On Solutions of the Derivative Nonlinear Schrödinger Equation, II.

By

Masayoshi TSUTSUMI and Isamu FUKUDA

(Waseda University and Kokushikan University, Japan)

§ 1. Introduction

This is a continuation of an earlier paper "on solutions of the derivative nonlinear Schrödinger equation. Existence and Uniqueness Theorem" [4] concerning the existence and uniqueness of local smooth solutions and global weak solutions for two initial value problems to the derivative nonlinear Schrödinger (DNLS) equation:

\[
\frac{\partial u}{\partial t} = i \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}(|u|^2 u) + g(x, t), \quad (x, t) \in R \times R^+, \tag{1.1}
\]

\[
\frac{\partial^k u}{\partial x^k}(x, t) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty, \quad k=0, 1, 2, \ldots, \tag{1.2}
\]
or

\[
\frac{\partial^k u}{\partial x^k}(x+1, t) = \frac{\partial^k u}{\partial x^k}(x, t), \quad k=0, 1, 2, \ldots, \tag{1.2}'
\]

\[
u(x, 0) = u_0(x). \tag{1.3}
\]

Here we investigate the global existence of strong and smooth solutions for small data and continuous-dependence-on-data. The notations and definitions introduced in [4] will be used throughout this paper.

Let \( A = -\partial^2/\partial x^2 \) with domain \( D(A) = H^2 \), its fractional power \( A^{\alpha} (\alpha \geq 0) \) with domain \( D(A^{\alpha}) = H^{2\alpha} \) and \( J_u = (\partial/\partial x)(|u|^2 u) \). We denote \( [u]_{\alpha} = (\|u\|^2 + \|A^{\alpha} u\|^2)^{1/2} \). We can rewrite (1.1), (1.2) or (1.2)', (1.3) as

\[
u_t = -iAu + Ju + g, \tag{1.4}
\]

\[
u(0) = u_0. \tag{1.5}
\]

For convenience we propose the local existence result for smooth solutions to the problem (1.4)--(1.5) which is obtained in [4].

Theorem 1 (Local smooth solution). Let \( \alpha \) be any real number \( >3/4 \) and \( T>0 \).
Let $u_0 \in D(A^\alpha)$ and $g \in L^1(0, T; D(A^\alpha))$. Then there exists a positive constant $T_0$ with $0 < T_0 < T$ such that the initial value problem (1.4)-(1.5) has a unique solution $u=u(t)$ in the interval $[0, T_0]$ with the following property;

$$u \in L^\infty(0, T_0; D(A^\alpha)) \cap C([0, T_0]; D(A^{\alpha-1}))$$

and

$$u_t \in L^1(0, T_0; D(A^{\alpha-1})).$$

**Remark 1.** $T_0$ depends only on $[u_0]$, and

$$\int_0^r [g(t)]dt,$$

where $\gamma$ is an arbitrary fixed number such that $3/2 < \gamma < 2\alpha$. Moreover for each bounded set of the form

$$\left\{u, g : [u] \leq a, \int_0^r [g(t)]dt \leq b \right\}$$

$T_0$ can be chosen uniformly for all $\{u_0, g\}$ in the set. Hence, as $\alpha \to +\infty$, the existence interval $[0, T_0]$ does not shrink to a point $\{0\}$. If $T_0 < T$, then $\lim_{t \to T_0} [u(t)]_{2\alpha} = +\infty$.

**Remark 2.** In addition to the assumption in Theorem 1, if we assume that $\alpha \geq 1$ and $g \in L^1(0, T; D(A^\alpha)) \cap C([0, T]; D(A^{\alpha-1}))$, then (1.4)-(1.5) has a continuously differentiable (strong) solution for $t \in [0, T]$. Moreover, if $u_0 \in D(A^\alpha)$ and $g \in C^\infty([0, T]; D(A^\alpha))$, then $u \in C^\infty([0, T_0]; D(A^\alpha))$.

§ 2. Results

**Theorem 2** (Global strong solution). Let $T > 0$, $u_0 \in D(A)$ and $g \in L^\infty(0, T; L^3)$ \cap $L^3(0, T; D(A))$. Then there exists a $\kappa > 0$ with the following property; if $[u_0] \leq \kappa$ and $\|g\|_{L^1(0, T; L^2)} \leq \kappa$, the initial value problem (1.4)-(1.5) has a unique strong solution $u = u(t)$ in the large such that

$$u \in L^\infty(0, T; D(A)) \cap C([0, T]; L^3)$$

and

$$u_t \in L^\infty(0, T; L^3).$$

**Theorem 3** (Global smooth solution). Let $T > 0$ and $\alpha \geq 1$. Assume that $u_0 \in D(A^\alpha)$ and $g \in L^\infty(0, T; L^3) \cap L^3(0, T; D(A)) \cap L^1(0, T; D(A^\alpha))$ such that $[u_0] \leq \kappa$ and $\|g\|_{L^1(0, T; L^2)} \leq \kappa$ where $\kappa$ is as in Theorem 2. Then the initial value problem (1.4)-(1.5) has a unique solution $u = u(t)$ so that
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$$u \in L^\infty(0, T; D(A^\alpha)) \cap C([0, T]; D(A^{\alpha-1}))$$

and

$$u_t \in L'(0, T; D(A^{\alpha-1})) \cap L^\infty(0, T; L^2).$$

Remark 3. Due to Strauss [3], $u(t)$ is weakly continuous on $[0, T]$ with values in $D(A^\alpha)$.

Corollary. If $u_0 \in D(A^\alpha)$ and $g \in C^\infty([0, T]; D(A^\alpha))$ with $[u_0] \leq \kappa$ and $\|g\|_{L^1(0, T; L^2)} \leq \kappa$, then $u \in C^\infty([0, T]; D(A^\alpha))$.

We shall consider the initial value problem (1.4)–(1.5) with $g=0$. Under the assumption of Theorem 3, let us define the mapping $S_t$:

$$u(0)=u_0 \mapsto u(t)$$

which is well defined from $D(A^\alpha)$ into itself ($\alpha \geq 1$).

Theorem 4 (continuous dependence). $S_t$ is locally Hölder continuous with exponent $1/2$ in the following sense: there exists a continuous function $L$ from $R^+ \times R^+$ into $R^+$ such that

$$[S_t u_0 - S_t v_0]_{2a} \leq L([u_0]_{2a+1/2}, [v_0]_{2a+1/2}) [u_0 - v_0]_{2a}^{1/2}.$$

Remark 4. We regard $S_t$ as an operator from $D(A^{\alpha+1/2})$ into $D(A^\alpha)$ ($\alpha \geq 1$). Then $S_t$ is locally Lipschitz continuous. Indeed, the estimate

$$[S_t u_0 - S_t v_0]_{2a} \leq L([u_0]_{2a+1}, [v_0]_{2a+1}) [u_0 - v_0]_{2a}$$

holds.

Remark 5. Let $g_i \in L^\infty(0, T; L^2) \cap L^2(0, T; D(A^\alpha))$ ($i=1, 2$). Let $u$ and $v$ be the solutions of (1.4)–(1.5) with $g=g_1$ and $g=g_2$, respectively. Then we have

$$[u(t) - v(t)]_{2a} \leq L([u_0]_{2a+1/2}, [v_0]_{2a+1/2}) [u_0 - v_0]_{2a}^{1/2} + c \|g_1 - g_2\|_{L^2(0, T; D(A^\alpha))}$$

and

$$[u(t) - v(t)]_{2a} \leq L([u_0]_{2a+1}, [v_0]_{2a+1}) [u_0 - v_0]_{2a} + c \|g_1 - g_2\|_{L^2(0, T; D(A^\alpha))}.$$

Theorem 4 is obtained by much the same way as in Saut and Temam [2] and its proof is omitted.

§ 3. Proof of Theorem 2 and 3

We begin with a statement of a lemma concerning with conserved functionals associated with a solution of (1.1) with $g \equiv 0$. 
Lemma 1. Let \( v(x, t) \) be a smooth solution of (1.1) with \( g \equiv 0 \). Then the following quantities are independent of \( t \);

\[
I_0(v(t)) = \| v(t) \|^2
\]

\[
I_2(v(t)) = \| Dv(t) \|^2 + \frac{1}{2} |v(t)|_6^6 - \frac{3}{2} \text{Im} (|v(t)|^2 Du(t), v(t)),
\]

\[
I_4(v(t)) = \| D^2v(t) \|^2 + \frac{7}{8} |v(t)|_{10}^{10} + \frac{25}{2} \text{Re} (|v(t)|^4 v(t), (Dv(t))^2) + 5 \text{Re} (|v(t)|^4 v(t), Dv(t)) - \frac{35}{8} \text{Im} (|v(t)|^6 v(t), Dv(t)).
\]

Proof. (3.1) is easily obtained. After long and tedious calculations, (4.14) and (4.16) in section 4 yield (3.2) and (3.3), respectively.

Proof of Theorem 2. Let \( u \) be a smooth solution of (1.4)–(1.5) for \( t \in [0, T_0] \). We multiply (1.4) by \( u \), integrate with respect to \( x \) and take the real part to obtain

\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|^2 = \text{Re} (g(t), u(t)) \leq \| g(t) \| \| u(t) \|.
\]

form which it follows that

\[
\| u(t) \| \leq \| u_0 \| + \int_0^T \| g(t) \| dt \text{ for } 0 < t < T_0.
\]

Thus we have

\[
\| u \|_{L^\infty(0, T_0; L^2)} \leq c_1.
\]

Here and in the sequel, \( c_k \) \((k = 1, 2, \ldots)\) and \( c \) denote various positive constants independent of \( T_0 \).

Differentiating \( I_2(u(t)) \) with respect to \( t \), using the eq. (1.4) and integrating by parts with respect to \( x \), we have

\[
\frac{d}{dt} I_2(u(t)) = 2 \text{Re} (Du, Dg) + 3 \text{Re} (|u|^2 u, g) - 6 \text{Im} (|u|^2 Du, g) \leq c \| Du \|^2 + c \| g \|^2 + c \text{ for } 0 \leq t \leq T_0
\]

where the time variable is suppressed.

Integration of (3.6) on \([0, t]\) gives

\[
\| Du \|^2 + \frac{1}{6} |u|^6 - \frac{3}{8} \text{Im} (|u|^2 Du, u)
\]

\[
\leq c \int_0^t \| Du(s) \|^2 ds + c \int_0^t \| g(s) \|^2 ds + cT + \text{(initial value)} \text{ for } 0 \leq t \leq T_0 < T.
\]
The third term of the left hand side of (3.7) is bounded in absolute value by

$$\frac{3}{2} |u|_{6}^{2} \|Du\| \leq \frac{3}{4} |u|_{6}^{2} + \frac{3}{4} \|Du\|^{2}.$$

Since $|u|_{6} \leq K(\|Du\| + \|u\|)^{1/3} \|u\|^{2/3}$, we have

$$\tag{3.8} (1 - c_{1}^{3}K^{3}) \|Du(t)\|^{2} \leq c \int_{0}^{T} \|Du(s)\| \, ds + c$$

for $0 \leq t \leq T_{0} < T$. We choose $\kappa$ so that

$$1 - c_{1}^{3}K^{3} > \delta > 0$$

provided $\|u_{0}\| \leq \kappa$ and $\|g\|_{L^{1}(0,T;L^{2})} \leq \kappa$. Then, using Gronwall's inequality and (3.8) we get

$$\tag{3.9} \|Du\|_{L^{\infty}(0,T_{0};L^{2})} \leq c_{2}.$$
Hence we have from (3.12)

$$\|D^2 u(t)\|^2 \leq c \int_0^t \|D^2 u(s)\|^2 \, ds + c \|g\|_{L^2(0,T; D(A))} + c$$

for $0 \leq t \leq T_0 < T$, from which it follows that

(3.13) \[ \|D^2 u\|_{L^2(0,T; L^2)} \leq c_3 \]

where $c_3$ depends on the initial data $u_0$ and the forcing term $g$. Thus by virtue of the last statement of Remark 1, we conclude that the solution $u(t)$ of the initial value problem (1.4)–(1.5) continues globally up to $T$.

**Proof of Theorem 3.** Let $u$ be a smooth solution of (1.4)–(1.5) for $t \in [0, T_0]$. We apply $A^s$ to each member of (1.4) and take the $L^2$-scalar product of the resultant equation with $A^s u$. The real part of the result yields

(3.14) \[ \frac{1}{2} \frac{d}{dt} \|A^s u\|^2 = \text{Re} (A^s Ju, A^s u) + \text{Re} (A^s g, A^s u). \]

We rewrite the first term of the right hand side of (3.14) in the form

(3.15) \[ 2 \text{Re} (A^s |u|^2 Du) - |u|^2 A^s Du, A^s u) + 2 \text{Re} (|u|^2 A^s Du, A^s u) \]

\[+ \text{Re} (A^s (u^2 Du) - u^2 A^s Du, A^s u) + \text{Re} (u^2 A^s Du, A^s u). \]

If $s > 1/2$, then the first and third terms of (3.15) are bounded by

$$c \|u\|_r [u]_s \quad \text{for any } r > \frac{1}{2}. \]

Here we have used lemma 2.3–2.5 introduced in [4]. Since $A^s$ and $D$ commute, integration by parts gives

$$2 \text{Re} (|u|^2 A^s Du, A^s u) = - \text{Re} (A^s u \cdot D(|u|^2), A^s u)$$

which is bounded in absolute value by

$$c \|u\|_r [u]_s \quad \text{for any } r > \frac{1}{2}. \]

Thus we have

(3.16) \[ \frac{1}{2} \frac{d}{dt} \|A^s u\|^2 \leq c \|u\|_r [u]_s + \|A^s g\| \|A^s u\|. \]

Combining (3.5) and (3.16), we get

(3.17) \[ \frac{d}{dt} [u]_r \leq c \|u\|_r [u]_s + 2[g]_s [u]_s. \]
Since $s > 1/2$ and $r > 1/2$ are arbitrary number, we can choose $s$ and $r$ in (3.17) so that $s = \alpha$ and $r = 1$. Then using the fact that $\|u(t)\|_s \leq c_4$ obtained in the proof of Theorem 2, we have

$$\frac{d}{dt}[u(t)]_s \leq c[u(t)]_s + 2[g(t)]_s$$

which implies that

(3.18) $$[u(t)]_s \leq [u_0]_s + c \int_0^t [g(s)]_s ds$$

for $0 \leq t \leq T_0 < T$. Hence

$$\|u\|_{L^\infty(0,T_0; D(A^s))} \leq c_4.$$  

Since $c_4$ is independent of $T_0$, the solution $u(t)$ continues globally up to $T$.

§ 4. Conservation laws

This section is devoted to formal derivation of the recurrence formula of the infinite family of conserved functionals for (1.1) with $g \equiv 0$. Another approach has been done by Kaup-Newell [1]. Our recurrence formula (4.13) below is more convenient than Kaup-Newell’s.

Let $\zeta$ be a complex parameter. We consider a system of total differential equations

\begin{align*}
(4.1a) & \quad dv_1 = F_1dx + G_1dt, \\
(4.1b) & \quad dv_2 = F_2dx + G_2dt
\end{align*}

where $F_1 = -i\zeta^2v_1 + q(x, t)\zeta v_2$, $F_2 = i\zeta^2v_2 + r(x, t)\zeta v_1$, $G_1 = A(x, t, \zeta)v_1 + B(x, t, \zeta)v_2$ and $G_2 = C(x, t, \zeta)v_1 - A(x, t, \zeta)v_2$, which can be rewritten as

\begin{align*}
(4.2a) & \quad v_{1x} = -i\zeta^2v_1 + q(x, t)\zeta v_2, \\
(4.2b) & \quad v_{2x} = i\zeta^2v_2 + r(x, t)\zeta v_1, \\
(4.3a) & \quad v_{1t} = Av_1 + Bv_2 \\
& \quad v_{2t} = Cv_1 - Av_2.
\end{align*}

Because of the integrability condition for (4.1), we have

\begin{align*}
(4.4a) & \quad A_x = \zeta(qC - rB),
\end{align*}
\( B_{x} + 2i\zeta^{2}B = q_{t} - 2\zeta q A, \)
\( C_{x} - 2i\zeta^{2}C = r_{t} + 2\zeta r A. \)

As a special case, we choose
\[
A = -2i\zeta^{4} - iqr\zeta,
\]
\[
B = 2q\zeta^{4} + q^2r\zeta + iq_{x}\zeta,
\]
\[
C = 2r\zeta^{4} + qr\zeta - ir_{x}\zeta
\]
which are solutions of (4.4).

If we set \( r = \bar{q}, \) then from (4.4) and (4.5) we have the so-called DNLS equation:
\[
q_{t} = iq_{xx} + (q^2\bar{q})_{x}.
\]

We now present a method of a recurrence formula of the infinite family of conserved functionals. From (4.2a) we get
\[
v_{2} = \frac{1}{\zeta q}(v_{1x} + i\zeta^{2}v_{1}).
\]
Substitution (4.7) into (4.2b) yields
\[
q\left(\frac{1}{q}v_{1x}\right)_{x} - i\zeta^{4}\frac{q_{x}}{q}v_{1} + \zeta^{4}v_{1} = \zeta^{2}qr v_{1}.
\]
We find a solution of (4.8) in the form
\[
v_{1}(x, t, \zeta) = \exp(-i\zeta^{4}x + \int^{x}E(s, t, \zeta)ds).
\]
On the other hand, we substitute (4.7) into (4.3a) to obtain
\[
v_{1t} = A v_{1} + B\left(\frac{v_{1x}}{\zeta q} + i\frac{\zeta v_{1}}{q}\right).
\]
(4.9) and (4.10) imply that
\[
\int^{x}E_{1}(s, t, \zeta)ds \cdot v_{1} = \left( A + \frac{B}{\zeta q} \right) v_{1}
\]
from which it follows that
\[
\frac{\partial}{\partial t}E(x, t, \zeta) = \frac{\partial}{\partial x}\left( A + \frac{B}{\zeta q} \right)E(x, t, \zeta).
\]
Hence, the quantity \( \int E(x, t, \zeta)dx \) is time invariant.
From (4.8) and (4.9) we get the equation for $E(x, t, \zeta)$;

\begin{equation}
E(x, t, \zeta) - \frac{i}{2} q r = \frac{1}{2i\xi} \left( E(x, t, \zeta)^2 + q \left( \frac{1}{q} E(x, t, \zeta) \right)_x \right).
\end{equation}

Let $E - (i/2) q r = Z$ and $\zeta^2 = \xi$. We have

\begin{equation}
Z(x, t, \xi) = \frac{1}{2i\xi} \left( \left( Z(x, t, \xi) + \frac{i}{2} q r \right)^2 + q \left( \frac{1}{q} Z(x, t, \xi) \right) + \frac{i}{2} r \right)_x.
\end{equation}

Since the eq. (4.12) is a Riccati type differential equation, it is well known that $Z$ has the asymptotic expansion in $\xi^{-1}$ in the form $Z = \sum Z^{(n)}(x, t, \xi) \cdot (2i\xi)^{-n}$, where each $Z^{(i)} (i=0, 1, 2, \cdots)$ are determined by the following recurrence formula;

\begin{equation}
Z^{(i+1)} = \sum_{k=1}^{i-1} Z^{(k)} Z^{(i-k)} + i q r Z^{(i)} + q \frac{\partial}{\partial x} \left( \frac{1}{q} Z^{(i)} \right) \quad (i=1, 2, 3, \cdots)
\end{equation}

with $Z^{(0)} = qr$ and

\begin{equation}
Z^{(i)} = -\frac{1}{4} q r^2 + \frac{i}{2} q r_x.
\end{equation}

The first three $Z^{(i)}$'s are

\begin{equation}
Z^{(2)} = -\frac{1}{4} q q_x r^3 + q^2 r r_x - \frac{i}{4} q^3 r^3 + \frac{i}{2} q r_{xxx},
\end{equation}

\begin{equation}
Z^{(3)} = \frac{5}{16} q^3 r^4 - \frac{5}{4} q^2 r^2 - \frac{3}{2} q^3 r r_x - \frac{3}{2} q q_x r r_x
-\frac{1}{4} q q_x r^2 - 2 i q^3 r r_x - \frac{3}{4} i q^2 q_x r^3 + \frac{i}{2} q r_{xxx},
\end{equation}

\begin{equation}
Z^{(4)} = 4 q^4 r^3 r_x + \frac{29}{16} q^3 q_x r^4 - \frac{9}{4} q^3 r_x r x - 2 q^4 r r_{xxx}
-\frac{11}{4} q q_x r^2 - 3 q q_x r r_x - 2 q q_x r r_{xx} - \frac{1}{4} q q_{xxx} r^2
+ \frac{7}{16} i q^5 r^5 - \frac{15}{4} i q^4 r x r_x - \frac{25}{4} i q^4 r^2 r_x - 8 i q^3 r_x r_x r_x
- i q^2 q_x r^3 - \frac{3}{4} i q^2 q_x r^3 + \frac{i}{2} q r_{xxxx}.
\end{equation}

Noting that every $\int Z^{(i)}(x, t) dx$ is a conserved functional, we obtain (3.2) and (3.3) from (4.14) and (4.16), respectively.
For example, taking $r=\overline{q}$ in (4.14), we have

\begin{equation}
\int Z^{(2)}(q)dx=\int \frac{i}{2} q \overline{q}_{zx} dx - \frac{i}{4} \int |q|^4 dx - \int |q|^6 q_{x}dx - \frac{1}{4} \int |q|^4 q_{x} dx.
\end{equation}

Taking account of the relation

\begin{equation}
0=\int \frac{\partial}{\partial x} |q|^4 dx = 2 \int |q|^2 q_{x} \overline{q} dx + 2 \int |q|^2 q \overline{q}_{x} dx
\end{equation}

and using integration by parts, we can rewrite (4.17) to

\begin{equation}
\int Z^{(2)}(q)dx= -\frac{i}{2} \int |q|^2 dx - \frac{i}{4} \int |q|^6 dx + \frac{3}{8} \left( \int |q|^2 q_{x} \overline{q} dx - \int |q|^2 q \overline{q}_{x} dx \right)
\end{equation}

from which (3.2) is easily obtained.

In a similar way, we can get (3.3) from (4.16) using analogous manipulations to (4.18).

References


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M. Tsutsumi
Division of Mathematical Physics
Department of Applied Physics
Waseda University
Tokyo, Japan
I. Fukuda
Division of Mathematics
Faculty of Engineering
Kokushikan University
Tokyo, Japan

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