_1 > \tau_2$.

In order to satisfy the assumptions of theorem 3.1 we require $(H)$ there exists $\tau_1 > 0$ and $\delta > 0$ such that

(a) $\int_{t-\tau_1}^t a(s)ds \leq 2$ for all $t \in \set R$

(b) $\int_{t-\tau_1}^t a(s)b(s)ds \geq (1 + \delta)b(t)$ for all $t \in \set R$

where $b$ is a positive $\AP$ function.
Clearly (b) implies that \( \lambda (T'_\tau(0)) \geq 1 + \delta > 1 \). If (a) holds and \( \tau \leq \tau_1, 0 \leq x(t) \leq 1/2 \) then

\[
(T_\tau x)(t) \leq \int_{t-\tau}^{t} a(s)x(s)(1-x(s))ds \leq \frac{1}{4} \int_{t-\tau}^{t} a(s)ds \leq 1/2,
\]

i.e. \( T_\tau(0, 1/2) \subset (0, 1/2) \).

The assumption \((H)\) is a restrictive one not required in the periodic case [see e.g. 12]. It is satisfied, for example \((b(t) \equiv 1)\), if

\[
(1 + \delta) \leq \int_{t-\tau}^{t} a(s)ds \leq 2 \quad \text{for all } t \in \mathbb{R}.
\]

For example, if \( a(t) = 1 + Ap(t) \) where \( \|p\| = 1, 0 < A < 1 \), then

\[
(1 - A)\tau \leq \int_{t-\tau}^{t} a(s)ds \leq (1 + A)\tau.
\]

Provided \( A < 1/3 \) we can take \( \tau_1 = 2/(1 + A) \) and \((H)\) will be satisfied.

The main result of this section can now be stated.

**Theorem 4.2.** Let \((H)\) hold. Then there exists \( \tau_0 \in (0, \tau_1) \) such that \((4.1)\), has a unique almost periodic solution \( x_\tau \) in \( (0, 1/2) \cap \text{int } K \) for each \( \tau \in (\tau_0, \tau_1) \). This solution can be obtained by successive approximations: \( x_0 \in (0, 1/2) - \{0\}, x_1 = T_\tau x_0, \ldots, x_{n+1} = T_\tau x_n \). The map \( \tau \rightarrow x_\tau \) for \( \tau \in (\tau_0, \tau_1) \) is nondecreasing in \( K \) and \( C^1 \). There are no almost periodic solutions of \((4.1)\), if \( \tau \leq \tau_0 \) except the zero solution. Moreover \( \inf_{\tau > \tau_0} x_\tau = 0 \). Finally, mod \( x_\tau(t) \subseteq \text{mod } a(t) \) for each \( \tau \in (\tau_0, \tau_1) \).

**Proof of theorem 4.1.** It suffices to check the hypotheses of theorem 3.1. The only remaining nontrivial ones to check are \((5)\) and \((6)\). We begin with \((5)\). Let \( y \in (0, 1/2) \cap \text{int } K \) and consider

\[
T'_\tau(0)y - T_\tau y = \int_{t-\tau}^{t} [a(s)y(s) - a(s)y(s)(1-y(s))]ds
= \int_{t-\tau}^{t} a(s)y^2(s)ds \geq \tau(\inf a)(\inf y)^2 > 0.
\]

Hence \( \inf_{t \in a} (T'_\tau(0)y)(t) - (T_\tau y)(t) > 0 \) so \( T'_\tau(0)y - T_\tau y \in \text{int } K \). An entirely similar argument shows that \( T_\tau x - T'_\tau(x)x \in \text{int } K \) for \( x \in \text{int } K \cap (0, v) \). Note that \( T(\tau, x) = T_\tau x \) is \( C^1 \) in \( (\tau, x) \).

Let \( u \in \text{int } K \) and \( u(t) \leq 1/2 \) for \( t \in \mathbb{R}, \ [a, b] \subset (0, 1) \), \( \delta \in [a, b], \) and \( u \leq x \leq 1/2 \). Then

\[
[T'_\tau(\delta x)](t) = \int_{t-\tau}^{t} a(s)\delta x(s)(1-\delta x(s))ds
\]
\[
\begin{align*}
\delta \int_{t-\tau}^{t} a(s) x(s) \left(1 - \frac{\delta}{x(s)}\right) x(s) \left(1 - x(s)\right) ds \\
\geq \delta \left[ \frac{1 - b \inf u}{1 - \inf u} \right] \int_{t-\tau}^{t} a(s) x(s) \left(1 - x(s)\right) ds \\
\geq \delta(1 + \eta) T_{\varepsilon n}
\end{align*}
\]

where \( \eta = (\inf u(1 - b))/(1 - \inf u) > 0 \). Notice that \( \eta(a, b, u) = (\inf u(1 - b))/(1 - \inf u) \) is nondecreasing in \( u \). This verifies (6) and theorem 3.1 concludes the proof of all but the last assertion.

By theorem 4.5 in [3], mod \( (x_{n}) \subseteq \text{mod} \ (a) \), provided the following holds: if \( \{\alpha_{n}\}_{n=1}^{\infty} \) is a sequence of real numbers for which \( a(t + \alpha_{n}) \rightarrow a(t) \) uniformly on \( \mathcal{R} \) then there is a subsequence \( \{\alpha_{n_{i}}\} \) for which \( x_{i}(t + \alpha_{n_{i}}) \rightarrow x_{i}(t) \) uniformly on \( \mathcal{R} \). Since the hull of \( x_{r} \) is compact there is a subsequence \( \{\alpha_{n_{i}}\} \) and an almost periodic function \( y(\in \text{int} K) \) such that \( x_{i}(t + \alpha_{n_{i}}) \rightarrow y(t) \) uniformly on \( \mathcal{R} \) as \( i \rightarrow \infty \). From the equality,

\[ x_{i}(t + \alpha_{n_{i}}) = \int_{t-\tau}^{t} a(s + \alpha_{n_{i}}) x_{i}(s + \alpha_{n_{i}}) \left(1 - x_{i}(s + \alpha_{n_{i}})\right) ds, \]

we obtain immediately that

\[ y(t) = \int_{t-\tau}^{t} a(s) y(s) \left(1 - y(s)\right) ds. \]

By the uniqueness of \( x_{r} \) we must have \( y = x_{r} \) and this completes the proof of the last assertion of theorem 4.1.

It is possible, in this particular application of theorem 3.1, to obtain more information about the behavior of \( x_{r} \) as \( \tau \rightarrow \tau_{0} \). One easily sees, by differentiating both sides of (1.1), that \( \{x_{r}\}_{r>\tau_{0}} \) is a uniformly bounded and equicontinuous family of \( AP \) functions. Hence

\[ x(t) = \lim_{r \rightarrow \tau_{0}} x_{r}(t) \]

exists, the limit being uniform on compact sets. Thus \( x(t) \) is a bounded continuous function which satisfies (1.1) for \( \tau = \tau_{0} \). If \( x(t) \) could be shown to be almost periodic then it would follow that \( x(t) \equiv 0 \) since \( \inf_{r>\tau_{0}} x_{r}=0 \) and \( 0 \leq x(t) \leq x_{r}(t) \). In this case we could conclude that \( x_{r} \rightarrow 0 \) as \( \tau \rightarrow \tau_{0} \) the convergence being uniform on compact sets (not on \( R \) since Dini’s theorem does not extend in a straightforward way to \( AP \) functions). It is not difficult to show [using thm 1.7, 3] that sufficient condition for \( x(t) \) to be almost periodic is that for every \( c \in \text{Hull} \ (a) \) the equation (1.1) with a replaced by \( c, \tau = \tau_{0} \) has at most one nonzero, nonnegative bounded continuous solution. In any case, the assertion \( \inf_{r>\tau_{0}} x_{r}=0 \) implies that no nonnegative almost periodic function \( u(t) \) can be found other than the zero function which satisfies \( 0 \leq
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$u(t) \leq x(t)$. In particular $\inf_{t \in R} x(t) = 0$, but $\inf_{t > t_0} x(t) = 0$ is a much stronger assertion than this. Exactly how much stronger is not clear to us.

References


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