Fundamental Systems of Analytic Solutions of Fuchsian Type Partial Differential Equations

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In this paper, we discuss the existence of fundamental systems of analytic solutions for Fuchsian type partial differential equations treated in Tahara [3][4]. If the characteristic exponents of the equation do not differ by positive integers, the existence of fundamental systems is established in [4]. Here, we generalize this result to some more general case where the characteristic exponents may differ by positive integers. Throughout this paper, we use the same notations and terminologies as in [4].

§ 1. Fuchsian type equations

Let \((t, z) \in \mathbb{C} \times \mathbb{C}^n\) and let

\[ P(t, z, D_t, D_z) = t^m D_t^m + P_1(t, z, D_z) t^{m-1} D_t^{m-1} + \cdots + P_m(t, z, D_z) \]

be a linear partial differential operator of order \(m\) whose coefficients are holomorphic functions defined in a neighbourhood of the origin. We impose the following conditions on \(P\):

(A-1) the order of \(P_j(t, z, D_z)\) \(\leq j\) for \(1 \leq j \leq m\),
(A-2) the order of \(P_j(0, z, D_z)\) = 0 for \(1 \leq j \leq m\).

Then, the operator \(P\) is said to be of Fuchsian type with respect to \(t\) (Baouendi-Goulaouic [1], Tahara [3][4]). By (A-2), \(P_j(0, z, D_z)\) is a function of \(z\). We set \(P_j(0, z, D_z) = a_j(z)\). Then the indicial equation associated with \(P\) is defined by

\[ 0 = \lambda(\lambda - 1) \cdots (\lambda - m + 1) + a_1(z)\lambda(\lambda - 1) \cdots (\lambda - m + 2) + \cdots + a_m(z). \]

The roots, which we denote by \(\rho_1(z), \ldots, \rho_m(z)\), are called the characteristic exponents of \(P\). Further we assume the following condition:

(A-3) the order of \((D_t^l P_j)(0, z, D_z)\) < \(j\) for \(1 \leq j \leq m\) and \(0 \leq l \leq n_0\),

where \(n_0\) is the non negative integer defined by

\begin{equation}
1.1 \quad n_0 = \max (Z \cap [\rho_i(0) - \rho_j(0); 1 \leq i, j \leq m]).
\end{equation}

Under these assumptions (A-1), (A-2) and (A-3), we have
Theorem 1. Let $\tilde{\mathcal{O}}$ be the set of all germs of multivalued holomorphic functions on $C \times C^n - \{t=0\}$ near the origin. Then the equation $Pu=f$ is always solvable in $\tilde{\mathcal{O}}$. Moreover, there exist holomorphic functions $K_j(t, z, w) (1 \leq j \leq m)$ on

$$\{(t, z, w) \in (\tilde{C}-0) \times C^n \times C^n; |t|, |z|, |w| < \varepsilon,$$

$$0 < |t| < M|z_i - w_i|^{n-1} \text{ for } i = 1, \ldots, n\}$$

(where $\tilde{C}-0$ means the universal covering space of $C-0$)

which satisfy the following conditions:

1. For any holomorphic functions $\varphi_j(z) (1 \leq j \leq m)$ at the origin, we set

$$u(t, z) = \sum_{j=1}^{m} \int K_j(t, z, w)\varphi_j(w)dw.$$ 

Then $u(t, z)$ is a solution of the equation $Pu=f$ in $\tilde{\mathcal{O}}$.

2. If $u(t, z) \in \tilde{\mathcal{O}}$ and $Pu=0$ holds, then $u(t, z)$ is uniquely expressed in the form (1).

Remark 1. We call $\{K_j(t, z, w); 1 \leq j \leq m\}$ a fundamental system of solutions in $\tilde{\mathcal{O}}$. See Section 1.3 in [4].

Remark 2. If $\rho_i(0) - \rho_j(0) \notin Z - \{0\}$ holds for $1 \leq i, j \leq m$, then we have $n_0 = 0$. In this case, (A-3) is trivial from (A-2) and the above conditions coincide with those assumed in [3] and [4].

§ 2. Fuchsian systems

Let $(t, z) \in C \times C^n$ and let

$$P(t, z, D_t, D_z) = tD_t - A(t, z, D_z)$$

be an $m \times m$ matrix of differential operators whose coefficients are holomorphic functions defined in a neighbourhood of the origin. We impose the following conditions on $P$:

(B-1) the matrix order of $A(t, z, D_z) \leq 1$,

(B-2) the order of $A_{ij}(0, z, D_z) = 0$ for $1 \leq i, j \leq m$,

where $A_{ij}(t, z, D_z)$ is the $(i, j)$ component of the matrix $A(t, z, D_z)$. Then, the operator $P$ is said to be a Fuchsian system with respect to $t$ (Tahara [4]). By (B-2), $A(0, z, D_z)$ is a matrix of functions of $z$. We set $A(0, z, D_z) = A_0(z)$. Then the roots of the equation $\det (\lambda - A_0(0)) = 0$, that we denote by $\alpha_1, \ldots, \alpha_m$, are called the characteristic eigen-values of $P$. Further we assume the following condition:

(B-3) the matrix order of $(D_t^l A)(0, z, D_z) < 1$ for $0 \leq l \leq n_0$,

where $n_0$ is the non negative integer defined by
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(2.1) \[ n_{0} = \max (Z \cap \{\alpha_{i} - \alpha_{j}; 1 \leq i, j \leq m\}). \]

Under these assumptions (B-1), (B-2) and (B-3), we have

**Theorem 2.** The equation \( Pu = f \) is always solvable in \( \tilde{\mathcal{O}} \). Moreover, there exists a matrix \( K(t, z, w) \) of holomorphic functions on

\[ \{(t, z, w) \in (C-0) \times C^{n} \times C^{n}; |t|, |z|, |w| < \varepsilon, \]

\[ 0 < |t| < M |z_{i} - w_{i}|^{s-1} \text{ for } i = 1, \ldots, n \} \]

(where \( s \) is the \( F \)-degree of \( P \))

which satisfies the following conditions:

1. For any holomorphic function \( \varphi(z) \) at the origin, we set

\[ u(t, z) = \oint K(t, z, w)\varphi(w)dw. \]

Then \( u(t, z) \) is a solution of the equation \( Pu = 0 \) in \( \tilde{\mathcal{O}} \).

2. If \( u(t, z) \in \mathcal{O} \) and \( Pu = 0 \) holds, then \( u(t, z) \) is uniquely expressed in the form (1).

§ 3. Proofs of Theorems 1 and 2

Before the proofs of Theorems 1 and 2, we prepare some lemmas.

**Lemma 1.** Let \( P_{A} = tD_{t} - A(t, z, D_{z}) \), \( P_{B} = tD_{t} - B(t, z, D_{z}) \) be two Fuchsian systems (of size \( m \)) with respect to \( t \) and let \( s_{A}, s_{B} \) be the \( F \)-degrees of \( P_{A}, P_{B} \) respectively. Put

\[ \tilde{P}(t, z, w, D_{t}, D_{z}, D_{w}) = tD_{t} - I_{m} \otimes A(t, z, D_{z}) + B(t, w, D_{w}) \otimes I_{m}, \]

where \( \otimes \) means the Kronecker product of matrices. Then we have

1. \( \tilde{P} \) is also a Fuchsian system with respect to \( t \),

2. the \( F \)-degree of \( \tilde{P} \leq s_{A} + s_{B} - 1 \).

**Proof.** First, we will show (1). We set

\[ A(t, z, D_{2}) = (A_{ij}(t, z, D_{2}))_{1 \leq i, j \leq m}; \]

\[ B(t, z, D_{3}) = (B_{ij}(t, z, D_{3}))_{1 \leq i, j \leq m}; \]

\[ C(t, z, w, D_{w}) = I_{m} \otimes A(t, z, D_{2}) - B(t, w, D_{w}) \otimes I_{m} \]

\[ = (C_{ij,kl})_{1 \leq i, j, k, l \leq m}; \]

\[ C_{ij,kl} = \delta_{ik}A_{jl}(t, z, D_{2}) - \delta_{jl}B_{ik}(t, w, D_{w}). \]

Since the matrix orders of \( A(t, z, D_{2}) \) and \( B(t, z, D_{3}) \) are at most 1, we can take
vectors \((n_1, \ldots, n_m), (m_1, \ldots, m_m)\) of integers such that \(\text{ord} A_{ij}(t, z, D_z) \leq n_i - n_j + 1\) and \(\text{ord} B_{kl}(t, z, D_z) \leq m_k - m_l + 1\) for \(1 \leq i, j, k, l \leq m\). Then, by the definition of \(C(t, z, w, D_z, D_w)\) we have
\[
\text{ord} C_{ijkl}(t, z, w, D_z, D_w) \leq (n_j + m_i) - (n_i + m_j) + 1
\]
for \(1 \leq i, j, k, l \leq m\). Therefore, the matrix order of \(C(t, z, w, D_z, D_w)\) is also at most 1. On the other hand, it is clear that \(C(0, z, w, D_z, D_w)\) is a matrix of functions. Therefore, \(\tilde{P}\) is a Fuchsian system (of size \(m^2\)) with respect to \(t\). Thus, (1) is proved.

Next, we will show (2). Denote the \(F\)-degree of \(\tilde{P}\) by \(\tilde{s}\). If \(\tilde{s} = 1\), then (2) is clear because \(s_d \geq 1\) and \(s_B \geq 1\). If \(\tilde{s} > 1\), then by the definition of the \(F\)-degree we can find a cyclic permutation \(((i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q) = (k, l))\) of length \(p\) such that \(\tilde{s} = |(n_j + m_i) - (n_i + m_j)| + 1\) and
\[
\delta_{iv_{v+1}} A_{0,jv_{v+1}}(z) - \delta_{jv_{v+1}} B_{0,iv_{v+1}}(w) \equiv 0
\]
for \(1 \leq v \leq p - 1\). In this case, (2) is easily obtained from the following facts:
\[
|n_j - n_i| + 1 \leq s_d, \tag{3.2}
\]
\[
|m_i - m_k| + 1 \leq s_B. \tag{3.3}
\]
Therefore, we have only to show (3.2) and (3.3). (3.2) is verified as follows. If \(j = l\), then (3.2) is trivial because \(s_d \geq 1\). If \(j \neq l\), then by the same argument as in the proof of Lemma 1.3.7 in [4] we can choose a subset \(\{j_{a_1}, \ldots, j_{a_q}\}\) of \(\{j_1, \ldots, j_q\}\) which satisfies the following conditions:
(i) \(1 \leq a_1 < a_2 < \cdots < a_q \leq p\);
(ii) \(j_{a_1}, \ldots, j_{a_q}\) are distinct;
(iii) \(j = j_{a_1}, j_{a_{p+1} - 1} = 1 \leq \mu \leq g - 1), j_{a_q} = l\).
Then \((j_{a_1} = j, j_{a_2} = j, \ldots, j_{a_q} = l)\) is a cyclic permutation and for this cycle
\[
A_{0,ja_1(p-1) + 1}(z) = A_{0,ja_1(p+1) - 1}A_{0,ja_1(p+1)}(z) \neq 0
\]
where \(a(\mu) = a_\mu\) and \(a(\mu + 1) = a_{\mu + 1}\)
holds for \(1 \leq \mu \leq g - 1\) from (3.1). This implies \(|n_j - n_i| + 1 \leq s_d\). Thus, (3.2) is proved. The proof of (3.3) is the same as (3.2). Therefore, the proof of (2) is completed.

**Lemma 2.** Let \(P = tD_t - A(t, z, D_z)\) be a Fuchsian system (of size \(m\)) with respect to \(t\) and let \(s\) be the \(F\)-degree of \(P\). Then, there exists an invertible matrix \(U = U(t, z, D_z)\) in \(M(m, \mathcal{D}(2s - 1))\) which satisfies the equation
\[
\begin{align*}
U^{-1}PU &= tD_t - \sum_{k=0}^{m} t^k A_k(z, D_z), \\
U|_{t=0} &= I \quad \text{and} \quad D_t U|_{t=0} = 0 \quad \text{for} \quad 1 \leq l \leq n_0.
\end{align*}
\]
where \( A_{k}=(1/k!)(D^{k}A)_{|z=0} \) and \( n_{0} \) is the non negative integer defined by (2.1). Here, \( \mathcal{D}(2s-1) \) means the ring of formal differential operators of degree at most \( 2s-1 \).

**Proof.** Let

\[
\tilde{P} = tD_{t} - I_{m} \otimes A(t, z, D_{z}) + \sum_{k=0}^{n_{0}} t^{k}(\bar{A}_{k}(w, D_{w}) \otimes I_{m}),
\]

where \( \bar{A}_{k}(w, D_{w}) \) is the formal adjoint operator of \( A_{k}(w, D_{w}) \). Then, from Lemma 1 we have

1. \( \tilde{P} \) is also a Fuchsian system with respect to \( t \),
2. the F-degree of \( \tilde{P} \leq 2s-1 \).

Hence, using (1) and (2) instead of Lemma 1.3.7 in [4] we can easily obtain this lemma by the same argument as in the proof of Theorem 1.3.6 in [4]. Therefore we may omit the details.

**Lemma 3.** Let \( Q = tD_{t} - B(t, z, D_{z}) \) be an \( m \times m \) matrix of differential operators near the origin and assume that the matrix order of \( B(t, z, D_{z}) < 1 \). Then, there exists an invertible matrix \( V = V(t, z, D_{z}) \) of differential operators on \( \tilde{U} = \{(t, z) \in (\mathbb{C} - 0) \times C^{n}; 0 < |t| < \varepsilon, |z| < \varepsilon \} \) which satisfies the equation

\[
V^{-1}QV = tD_{t}
\]
on \( \tilde{U} \).

**Proof.** Take any \( t_{0} \neq 0 \). Then, making use of Remark 2 in pp. 447–448 of Sato-Kawai-Kashiwara [2] we can find an invertible matrix \( V = V(t, z, D_{z}) \) of differential operators on \( U_{0} = \{(t, z) \in C \times C^{n}; |t - t_{0}| < \delta_{0}, |z| < \varepsilon_{0} \} \) (where \( 0 < \delta_{0} < |t_{0}| \) and \( 0 < \varepsilon_{0} \)) which satisfies the equation

\[
\begin{cases}
QV = VtD_{t}, \\
V|_{z=t_{0}} = I
\end{cases}
\]
on \( U_{0} \). Since we can choose \( \varepsilon_{0} > 0 \) independent of \( t_{0} \), this immediately leads us to Lemma 3.

**Proof of Theorem 2.** From Lemma 2 we can find \( U = U(t, z, D_{z}) \) in \( GL(m, \mathcal{D}(2s-1)) \) such that \( U^{-1}(tD_{t} - A(t, z, D_{z}))U = tD_{t} - \sum_{k=0}^{n_{0}} t^{k}A_{k}(z, D_{z}) \). Further, applying Lemma 3 to the operator \( tD_{t} - \sum_{k=0}^{n_{0}} t^{k}A_{k}(z, D_{z}) \) we have \( V^{-1}(tD_{t} - \sum_{k=0}^{n_{0}} t^{k}A_{k}(z, D_{z}))V = tD_{t} \), for some invertible matrix \( V = V(t, z, D_{z}) \) of differential operators on \( \{(t, z) \in (\mathbb{C} - 0) \times C^{n}; 0 < |t| < \varepsilon, |z| < \varepsilon \} \). Since the mapping \( U V : \mathcal{E} \rightarrow \mathcal{E} \) is invertible, the equation \( Pu = f(u, f \in \mathcal{E}) \) is equivalent to the equation \( tD_{t}v = g(u, g \in \mathcal{E}) \) under the relations \( u = UVu \) and \( g = UVf \). Note that the operator \( UV \) can be expressed by an integral operator whose kernel function is holomorphic on
\[(t, z, w) \in (C - 0) \times C^n \times C^n; |t|, |z|, |w| < \varepsilon, 0 < |t| < M |z_i - w_i|^{2s-1} \text{ for } i = 1, \ldots, n\].

Therefore, Theorem 2 is clear. Q.E.D.

**Proof of Theorem 1.** Let \( P \) be the operator in Theorem 1. Then the equation \( Pu = f \) is equivalent to

\[
\begin{pmatrix}
0, & 1, & 1, & 2, & \cdots \\
1, & -P_m, & -P_m - 1, & \cdots, & -P_1 + m - 1 \\
1, & -P_{m-1}, & \cdots, & 1, & \cdots \\
2, & \cdots & \cdots & \cdots & \cdots \\
1, & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_m
\end{pmatrix}
= 
\begin{pmatrix}0 \\
0 \\
\vdots \\
f
\end{pmatrix}
\]

under the relations \( u_j = t^{j-1} D_t^{j-1} u \) for \( 1 \leq j \leq m \). Therefore, applying Theorem 2 to the above equation we can easily obtain Theorem 1. Q.E.D.

**References**


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