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Singular Perturbation Problems for Systems of Partial Differential Equations of Parabolic Type

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§ 1. Introduction

Singular perturbation for parabolic first boundary value problems (BVPs) have been discussed by Aronson [1] and Bobisud [2]. These authors consider a second order linear parabolic equation in one space dimension with a small parameter multiplying the second order space derivative subject to Dirichlet boundary conditions (BCs) prescribed on the parabolic boundary of the domain under consideration. They, in fact, construct asymptotic solutions to the problem and study the asymptotic behaviour of solution as the small parameter goes to zero.

Recently, Lam [3] has discussed a convection diffusion problem governed by a second order semilinear parabolic equation in one space dimension with a small parameter multiplying the second order space derivative subject to mixed type BCs prescribed on the parabolic boundary of the domain of the problem.

In any construction of an approximate solution of a singular perturbation problem, one is interested in knowing as to how close one is to the actual solution of the problem. This is the problem considered in this paper.

Van Harten [4, 5] has considered linear and nonlinear elliptic BVPs of the first kind for a single partial differential equation (PDE) with a small parameter multiplying the highest derivative. Supposing that an approximation to the solution of the BVP is given which satisfies the differential equation and the BCs up to a certain order in some norm, he has shown that, under appropriate assumptions, the distance between the exact solution and an approximation can be made small. His proofs are based on the maximum principle for linear and nonlinear elliptic equations and a contraction principle in a suitable Banach space.

The results of Van Harten have been generalised to systems of PDEs of elliptic type by Ramanujam and Srivastava [6] where the theory of differential inequalities is employed.

In this paper, we obtain results similar to that given in [6] for linear and nonlinear parabolic BVPs of the third kind for systems of second order PDEs in m-space dimensions with a small parameter multiplying the second order space derivatives.

In 'section 2', we give the relevant definitions and a differential inequality theorem which shall be used throughout this paper. 'Section 3 and 4' are concerned with the linear problems whereas 'section 5' deals with the nonlinear problems.

§ 2. Monotonicity theorem*

We begin this section by giving a few definitions which, though well known to a certain extent [7], are included here for the sake of completeness.

Consider the parabolic system defined by

$$\partial u_i/\partial t+F_i(t, x, u, u_{i,j}, u_{i,jk})=0, \qquad i=1, \cdots, n,$$

where

$(t, x) \in G_p: =(0, T]\times D$, $D\subset R^m$ being a bounded domain;

$x=(x_1, \cdots, x_m) \in D$, $u=(u_1, \cdots, u_n)$; $u_{i,j}: =\partial u_i/\partial x_j$, $u_{i,jk}: =\partial^2 u_i/\partial x_j \partial x_k$.

G_p is called the parabolic domain and its parabolic boundary

$R_p: =\{0\}\times \bar{D} \cup (0, T]\times \partial D$, $\bar{D}=D\cup \partial D$, ∂D being the boundary of D.

Finally R_n will stand for a part of the boundary $(0, T]\times \partial D$.

Definition 2.1. For any two symmetric matrices $r=(r_{jk})$, $\bar{r}=(\bar{r}_{jk})$ we have $r\leqslant \bar{r}$ if and only if the matrix $\bar{r}-r=(\bar{r}_{jk}-r_{jk})$ is positive semidefinite.

Definition 2.2. $F_i(t, x, u, p, q)$ is monotone decreasing in the matrix 'q' if and only if

$$F_i(t, x, u, p, q)\geqslant F(t, x, u, p, \bar{q}) \qquad \text{for } q\leqslant \bar{q}. \tag{2.1}$$

Definition 2.3. $F=(F_1, \cdots, F_n)$ is said to be quasimonotone decreasing in $u=(u_1, \cdots, u_n)$ if each F_i, $i=1(1)n$, is monotone decreasing with u_j, $j\neq i$, $i, j=1, \cdots, n$.

Since we are going to deal with BVPs of the third kind we need the following:

Definition 2.4. An interior normal at a point $(t_0, x_{(0)}) \in (0, T]\times \partial D$ is said to exist if there is a sequence of points $(t_0, x_{(k)}) \in G_p$ $(k=1, 2, \cdots)$ with $x_{(k)}\to x_{(0)}$ $(k\to\infty)$. The outer normal derivative relative to this normal is defined by [7]

$$\partial u/\partial n_a: =-\lim_{k\to\infty} [u(t_0, x_{(k)})-u(t_0, x_{(0)})]/\| x_{(k)}-x_{(0)}\|_e, \tag{2.2}$$

where $\|\cdot\|_e$ is the Euclidean distance.

Now consider the singular perturbation problem for a system of second order PDEs of parabolic type defined by

$$\begin{cases} P_i u: =\partial u_i/\partial t+F_i(t, x, u, u_{i,j}, \varepsilon^{\lambda_i} u_{i,jk})=f_i(t, x) & \text{in } G_p, \\ R_i u: =\begin{cases} u_i(t, x)=g_i(t, x) \text{ on } R_p-R_n, \\ \partial u_i/\partial n_a+H_i(t, x, u)=h_i(t, x) \quad \text{on } R_n, \end{cases} & i=1, \cdots, n, \end{cases} \tag{2.3}$$

* A:=B means that A is defined by B.

where

(i) $\varepsilon > 0$ is a small parameter; $\lambda_i \geqslant 0$; $f_i \in C(G_p)$, $g_i \in C(R_p - R_n)$, $h_i \in C(R_n)$; $u_i \in U: = C^{(1,2)}(G_p) \cap C(\bar{G})$;

(ii) $F_i(t, x, u, p, q)$, for fixed i, is monotone decreasing with respect to the $m \times m$ matrix 'q' in G_p;

(iii) it is assumed that an outer normal derivative exists at the points of R_n. Then we have the following monotonicity theorem for the BVP (2.3).

Theorem 2.1. *Consider the BVP (2.3) and assume that:*

(i) $F_i(t, x, u, p, q)$ and $H_i(t, x, u)$, for fixed i, are quasimonotone decreasing in $u = (u_1, \cdots, u_n)$;

(ii) there exists a vector function $v = (v_1, \cdots, v_n)$, known as a test function, such that

$$(2.4) \quad \begin{cases} v_i = v_i(t, x) > 0 \quad \text{on } \bar{G}, \quad v_i \in U, \\ P_i[u + cv] - P_i u > 0, \quad R_i[u + cv] - R_i u > 0, \\ cv = (cv_1, \cdots, cv_n), \text{ for every } c \in R^+ (\text{the set of all positive real numbers}), \text{ for} \\ \quad \text{every } u, u_i \in U, i = 1, \cdots, n. \end{cases}$$

Then the following implication is true for all $y = (y_1, \cdots, y_n)$, $z = (z_1, \cdots, z_n)$, y_i, $z_i \in U$, $i = 1, \cdots, n$:

$$(2.5) \quad \begin{cases} P_i z \leqslant P_i u \leqslant P_i y, \quad R_i z \leqslant R_i u \leqslant R_i y, \quad i = 1, \cdots, n \\ \Rightarrow \\ z_i(t, x) \leqslant u_i(t, x) \leqslant y_i(t, x) \qquad \text{on } \bar{G}, \ i = 1, \cdots, n. \end{cases}$$

The functions y and z are respectively called *superfunction* and *subfunction* with respect to a solution u of the BVP (2.3). A consequence of the above theorem is that the solution of the BVP (2.3) is unique (if it exists).

Proof. We give the proof for the superfunction. The proof for the subfunction can be given in a similar manner.

If the implication (2.5) is not true for the superfunction, then there exists at least one point $(t_1, x_{(1)}) \in G_p \cup R_n$ and a number $j \in \{1, \cdots, n\}$ such that

$$(2.6) \qquad u_j(t_1, x_{(1)}) - y_j(t_1, x_{(1)}) > 0.$$

Define $c > 0$ as

$$c: = \max_{\substack{(t,x) \in \bar{G} \\ i=1,\cdots,n}} [[u_i(t, x) - y_i(t, x)]/v_i(t, x)],$$

and hence there exists a point $(t_0, x_{(0)}) \in G_p \cup R_n$ and a number $l \in \{1, \cdots, n\}$ such that

$$(2.7)\qquad \begin{cases} cv_l(t_0, x_{(0)}) = u_l(t_0, x_{(0)}) - y_l(t_0, x_{(0)}), \\ cv_i(t, x) \geqslant u_i(t, x) - y_i(t, x) \qquad \text{on } \bar{G}. \end{cases}$$

Therefore, the function $cv_l - u_l + y_l$ attains an absolute minimum at $(t_0, x_{(0)})$. The point $(t_0, x_{(0)})$ should either belong to G_p or R_n.

Case i. $(t_0, x_{(0)}) \in G_p$.
At $(t, x) = (t_0, x_{(0)})$, we then have

$$(2.8)\qquad \begin{cases} cv_l - u_l + y_l = 0 = cv_{l,j} - u_{l,j} + y_{l,j}, \qquad j = 1, \cdots, n, \\ \partial[cv_l - u_l + y_l]/\partial t \leqslant 0 \quad \text{and} \quad (cv_{l,jk} - u_{l,jk} + y_{l,jk}) \geqslant 0.^* \end{cases}$$

Furthermore, the following inequalities are valid at $(t, x) = (t_0, x_{(0)})$:

$$\begin{aligned} 0 &\leqslant P_l y - P_l u = P_l y - \partial u_l/\partial t - F_l(t_0, x_{(0)}, u, u_{l,j}, \varepsilon^{\lambda_l} u_{l,jk}) \\ &\leqslant P_l y - \partial y_l/\partial t - \partial c v_l/\partial t - F_l(t_0, x_{(0)}, y + cv, y_{l,j} + cv_{l,j}, \varepsilon^{\lambda_l}[y_{l,jk} + cv_{l,jk}]) \\ &= P_l y - P_l[y + cv], \end{aligned}$$

by (2.5), (2.8), (2.7) and the quasimonotonicity of F_l. This is a contradiction to (2.4). Hence $(t_0, x_{(0)}) \notin G_p$.

Case ii. $(t_0, x_{(0)}) \in R_n$.
At $(t, x) = (t_0, x_{(0)})$, we then have

$$(2.9)\qquad \partial[cv_l - u_l + y_l]/\partial n_a \leqslant 0.$$

Therefore at $(t, x) = (t_0, x_{(0)})$ we have,

$$\begin{aligned} 0 &\leqslant R_l y - R_l u = R_l y - \partial u_l/\partial n_a - H_l(t_0, x_{(0)}, u) \\ &\leqslant R_l y - \partial[cv_l + y_l]/\partial n_a - H_l(t_0, x_{(0)}, y + cv), \\ &\qquad \text{by (2.7), (2.9) and the quasimonotonicity condition of } H_l, \\ &= R_l y - R_l[y + cv]. \end{aligned}$$

It is again a contradiction to (2.4) and hence $(t_0, x_{(0)}) \notin R_n$. That is, such a point $(t_1, x_{(1)}) \in G_p \cup R_n$ cannot exist and hence the proof of the theorem.

Remark. Apparently the above theorem is similar to the one given in [8, 9] but the proof provided here is rigorous and the form of the test function is simpler.

We conclude this section with a few more definitions.

Definition 2.5. For $f = (f_1, \cdots, f_n)$, $g = (g_1, \cdots, g_n)$ and $h = (h_1, \cdots, h_n)$ defined respectively in G_p, $R_p - R_n$ and R_n, we define their norms by†

* The meaning of this inequality is given in 'definition 2.1'.

† One can also consider $\|f\| := \operatorname{Sup}\{|f_i(t, x)| \mid (t, x) \in G_p, i = 1\ (1) n\}$ with similar definitions for $\|g\|'$ and $\|h\|''$ and pursue analysis.

$$(2.10)\qquad \|f\|: =\operatorname{Sup}\left\{\left[\sum_{i=1}^{n}|f_i(t, x)|^2\right]^{1/2}\middle|(t, x)\in G_p\right\},$$

$$(2.11)\qquad \|g\|': =\operatorname{Sup}\left\{\left[\sum_{i=1}^{n}|g_i(t, x)|^2\right]^{1/2}\middle|(t, x)\in R_p-R_n\right\},$$

$$(2.12)\qquad \|h\|'': =\operatorname{Sup}\left\{\left[\sum_{i=1}^{n}|h_i(t, x)|^2\right]^{1/2}\middle|(t, x)\in R_n\right\}.$$

Definition 2.6. Suppose a vector function $Z=(Z_1, \cdots, Z_n)$ is given satisfying the following equations

$$(2.13)\qquad P_iu-P_iZ=q_i \qquad \text{in } G_p,$$

$$(2.14)\qquad R_iu-R_iZ=r_i \qquad \text{on } R_p-R_n,$$

$$(2.15)\qquad R_iu-R_iZ=s_i \qquad \text{on } R_n,\ i=1, \cdots, n.$$

Then the function Z is called a formal approximation of the solution u of the BVP (2.3) in the norms $\|\cdot\|$, $\|\cdot\|'$ and $\|\cdot\|''$ if

$$(2.16)\qquad \|q\|=O(\delta(\varepsilon))=o(1), \qquad q=(q_1, \cdots, q_n),$$

$$(2.17)\qquad \|r\|'=O(\delta'(\varepsilon))=o(1), \qquad r=(r_1, \cdots, r_n),$$

$$(2.18)\qquad \|s\|''=O(\delta''(\varepsilon))=o(1)^*, \qquad s=(s_1, \cdots, s_n),$$

where δ, δ', δ'' are order functions.

§ 3. Linear boundary value problems

In this section we consider a special case of the BVP (2.3). In fact, we focus our attention on the linear parabolic BVP (3.1) and obtain estimates for its solution. In turn, since the BVP is linear, these yield estimates for the difference between the solution and a formal approximation.

Consider the linear parabolic BVP for $u=(u_1, \cdots, u_n)$ defined by

$$(3.1)\qquad \begin{cases} P_iu: =\partial u_i/\partial t+\varepsilon^{\lambda_i}P_{i2}u+P_{i1}u=f_i(t,x) & \text{in } G_p,\\ R_iu: =\begin{cases} u_i(t, x)=g_i(t, x) & \text{on } R_p-R_n,\\ \partial u_i/\partial n_a+\sum_{j=1}^{n} d_{ij}u_j=h_i(t, x) & \text{on } R_n,\ i=1, \cdots, n,\end{cases}\end{cases}$$

where

$$P_{i2}u: =\sum_{j=1}^{n} a_{ij}(t, x)u_j+\sum_{k=1}^{m} b_{ik}(t, x)u_{i,k}-\sum_{j,k=1}^{m} c_{ijk}(t, x)u_{i,jk},$$

* The symbols O and o denote the Landau order symbols.

$$P_{i1}u: =\sum_{j=1}^{n} \alpha_{ij}(t, x)u_j+\sum_{k=1}^{m} \beta_{ik}(t, x)u_{i,k}, \qquad i=1, \cdots, n.$$

Further assume that:

(i) the coefficients a_{ij}, b_{ik}, c_{ijk}, α_{ij} and β_{ik} are defined and continuous in their arguments;

(ii) for fixed i, the matrix with elements c_{ijk} is symmetric and positive semi-definite with respect to j and k.

According to 'theorem 2.1', whenever there exists a test function for the BVP (3.1) then the implication (2.5) is true. We now give two sets of sufficient conditions which yield test functions for the BVP (3.1).

Set I. Consider the BVP (3.1) and assume that:

$$\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij}\leqslant 0 \quad \text{and} \quad d_{ij}\leqslant 0, j\neq i, \quad i, j=1, \cdots, n; \tag{3.2}$$
$$\text{(the quasimonotonicity condition)}$$

$$\begin{cases} \sum_{j=1}^{n} [\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij}]\geqslant a_0 \in R \text{ (the set of all real numbers)} \quad \text{and} \\ \sum_{j=1}^{n} d_{ij}\geqslant d_0 \in R^+, \qquad i=1, \cdots, \text{n}. \end{cases} \tag{3.3}$$

Then a test function in the present case is given by $v=(v_1, \cdots, v_n)$ where

$$v_i(t)=e^{\eta t}, \ \eta \in R^+, \ t \in [0, T], \qquad i=1, \cdots, n.$$

Set II. Consider the BVP (3.1) and assume that the quasimonotonicity condition (3.2) is true. Further assume that:

(i) the coefficients a_{ij}, b_{ik}, c_{ijk}, d_{ij}, α_{ij} and β_{ij} are bounded, $\sum_{j=1}^{n} d_{ij}\geqslant 0$, $i=1, \cdots, n$;

(ii) there exists a vector function $\gamma=(\gamma_1, \cdots, \gamma_n)$, $\gamma_i \in U$ with bounded first and second derivatives and independent of 't' with the property

$$\partial\gamma_0/\partial n_a>0 \quad \text{on } R_n, \ \gamma_0(x)\equiv\gamma_i(x), \quad i=1, \cdots, n.$$

Then a test function for this problem is given by

$$v=(v_1, \cdots, v_n), \ v_i(t, x)=m_1e^{\eta t}+\gamma_0(x)>0 \qquad \text{on } \bar{G}, \ \eta, m_1 \in R^+ \quad i=1, \cdots, n.$$

Remarks. (i) There is a difference between the above sets of conditions. Set I rules out any consideration of second BVPs whereas set II allows us to consider second BVPs also.

(ii) The existence of a γ function defined in the set II has been discussed in certain cases by Walter [7, p. 249]. Also the significance of the condition $\sum_{j=1}^{n} d_{ij}\geqslant d_0>0$, $i=1, \cdots, n$ is mentioned in [7, p. 244].

The constants m_i, $i=1, \cdots, 14$, appearing in the rest of the paper are independent of the parameter ε.

We now obtain estimates for solution of the BVP (3.1)

Theorem 3.1. *Consider the BVP* (3.1) *and assume that the quasimonotonicity condition* (3.2) *is true. Further assume that there exists a vector function* $w=(w_1, \cdots, w_n)$ *such that*

$$\begin{cases} w_i=w_i(t,x)>0 & \text{on } \bar{G},\ w_i \in U, \\ P_i w \geqslant m_1 \in R,\ R_i w \geqslant m_2 \in R^+, & i=1, \cdots, n. \end{cases} \tag{3.4}$$

Then we have

$$|u_i(t,x)| \leqslant m_3\{\|f\|+\|g\|'+\|h\|''\} \text{ on } \bar{G},\ m_3 \in R^+,\ i=1, \cdots, n. \tag{3.5}$$

Proof. The analysis carried out in the following is similar to that given in [6].

Consider the linear BVP (3.1). We set

$$v=(v_1, \cdots, v_n),\ v_i=u_i/w_i \quad \text{on } \bar{G}, \quad i=1, \cdots, n, \tag{3.6}$$

and find that v must satisfy

$$\begin{aligned} & w_i \partial v_i/\partial t+\sum_{j=1}^{n} \delta_{ij}[\partial w_i/\partial t]v_j+\sum_{j=1}^{n}[\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij}]v_j w_j \\ & \quad +\sum_{j=1}^{n}\delta_{ij}\Big(\sum_{k=1}^{m}[\varepsilon^{\lambda_i}b_{ik}+\beta_{ik}]w_{i,k}\Big)v_j-\sum_{j=1}^{n}\delta_{ij}\Big(\sum_{j,k=1}^{m}\varepsilon^{\lambda_i}c_{ijk}w_{i,jk}\Big)v_j \\ & \quad +\sum_{k=1}^{m}\Big(-2\sum_{j=1}^{m}\varepsilon^{\lambda_i}c_{ijk}w_{i,j}+(\varepsilon^{\lambda_i}b_{ik}+\beta_{ik})w_i\Big)v_{i,k} \\ & \quad -\sum_{j,k=1}^{m}\varepsilon^{\lambda_i}c_{ijk}v_{i,jk}w_i=f_i(t,x) \quad \text{in } G_p, \quad i=1, \cdots, n, \\ & v_i w_i=g_i(t,x) \qquad \text{on } R_p-R_n, \\ & w_i\partial v_i/\partial n_a+\sum_{j=1}^{n}[d_{ij}w_j+\delta_{ij}\partial w_i/\partial n_a]v_j=h_i(t,x) \qquad \text{on } R_n, \end{aligned}$$

where $i=1, \cdots, n$ and δ_{ij} is the Kronecker Symbol. The above problem can be written as a BVP for $v=(v_1, \cdots, v_n)$ in the following form similar to that of (3.1):

$$\begin{cases} P_i'v:=\partial v_i/\partial t+\sum_{j=1}^{n}A_{ij}v_j+\sum_{k=1}^{m}B_{ik}v_{i,k}-\sum_{j,k=1}^{m}c_{ijk}\varepsilon^{\lambda_i}v_{i,jk} \\ \qquad =f_i(t,x)/w_i(t,x) \qquad \text{in } G_p, \\ R_i'v:=\begin{cases} v_i(t,x)=g_i(t,x)/w_i(t,x) & \text{on } R_p-R_n, \\ \partial v_i/\partial n_a+\sum_{j=1}^{n}D_{ij}v_j=h_i(t,x)/w_i(t,x) & \text{on } R_n, \quad i=1, \cdots, n, \end{cases} \end{cases} \tag{3.7}$$

where

$$A_{ij}=\left[\delta_{ij}\partial w_i/\partial t+(\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij})w_j+\delta_{ij}\sum_{k=1}^{m}(\varepsilon^{\lambda_i}b_{ik}+\beta_{ik})w_{i,k}\right.$$
$$\left.-\delta_{ij}\sum_{j,k=1}^{m}\varepsilon^{\lambda_i}c_{ijk}w_{i,jk}\right]\Big/w_i,\qquad i,j=1,\cdots,n,$$
$$B_{ik}=-(2/w_i)\sum_{j=1}^{m}\varepsilon^{\lambda_i}c_{ijk}w_{i,jk}+(\varepsilon^{\lambda_i}b_{ik}+\beta_{ik}),\quad k=1,\cdots,m,\quad \text{and}$$
$$D_{ij}=[d_{ij}w_j+\delta_{ij}\partial w_i/\partial n_a]/w_i,\qquad i,j=1,\cdots,n.$$

It is easy to verify the following inequalities:

$$A_{ij}=[\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij}]w_j/w_i\leqslant 0,\quad j\neq i,\ i,j=1,\cdots,n,\quad \text{by (3.2) and (3.4)},\tag{3.8}$$

$$D_{ij}=d_{ij}w_j/w_i\leqslant 0,\quad j\neq i,\ i,j=1,\cdots,n,\quad \text{by (3.2) and (3.4)},\tag{3.9}$$

$$\sum_{j=1}^{n}A_{ij}=[P_iw]/w_i\geqslant m_4\in R,\quad i=1,\cdots,n,\quad \text{by (3.4)},\tag{3.10}$$

$$\sum_{j=1}^{n}D_{ij}=[R_iw]/w_i\geqslant m_2/w_i\quad \text{on } R_n,\quad i=1,\cdots,n,\quad \text{by (3.4)}.\tag{3.11}$$

The above inequalities (3.8)–(3.11) show that, by the set I of sufficient conditions, the implication (2.5) is true for the BVP (3.7).

To obtain the required estimate, we consider the vector function $y=(y_1,\cdots,y_n)$ defined by

$$y_i(t,x)=m_5e^{\eta t}[\|f\|+\|g\|'+\|h\|'']\quad \text{on } \bar{G},\quad i=1,\cdots,n,\tag{3.12}$$

where m_5, $\eta\in R^+$ are yet to be determined.

We now show that, by a proper choice of m_5 and η, the vector function y defined by (3.12) is nothing but a superfunction with respect to a solution v of the BVP (3.7). For,

$$\begin{cases} P_i'y=\left[\eta+\sum_{j=1}^{n}A_{ij}\right]y_i\geqslant f_i/w_i & \text{in } G_p,\\ R_i'y=\begin{cases} y_i\geqslant g_i/w_i & \text{on } R_p-R_n,\\ \partial y_i/\partial n_a+\sum_{j=1}^{n}D_{ij}y_j\geqslant h_i/w_i & \text{on } R_n,\quad i=1,\cdots,n,\end{cases}\end{cases}\tag{3.13}$$

by a proper choice of m_5, $\eta\in R^+$. Now (3.13) yields

$$P_i'y\geqslant P_i'v\quad \text{and}\quad R_i'y\geqslant R_i'v,\quad i=1,\cdots,n,\tag{3.14}$$

which gives

$$v_i(t, x) \leqslant y_i(t, x) \quad \text{on } \bar{G}, \quad i=1, \cdots, n.$$

Since the BVP (3.7) is linear we have,

$$|v_i(t,x)| \leqslant y_i(t, x) \quad \text{on } \bar{G}, \quad i=1, \cdots, n.$$

Hence

$$|u_i(t, x)| \leqslant m_6 y_i(t, x) \quad \text{on } \bar{G}, \quad i=1, \cdots, n,$$

where $u=(u_1, \cdots, u_n)$ is the solution of the BVP (3.1). Therefore,

$$|u_i(t, x)| \leqslant m_3[\|f\|+\|g\|'+\|h\|''] \quad \text{on } \bar{G}, \quad i=1, \cdots, n, \quad m_3 \in R^+.$$

Remark. In the above theorem, it has been assumed that the condition $R_i w > 0$, $i=1, \cdots, n$ holds good. This condition can be further relaxed to $R_i w \geqslant 0$.

Theorem 3.2. *Assume that all the conditions of 'theorem* 3.1*' are satisfied. Further, let Z be a formal approximation to the solution u of the BVP* (3.1) *such that it satisfies* (2.13)–(2.18). *Then we have*

$$(3.15) \qquad |u_i(t, x) - Z_i(t, x)| \leqslant O\,(\max(\delta, \delta', \delta'')), \quad \text{on } \bar{G}, \quad i=1, \cdots, n.$$

Proof. To establish (3.15), we consider the linear parabolic BVP for $u-Z$ defined by

$$(3.16) \qquad \begin{cases} P_i(u-Z)=q_i & \text{in } G_p, \\ R_i(u-Z)=\begin{cases} r_i & \text{on } R_p-R_n, \\ s_i & \text{on } R_n, \quad i=1, \cdots, n. \end{cases} \end{cases}$$

We now apply 'theorem 3.1' to the BVP (3.16) which ensures the existence of a constant m_8 such that

$$(3.17) \qquad |u_i(t, x) - Z_i(t, x)| \leqslant m_8\{\|q\|+\|r\|'+\|s\|''\}.$$

The result (3.15) now follows from (3.17) and (2.16)–(2.18).

§ 4. Auxiliary linear boundary value problems

Throughout the 'section 3' the validity of the quasimonotonicity condition (3.2) has been assumed to obtain estimates for solution of the linear BVP (3.1). However, in general, this may not be the case. To handle the general case one can introduce an extended auxiliary problem (4.1) with respect to the BVP (3.1) as in [8, 9, 10] and use it to obtain results for the latter BVP.

With

$$a_{ij}^{+}:=a_{ij}, \quad \text{if } a_{ij}\geqslant 0 \quad \text{and '0' otherwise,}$$
$$a_{ij}^{-}:=a_{ij}-a_{ij}^{+} \qquad \text{in } G_p,$$

and similar definitions for d_{ij}^{+}, d_{ij}^{-} on R_n, the quasimonotone extended auxiliary problem $(\hat{P}, \hat{R})$ with respect to (P, R) of (3.1) is defined by

$$(4.1)\quad \begin{cases} \hat{P}_i\hat{u}:=\partial\hat{u}_i/\partial t+(\varepsilon^{\lambda_i}a_{ii}+\alpha_{ii})\hat{u}_i-\sum\limits_{j=1,j\neq i}^{n}[(\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij})^{+}\hat{u}_{j+n}-(\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij})^{-}\hat{u}_j] \\ \qquad +\sum\limits_{k=1}^{m}[\varepsilon^{\lambda_i}b_{ik}+\beta_{ik}]\hat{u}_{i,k}-\sum\limits_{j,k=1}^{m}\varepsilon^{\lambda_i}c_{ijk}\hat{u}_{i,jk}=-g_i(t,x) \quad \text{in } G_p, \\ \hat{P}_{i+n}\hat{u}:=\partial\hat{u}_{i+n}/\partial t+(\varepsilon^{\lambda_i}a_{ii}+\alpha_{ii})\hat{u}_{i+n}-\sum\limits_{j=1,j\neq i}^{n}[(\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij})^{+}\hat{u}_j-(\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij})^{-}\hat{u}_{j+n}] \\ \qquad +\sum\limits_{k=1}^{m}[\varepsilon^{\lambda_i}b_{ik}+\beta_{ik}]\hat{u}_{i+n,k}-\sum\limits_{j,k=1}^{m}\varepsilon^{\lambda_i}c_{ijk}\hat{u}_{i+n,jk}=g_i(t,x) \quad \text{in } G_p, \\ \hat{R}_i\hat{u}:=\begin{cases}\hat{u}_i=-f_i(t,x) \quad \text{on } R_p-R_n, \\ \partial\hat{u}_i/\partial n_a+d_{ii}\hat{u}_i-\sum\limits_{j=1,j\neq i}^{n}[d_{ij}^{+}\hat{u}_{j+n}-d_{ij}^{-}\hat{u}_j]=-h_i(t,x) \quad \text{on } R_n,\end{cases} \\ \hat{R}_{i+n}\hat{u}:=\begin{cases}\hat{u}_{i+n}=f_i(t,x) \quad \text{on } R_p-R_n, \\ \partial\hat{u}_{i+n}/\partial n_a+d_{ii}\hat{u}_{i+n}-\sum\limits_{j=1,j\neq i}^{n}[d_{ij}^{+}\hat{u}_j-d_{ij}^{-}\hat{u}_{j+n}] \\ \qquad =h_i(t,x) \quad \text{on } R_n, \quad i=1,\cdots,n,\end{cases} \\ \text{where } \hat{u}=(\hat{u}_1,\cdots,\hat{u}_{2n}). \end{cases}$$

The following observations are made regarding the BVP (4.1):

(i) it satisfies the quasimonotonicity condition similar to that of (3.2), i.e., as defined in 'definition 2.3';

(ii) if $(u_1,\cdots,u_n)$ is a solution of the BVP (3.1) then $(-u_1,\cdots,-u_n, u_1,\cdots,u_n)$ is a solution of the BVP (4.1).

The following theorem, when the BVP (3.1) is not quasimonotone, gives estimates for the solution of the BVP (3.1) by making use of the extended auxiliary BVP (4.1).

Theorem 4.1. *If the BVP* (3.1) *does not satisfy the quasi-monotonicity condition* (3.2), *we introduce the auxiliary BVP* (4.1) *and assume that there exists a vector function* $\hat{w}=(\hat{w}_1,\cdots,\hat{w}_{2n})$ *such that*

$$(4.2)\quad \begin{cases}\hat{w}_i(t,x)>0 \quad \text{on } \bar{G},\ \hat{w}_i\in U, \quad i=1,\cdots,2n, \\ \hat{P}_i\hat{w}\geqslant m_8, \quad \hat{P}_{i+n}\hat{w}\geqslant m_8\in R, \\ \hat{R}_i\hat{w}\geqslant m_9, \quad \hat{R}_{i+n}\hat{w}\geqslant m_9\in R^{+}, \quad i=1,\cdots,n.\end{cases}$$

Then we have

(4.3) $\quad |u_i(t, x)| \leqslant m_{10}[\|f\| + \|g\|' + \|h\|''] \quad on\ \bar{G}, \quad i=1, \cdots, n, \quad m_{10} \in R^+.$

Proof. Consider the BVP (4.1). We set $\hat{v}=(\hat{v}_1, \cdots, \hat{v}_{2n})$, $\hat{v}_i=\hat{u}_i/\hat{w}_i$ on $\bar{G}$, $i=1, \cdots, 2n$, and find that, following a similar procedure as in the proof of 'theorem 3.1', $\hat{v}$ must satisfy

$$\begin{cases} \hat{P}'_i\hat{v}=-f_i/\hat{w}_i \quad \text{in } G_p, & \hat{P}'_{i+n}\hat{v}=f_i/\hat{w}_{i+n} \quad \text{in } G_p, \\ \hat{R}'_i\hat{v}=\begin{Bmatrix} -g_i/\hat{w}_i & \text{on } R_p-R_n \\ -h_i/\hat{w}_i & \text{on } R_n \end{Bmatrix}, & \hat{R}'_{i+n}\hat{v}=\begin{cases} g_i/\hat{w}_{i+n} & \text{on } R_p-R_n, \\ h_i/\hat{w}_{i+n} & \text{on } R_n, \quad i=1, \cdots, n. \end{cases} \end{cases}$$

Then, as in the case of 'theorem 3.1', we conclude that $|\hat{u}_i(t, x)| \leqslant m_{11}\{\|f\| + \|g\|' + \|h\|''\}$ on $\bar{G}$, $i=1, \cdots, 2n$. Since $(-u_1, \cdots, -u_n, u_1, \cdots, u_n)$ is a solution of the BVP (4.1) we have the estimate (4.3).

Theorem 4.2. *Assume that all the conditions of 'theorem* 4.1*' are satisfied. Further, let Z be a formal approximation to the solution u of the BVP* (3.1) *and satisfy* (2.13)–(2.18). *Then we have*

$$|u_i(t, x)-Z_i(t, x)| \leqslant O\,(\max(\delta, \delta', \delta'')) \quad on\ \bar{G}, \quad i=1, \cdots, n.$$

The proof follows in a straight forward manner if one carries out the steps as in the proof of 'theorem 3.2' and makes use of 'theorem 4.1'.

Remark. A set of sufficient conditions underwhich a function $\hat{w}=(\hat{w}_1, \cdots, \hat{w}_{2n})$ of 'theorem 4.1' exists is the following:

$$(\varepsilon^{\lambda_i}a_{ii}+\alpha_{ii}) - \sum_{j=1, j\neq i}^{n} |\varepsilon^{\lambda_i}a_{ij}+\alpha_{ij}| \geqslant m_{12} \in R;$$

$$d_{ii} - \sum_{j=1, j\neq i}^{n} |d_{ij}| \geqslant m_{13} \in R^+, \qquad i=1, \cdots, n.$$

In fact, one can take $\hat{w}$ as

$$\hat{w}_i = m_{14}e^{\gamma t}, \quad i=1, \cdots, 2n, \quad t \in [0, T], \quad m_{14} \in R^+.$$

§ 5. Nonlinear boundary value problems

We now consider the nonlinear BVP (2.3).

The constants n_i, $i=1, \cdots, 6$, appearing in this section are independent of the parameter ε.

The implication (2.5) is true for the BVP (2.3) under any one of the following two sets of conditions.

Set I. Consider the BVP (2.3) and assume that:

(i) $F_i(t, x, u, p, q)$ and $H_i(t, x, u)$, for fixed i, are quasimonotone decreasing in u according to the 'definition 2.3';

(ii) $F_i(t, x, u+\alpha, p, q)-F_i(t, x, u, p, q)\geqslant n_1\alpha_0,$

$n_1 \in R,\ \alpha=(\alpha_1, \cdots, \alpha_n),\ \alpha_i(t)=\alpha_0(t)>0, \qquad i=1, \cdots, n;$

(iii) $H_i(t, x, u+\alpha)-H_i(t, x, u)>0, \qquad i=1, \cdots, n.$

The test function in the present case is given by

$$v=(v_1, \cdots, v_n), \quad v_i(t)=e^{\eta t}, \quad \eta \in R^+, \quad t \in [0, T], \quad i=1, \cdots, n.$$

Set II. Consider the BVP (2.3) and assume that $F_i(t, x, u, p, q)$ and $H_i(t, x, u)$, for fixed i, are quasimonotone decreasing in u. Further, suppose that F_i and H_i, for fixed i, satisfy Lipschitz conditions in G_p and R_n respectively with respect to every $u=(u_1, \cdots, u_n)$, $u_i \in U$ and its derivatives. Then, as in [10], one can construct a linear auxiliary Lipschitz problem for the BVP (2.3). Consequently, if the implication (2.5) is true for the Lipschitz problem then so is for the BVP (2.3) [10].

Theorem 5.1. *Consider the nonlinear BVP* (2.3) *and assume that the implication* (2.5) *is true for it. Further assume that*:

$$\pm F_i(t, x, Z\pm X, Z_{i,j}, \varepsilon^{\lambda_i}Z_{i,jk})\mp F_i(t, x, Z, Z_{i,j}, \varepsilon^{\lambda_i}Z_{i,jk})\geqslant n_1X_i, \tag{5.1}$$

$$\pm H_i(t, x, Z\pm X)\mp H_i(t, x, Z)\geqslant n_2X_i, \qquad i=1, \cdots, n, \tag{5.2}$$

where $n_1 \in R$, $n_2 \in R^+$, Z *is a formal approximation to the solution* u *of the BVP* (2.3); X *is a vector function whose components are* $X_i(t)=n_3e^{\eta t}\gamma(\varepsilon)$, n_3 *and* η *belong to* R^+, $\gamma(\varepsilon)=\max(\delta, \delta', \delta'')$. *Then we have*

$$|u_i(t, x)-Z_i(t, x)|=O(\gamma(\varepsilon))=o(1) \quad \text{on } \bar{G}, \quad i=1, \cdots, n. \tag{5.3}$$

Proof. Using 'theorem 2.1', we shall show that the vector functions y, z defined by (5.4) below are respectively superfunction and subfunction with respect to the solution u of the B*VP* (2.3). Then the theorem follows immediately. Define

$$\begin{cases} y=(y_1, \cdots, y_n), & y_i=Z_i+X_i, \\ z=(z_1, \cdots, z_n), & z_i=Z_i-X_i, \quad i=1, \cdots, n, \end{cases} \tag{5.4}$$

where $X_i(t)=n_3e^{\eta t}\gamma(\varepsilon)$, n_3 and η belong to R^+ and are yet to be determined. Substituting y in the place of u in (2.3), we have

$$\begin{aligned} P_iy&=\partial Z_i/\partial t+\eta n_3e^{\eta t}\gamma(\varepsilon)+F_i(t, x, y, Z_{i,j}, \varepsilon^{\lambda_i}Z_{i,jk}) \\ &=\eta n_3e^{\eta t}\gamma(\varepsilon)+F_i(t, x, y, Z_{i,j}, \varepsilon^{\lambda_i}Z_{i,jk})-F_i(t, x, Z, Z_{i,j}, \varepsilon^{\lambda_i}Z_{i,jk})+P_iZ \\ &\geqslant n_3e^{\eta t}\gamma(\varepsilon)[\eta+n_1]+f_i-q_i, \qquad \text{by (5.1) and (2.13)}, \\ &\geqslant n_3e^{\eta t}\gamma(\varepsilon)[\eta+n_1]-K\gamma(\varepsilon)+f_i, \quad \text{where } q_i\leqslant K\gamma(\varepsilon) \quad \text{by (2.16)}, \quad K \in R^+, \\ &\geqslant \gamma(\varepsilon)[n_3(\eta+n_1)-K]+f_i\geqslant f_i, \end{aligned}$$

by a proper choice of η and n_3.

$$R_i y=\begin{cases} Z_i+n_3e^{\eta t}\gamma(\varepsilon)=g_i-r_i+n_3e^{\eta t}\gamma(\varepsilon), \quad \text{by (2.14)}, \\ \qquad \geqslant\gamma(\varepsilon)[n_3-K]+g_i\geqslant g_i, \quad \text{by (2.17)}, \quad \text{on } R_p-R_n. \\ \partial Z_i/\partial n_a+H_i(t, x, y)-H_i(t, x, Z)+H_i(t, x, Z) \\ \qquad \geqslant h_i-s_i+n_2n_3e^{\eta t}\gamma(\varepsilon)\geqslant h_i+\gamma(\varepsilon)[n_2n_3-K] \\ \qquad \geqslant h_i \quad \text{on } R_n \quad \text{by (5.2), (2.15), (2.18)}, \end{cases}$$

and also a proper choice of n_3.

i.e.,

$$\begin{cases} P_i y\geqslant f_i=P_i u & \text{in } G_p, \\ R_i y\geqslant g_i=R_i u & \text{on } R_p-R_n, \\ R_i y\geqslant h_i=R_i u & \text{on } R_n, \quad i=1, \cdots, n, \end{cases}$$

which, by 'theorem 2.1', yields

$$u_i(t, x)\leqslant Z_i(t, x)+X_i(t) \quad \text{on } \bar{G}, \quad i=1, \cdots, n.$$

Similarly one can show that

$$Z_i(t, x)-X_i(t)\leqslant u_i(t, x) \quad \text{on } \bar{G}, \quad i=1, \cdots, n.$$

Hence we have

$$|u_i(t, x)-Z_i(t, x)|\leqslant X_i(t)\leqslant X_i(T)=n_4\gamma(\varepsilon), \qquad n_4\in R^+,$$

or

$$|u_i(t, x)-Z_i(t, x)|=O(\gamma(\varepsilon))=o(1) \quad \text{on } \bar{G}, \quad i=1, \cdots, n.$$

One can also state the following theorem the proof of which is similar to the above theorem.

Theorem 5.2. *Consider the nonlinear BVP* (2.3) *and assume that the implication* (2.5) *is true for it. Further assume that:*

$$\pm F_i(t, X, Z\pm X, Z_{i,j}\pm X_{i,j}, \varepsilon^{\lambda_i}[Z_{i,jk}\pm X_{i,jk}])\mp F_i(t, x, Z, Z_{i,j}, \varepsilon^{\lambda_i}Z_{i,jk})\geqslant n_4X_i, \tag{5.5}$$

$$\pm H_i(t, x, Z\pm X)\mp H_i(t, x, Z)\geqslant n_5X_i, \qquad i=1, \cdots, n, \tag{5.6}$$

where $n_4\in R$, $n_5\in R^+$ and Z is a formal approximation to the solution u of the BVP (2.3); *X is a vector function with components $X_i(t, x)=\gamma(\varepsilon)[n_6e^{\eta t}+e_i(x)]$, where $e_i(x)$ is a bounded function with bounded first and second derivatives and independent of t with the property $\partial e_i/\partial n_a>0$ on R_n, $i=1(1)n$; $n_6\in R^+$ such that $n_6e^{\eta t}+e_i(x)>0$, $i=1, \cdots, n$.*

Then we have the estimate (5. 3) *for the BVP* (2.3).

Example [7, p 268].

If the system of differential equations in (2.3) is of the form

$$\partial u_i/\partial t = K_i \nabla^2 u_i + \phi_i(t, x, u_1, \cdots, u_n), \qquad i=1(1)n,$$

and $K_i \geqslant 0$, $i=1, \cdots, n$, are sufficiently small, then the nonlinear BVP (2.3) represents singular perturbation problems which arise in electrochemistry. In that case the above equations describe an n-component mixture with u_i as the concentration of the i^{th} component.

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