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On the Parallelizability of Regions of Asymptotic Stability

By

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It is well known that if a compact invariant subset M of a locally compact metric space X is globally asymptotically stable, then the dynamical system restricted to $X-M$ is parallelizable. In this paper we will generalize this result to dynamical systems whose phase spaces are merely metric. In the process of doing this we introduce the concept of local uniform asymptotic stability and show that if a compact invariant subset M of a metric space X is locally uniformly asymptotically stable, then the orbit space of the dynamical system restricted to $X-M$ is also metric.

A dynamical system on a topological space X is a continuous mapping $\pi: X\times R\rightarrow X$ such that (where $x\pi t=\pi(x, t)$)

(1) $x\pi 0=x$ for every $x\in X$,

(2) $(x\pi t)\pi s=x\pi(t+s)$ for every $x\in X$ and $s, t\in R$.

For $A\subset X$ and $B\subset R$, $A\pi B$ will denote the set $\{x\pi t: x\in A, t\in B\}$. In the special cases $B=R$ or $B=R^{+}$ we will write $C(A)$ and $C^{+}(A)$ instead of $A\pi R$ and $A\pi R^{+}$ respectively. A subset A of X is *invariant* if $C(A)=A$. The dynamical system π is called

(i) *dispersive*, if for every $x, y\in X$ there exist neighborhoods U of x and V of y such that $(U\pi[T, \infty))\cap V=\phi$ for some $T\in R$,

(ii) *parallelizable*, if there is a global section S for π, i.e., a subset S of X such that for every $x\in X$, there is a unique $t_{x}\in R$ for which $x\pi t_{x}\in S$ and the mapping $x\rightarrow t_{x}$ is continuous.

A compact subset M of X is called *stable* if for any neighborhood U of M there is a neighborhood V of M such that $C^{+}(V)\subset U$. A stable subset M of X is called

(i) *asymptotically stable* if for any neighborhood U of M and any $x\in X$, there is a $T\in R$ such that $x\pi[T, \infty)\subset U$,

(ii) *locally uniformly asymptotically stable* if for any $x\in X-M$, there is a neighborhood V of x such that for any neighborhood U of M there exists $T\in R$ such that $V\pi[T, \infty)\subset U$.

A continuous function $L: X\rightarrow R^{+}$ is called a Liapunov function for a subset M of X if

(i) $L(x)=0$ if and only if $x\in M$,

(ii) $L(x\pi t) < L(x)$ for every $x \in X - M$ and $0 < t$,

(iii) $L(x\pi t) \to 0$ as $t \to \infty$ for every $x \in X$.

If X is a metric space with metric d and $\varepsilon > 0$, then $B(x, \varepsilon)$ will denote the set $\{y \in X : d(x, y) \leq \varepsilon\}$. If $M \subset X$, then $B(M, \varepsilon)$ will denote the set $\cup \{d(x, \varepsilon) | x \in M\}$.

Throughout this paper M will denote a compact subset of a metric space X which is globally asymptotically stable with respect to a dynamical system π. The metric on X will be denoted by d and we will assume that $d(x, y) \leq 1$ for every $x, y \in X$.

We begin by constructing a Liapunov function for M. The following lemma is easily proved.

Lemma 1. *Let $x \in X - M$. If U is any neighborhood of M, then there are a neighborhood V of x and a $T > 0$ such that $V\pi[T, \infty) \subset U$. Hence π restricted to $X - M$ is dispersive.*

With this lemma the proof of the following theorem is essentially identical with that of Theorem 10 in [1].

Theorem 2. *Let X be a metric space. A compact subset M of X is globally asymptotically stable if and only if there is a Liapunov function for M.*

With a locally compact phase space a dynamical system π is dispersive if and only if π is parallelizable, [2]. For arbitrary phase spaces this is not necessarily the case. The following example is a slight modification of one found in [6, page 44]. Let π_1 be the planar dynamical system determined by the system of differential equations in polar coordinates,

$$\frac{dr}{dt} = f(r, \theta), \qquad \frac{d\theta}{dt} = 0$$

where f is a continuously differentiable function such that $f(n^{-1}, n^{-1}) = 0$ for every positive integer n, $f(0, 0) = 0$, and $f(r, \theta) < 0$ otherwise. Let X denote theplane with the half lines $\{(r, \theta) : \theta = n^{-1}, r \geq n^{-1}\}$ deleted and let π denote π_1 restricted to X. Then π restricted to $X - \{(0, 0)\}$ is dispersive, but not parallelizable. Note that the trajectories of π are rays or portions of rays emanating from the origin. The motion along these trajectories is toward the origin. Hence, the origin is asymptotically stable, but π restricted to $X - \{(0, 0)\}$ is not parallelizable.

The following lemma can be verified directly.

Lemma 3. *Let $L : X \to R^+$ be a Liapunov function for M. If $x \in X - M$, then*

(i) $L^{-1}(L(x))$ is a section for π restricted to $C(L^{-1}(L(x)))$,

(ii) $C(L^{-1}(L(x)))$ is an open subset of X,

(iii) $C(L^{-1}(L(x))) \subset C(L^{-1}(L(y)))$ whenever $0 \neq L(y) < L(x)$.

Lemma 4. *Let M be compact and globally locally uniformly asymptotically stable, let $x \in X-M$, and let U be an invariant neighborhood of x. Then there is a closed invariant neighborhood V of x such that $V \subset U \cup M$.*

Proof. If $A \subset X$ is closed and $B \subset R$ is compact, then $A\pi B$ is closed, [4, page 98]. Let U_x be a neighborhood of x such that for every neighborhood W of M, there is a T so that $U_x\pi[T, \infty) \subset W$. Without loss of generality, we may suppose that U_x is closed, $U_x \subset U$, and $U_x \cap M = \phi$. Then $U_x\pi[-t, t]$ is closed for every $t>0$. Let $\{x_i\}$ and $\{t_i\}$ be sequences in U_x and R, respectively, such that $x_i\pi t_i \to y$ for some $y \in X$. Since M is locally asymptotically stable there is an open neighborhood W_1 of y and a $T>0$ such that $W_1\pi[T, \infty) \subset B(M, 1/2d(M, U_x))$. Hence, $-T \leq t_i$ for all i sufficiently large. If $\{t_i\}$ has an accumulation point, then $y \in U_x\pi R$ since $U_x\pi[-t, t]$ is closed for every $t>0$. Suppose $t_i \to \infty$. Due to our choice of U_x, it is easy to show that $x_i\pi t_i \to M$. It follows that $\overline{U_x\pi R} = (U_x\pi R) \cup M \subset U \cup M$.

Set $Y = X - M$. Evidently Y is an open invariant subset of X. Define an equivalence relation C on Y by xCy if and only if $x \in C(y)$. The topology of Y/C will be the induced topology. It is known (e.g., see Lemma 1 of [5]) that the natural mapping e of Y onto Y/C is an open mapping.

Recalling that $L^{-1}(\lambda)$ is a section for π restricted to $C(L^{-1}(\lambda))$ and that $C(L^{-1}(\lambda))$ is an open subset of X for each λ in the range of a Liapunov function L for an asymptotically stable set M, (Lemma 3), the following lemma can be verified directly.

Lemma 5. *Let $U \subset X-M$ be an open subset of X, L a Liapunov function for a globally asymptotically stable compact set M, and λ a number in the range of L. The $C(U \cap L^{-1}(\lambda))$ is an open subset of X.*

Theorem 6. *If X is a metric space and M is a compact subset of X which is globally uniformly asymptotically stable, then $(X-M)/C$ is metrizable.*

Proof. We will first show that $(X-M)/C$ is regular. Set $Y = X-M$. Let A be a closed subset of Y/C and let $e(x) \in Y/C - A$. Then $e^{-1}(A)$ is a closed invariant subset of Y and $e^{-1}e(x) = C(x) \subset Y - A$. By Lemma 4 there is a closed invariant neighborhood V, in X, of x such that $V \subset (X - e^{-1}(A)) \cup M$. Set $U_1 = (\text{int } V) \cap Y$ and $U_2 = (X - V) \cap Y$. Then U_1 and U_2 are disjoint, invariant, open neighborhoods in Y of x and $e^{-1}(A)$ respectively. Hence, $e(U_1)$ and $e(U_2)$ are disjoint open neighborhoods of $e(x)$ and A respectively. Thus, Y/C is regular. The Nagata-Smirnov metrization theorem, [3, page 194], states that a topological space is metrizable if and only if it is regular and has a basis that can be decomposed into an at most countable collection of neighborhood-finite families. Let L be a Liapunov function for M. Without loss of generality we may suppose that the interval [0, 1] is contained in the range of L. Then for each positive integer n, the set $L^{-1}(n^{-1})$ is a metric space in the relative topology. Therefore there is an at most countable collection $\{U_{n,i}\}$ of neighborhood finite families such that for each n, $\{U_{n,i}\}$ forms a basis for the topology on $L^{-1}(n^{-1})$. Note that $e(\bigcup_{n=1}^{\infty} L^{-1}(n^{-1}))=e(Y)$. Since Y/C has the induced topology and e is an open mapping, Lemma 4 yields that for each n, $\{e(U_{n,i})\}$ is an at most countable collection of neighborhood finite families such that $\bigcup\{e(U_{n,i})\}$ forms a basis for the topology on Y/C. Hence, Y/C is metrizable.

The following Theorem is now an immediate consequence of Lemma 1, Theorem 6, and Theorem 7 of [5] which states "Let π be a dynamical system on a Tichonov space X, and assume that X/C is paracompact. Then π is parallelizable if and only if it is completely unstable."

Theorem 7. *Let X be a metric space. If a compact subset M of X is globally locally uniformly asymptotically stable with respect to a dynamical system π, then π restricted to $X-M$ is parallelizable.*

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