Singular Perturbation Problems for Systems of Partial Differential Equations of Parabolic Type

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§ 1. Introduction

Singular perturbation for parabolic first boundary value problems (BVPs) have been discussed by Aronson [1] and Bobisud [2]. These authors consider a second order linear parabolic equation in one space dimension with a small parameter multiplying the second order space derivative subject to Dirichlet boundary conditions (BCs) prescribed on the parabolic boundary of the domain under consideration. They, in fact, construct asymptotic solutions to the problem and study the asymptotic behaviour of solution as the small parameter goes to zero.

Recently, Lam [3] has discussed a convection diffusion problem governed by a second order semilinear parabolic equation in one space dimension with a small parameter multiplying the second order space derivative subject to mixed type BCs prescribed on the parabolic boundary of the domain of the problem.

In any construction of an approximate solution of a singular perturbation problem, one is interested in knowing as to how close one is to the actual solution of the problem. This is the problem considered in this paper.

Van Harten [4, 5] has considered linear and nonlinear elliptic BVPs of the first kind for a single partial differential equation (PDE) with a small parameter multiplying the highest derivative. Supposing that an approximation to the solution of the BVP is given which satisfies the differential equation and the BCs up to a certain order in some norm, he has shown that, under appropriate assumptions, the distance between the exact solution and an approximation can be made small. His proofs are based on the maximum principle for linear and nonlinear elliptic equations and a contraction principle in a suitable Banach space.

The results of Van Harten have been generalised to systems of PDEs of elliptic type by Ramanujam and Srivastava [6] where the theory of differential inequalities is employed.

In this paper, we obtain results similar to that given in [6] for linear and nonlinear parabolic BVPs of the third kind for systems of second order PDEs in $m$-space dimensions with a small parameter multiplying the second order space derivatives.

In 'section 2', we give the relevant definitions and a differential inequality theo-
rem which shall be used throughout this paper. 'Section 3 and 4' are concerned with the linear problems whereas 'section 5' deals with the nonlinear problems.

§ 2. Monotonicity theorem*

We begin this section by giving a few definitions which, though well known to a certain extent [7], are included here for the sake of completeness.

Consider the parabolic system defined by

$$\partial u_i/\partial t + F_i(t, x, u, u_{i,j}, u_{i,jk}) = 0, \quad i = 1, \ldots, n,$$

where

$$(t, x) \in G_p := (0, T] \times D, \quad D \subset R^n$$ being a bounded domain;

$x = (x_1, \ldots, x_m) \in D, \quad u = (u_1, \ldots, u_n); \quad u_{i,j} = \partial u_i/\partial x_j, \quad u_{i,jk} = \partial^2 u_i/\partial x_j \partial x_k.$

$G_p$ is called the parabolic domain and its parabolic boundary

$R_p := [0] \times \overline{D} \cup (0, T] \times \partial D, \quad \overline{D} \cup \partial D$ being the boundary of $D$.

Finally $R_n$ will stand for a part of the boundary $(0, T] \times \partial D$.

**Definition 2.1.** For any two symmetric matrices $r = (r_{jk}), \bar{r} = (\bar{r}_{jk})$ we have $r \leq \bar{r}$ if and only if the matrix $\bar{r} - r = (\bar{r}_{jk} - r_{jk})$ is positive semidefinite.

**Definition 2.2.** $F_i(t, x, u, p, q)$ is monotone decreasing in the matrix ‘$q$’ if and only if

$$F_i(t, x, u, p, q) \geq F(t, x, u, p, \bar{q}) \quad \text{for} \quad q \leq \bar{q}.$$

**Definition 2.3.** $F = (F_1, \ldots, F_n)$ is said to be quasimonotone decreasing in $u = (u_1, \ldots, u_n)$ if each $F_i, i = 1(1)n$, is monotone decreasing with $u_j, j \neq i, i, j = 1, \ldots, n$.

Since we are going to deal with BVPs of the third kind we need the following:

**Definition 2.4.** An interior normal at a point $(t_0, x_{(0)}) \in (0, T] \times \partial D$ is said to exist if there is a sequence of points $(t_0, x_{(k)}) \in G_p (k=1, 2, \ldots)$ with $x_{(k)} \rightarrow x_{(0)} (k \rightarrow \infty)$. The outer normal derivative relative to this normal is defined by [7]

$$\partial u/\partial n : = \lim_{k \rightarrow \infty} [u(t_0, x_{(k)}) - u(t_0, x_{(0)})]/||x_{(k)} - x_{(0)}||_e,$$

where $|| \cdot ||_e$ is the Euclidean distance.

Now consider the singular perturbation problem for a system of second order PDEs of parabolic type defined by

$$\begin{cases}
P_i u := \partial u_i/\partial t + F_i(t, x, u, u_{i,j}, \varepsilon^{i} u_{i,jk}) = f_i(t, x) \quad \text{in} \quad G_p, \\
R_i u := \begin{cases}
u(t, x) = g_i(t, x) \quad \text{on} \quad R_p - R_n, \\
\partial u_i/\partial n + H_i(t, x, u) = h_i(t, x) \quad \text{on} \quad R_n, \quad i = 1, \ldots, n,
\end{cases}
\end{cases}$$

* A := B means that A is defined by B.
where

(i) \( \epsilon > 0 \) is a small parameter; \( \lambda_i \geq 0 \); \( f_i \in C(G_p) \), \( g_i \in C(R_p - R_n) \), \( h_i \in C(R_n) \);
(ii) \( F_i(t, x, u, p, q) \), for fixed \( i \), is monotone decreasing with respect to the \( m \times m \) matrix \( q \) in \( G_p \);
(iii) it is assumed that an outer normal derivative exists at the points of \( R_n \).

Then we have the following monotonicity theorem for the BVP (2.3).

**Theorem 2.1.** Consider the BVP (2.3) and assume that:

(i) \( F_i(t, x, u, p, q) \) and \( H_i(t, x, u) \), for fixed \( i \), are quasimonotone decreasing in \( u = (u_1, \ldots, u_n) \);
(ii) there exists a vector function \( v = (v_1, \ldots, v_n) \), known as a test function, such that

\[
\begin{align*}
  & P_i [u + cv] - P_i u > 0, \quad R_i [u + cv] - R_i u > 0, \\
  & cv = (cv_1, \ldots, cv_n), \text{ for every } c \in R^* (\text{the set of all positive real numbers}), \text{ for every } u, u_i \in U, i = 1, \ldots, n.
\end{align*}
\]

Then the following implication is true for all \( y = (y_1, \ldots, y_n) \), \( z = (z_1, \ldots, z_n) \), \( y_i, z_i \in U, i = 1, \ldots, n \):

\[
\begin{align*}
  & P_i z \leq P_i u \leq P_i y, \quad R_i z \leq R_i u \leq R_i y, \quad i = 1, \ldots, n \\
  & \Rightarrow \\
  & P_i z \leq P_i u \leq P_i y, \quad R_i z \leq R_i u \leq R_i y, \quad i = 1, \ldots, n \\
  & z_i(t, x) \leq u_i(t, x) \leq y_i(t, x) \quad \text{on } \mathcal{G}, i = 1, \ldots, n.
\end{align*}
\]

The functions \( y \) and \( z \) are respectively called superfunction and subfunction with respect to a solution \( u \) of the BVP (2.3). A consequence of the above theorem is that the solution of the BVP (2.3) is unique (if it exists).

**Proof.** We give the proof for the superfunction. The proof for the subfunction can be given in a similar manner.

If the implication (2.5) is not true for the superfunction, then there exists at least one point \( (t, x_{(i)}) \) \( \in G_p \cup R_n \) and a number \( j \in \{1, \ldots, n\} \) such that

\[
(2.6) \quad u_j(t, x_{(i)}) - y_j(t, x_{(i)}) > 0.
\]

Define \( c > 0 \) as

\[
(2.7) \quad c := \max_{(t, x) \in \mathcal{G}} [u_i(t, x) - y_i(t, x)] / u_i(t, x),
\]

and hence there exists a point \( (t_0, x_{(0)}) \) \( \in G_p \cup R_n \) and a number \( l \in \{1, \ldots, n\} \) such that

\[
(2.8) \quad u_l(t_0, x_{(0)}) - y_l(t_0, x_{(0)}) > 0.
\]
(2.7) \[
\begin{cases} 
(cv_i(t_0, x_{(0)}) = u_i(t_0, x_{(0)}) - y_i(t_0, x_{(0)}), \\
(cv_i(t, x) \geq u_i(t, x) - y_i(t, x) \quad \text{on } G. 
\end{cases}
\]

Therefore, the function \(cv_i - u_i + y_i\) attains an absolute minimum at \((t_0, x_{(0)})\). The point \((t_0, x_{(0)})\) should either belong to \(G_p\) or \(R_n\).

**Case i.** \((t_0, x_{(0)}) \in G_p\).

At \((t, x) = (t_0, x_{(0)})\), we then have
\[
\begin{cases} 
(cv_i - u_i + y_i = 0 = cv_{i, j} - u_{i, j} + y_{i, j}, \quad j = 1, \ldots, n, \\
\partial [cv_i - u_i + y_i]/\partial t \leq 0 \quad \text{and } (cv_{i, jk} - u_{i, jk} + y_{i, jk}) \geq 0. \end{cases}
\]

Furthermore, the following inequalities are valid at \((t, x) = (t_0, x_{(0)})\):
\[
0 \leq P_i y - P_i u = P_i y - \partial u_i/\partial t - F_i(t_0, x_{(0)}, u, u_{i, j}, \epsilon^{i} u_{i, jk}) \\
\leq P_i y - \partial y_i/\partial t - \partial cv_i/\partial t - F_i(t_0, x_{(0)}, y + cv, y_{i, j} + cv_{i, j}, \epsilon^{i}[y_{i, jk} + cv_{i, jk}]) \\
= P_i y - P_i[y + cv],
\]

by (2.5), (2.8), (2.7) and the quasimonotonicity of \(F_i\). This is a contradiction to (2.4). Hence \((t_0, x_{(0)}) \notin G_p\).

**Case ii.** \((t_0, x_{(0)}) \in R_n\).

At \((t, x) = (t_0, x_{(0)})\), we then have
\[
\partial [cv_i - u_i + y_i]/\partial n \leq 0.
\]

Therefore at \((t, x) = (t_0, x_{(0)})\) we have,
\[
0 \leq R_i y - R_i u = R_i y - \partial u_i/\partial n - H_i(t_0, x_{(0)}, u) \\
\leq R_i y - \partial [cv_i + y_i]/\partial n - H_i(t_0, x_{(0)}, y + cv),
\]

by (2.7), (2.9) and the quasimonotonicity condition of \(H_i\),
\[
= R_i y - R_i[y + cv].
\]

It is again a contradiction to (2.4) and hence \((t_0, x_{(0)}) \notin R_n\). That is, such a point \((t_0, x_{(0)}) \notin G_p \cup R_n\) cannot exist and hence the proof of the theorem.

**Remark.** Apparently the above theorem is similar to the one given in [8, 9] but the proof provided here is rigorous and the form of the test function is simpler.

We conclude this section with a few more definitions.

**Definition 2.5.** For \(f = (f_1, \ldots, f_n)\), \(g = (g_1, \ldots, g_n)\) and \(h = (h_1, \ldots, h_n)\) defined respectively in \(G_p, R_p - R_n\) and \(R_n\), we define their norms by

* The meaning of this inequality is given in 'definition 2.1'.
† One can also consider \(||f|| = \text{Sup} ||f_i(t, x)|| (t, x) \in G_p, i = 1 (1)n\) with similar definitions for \(||g||^*\) and \(||h||^{**}\) and pursue analysis.
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\[ \| f \| = \text{Sup} \left\{ \left[ \sum_{i=1}^{n} |f_i(t, x)|^2 \right]^{1/2} \mid (t, x) \in G_p \right\}, \]

\[ \| g \|' = \text{Sup} \left\{ \left[ \sum_{i=1}^{n} |g_i(t, x)|^2 \right]^{1/2} \mid (t, x) \in R_p - R_n \right\}, \]

\[ \| h \|'' = \text{Sup} \left\{ \left[ \sum_{i=1}^{n} |h_i(t, x)|^2 \right]^{1/2} \mid (t, x) \in R_n \right\}. \]

**Definition 2.6.** Suppose a vector function \( Z = (Z_1, \ldots, Z_n) \) is given satisfying the following equations

\[ P_i u - P_i Z = q_i \quad \text{in} \ G_p, \]
\[ R_i u - R_i Z = r_i \quad \text{on} \ R_p - R_n, \]
\[ R_i u - R_i Z = s_i \quad \text{on} \ R_n, \quad i = 1, \ldots, n. \]

Then the function \( Z \) is called a formal approximation of the solution \( u \) of the BVP \((2.3)\) in the norms \( \| \cdot \|, \| \cdot \|' \) and \( \| \cdot \|'' \) if

\[ \| q \| = O(\delta(\epsilon)) = o(1), \quad q = (q_1, \ldots, q_n), \]
\[ \| r \|' = O(\delta'(\epsilon)) = o(1), \quad r = (r_1, \ldots, r_n), \]
\[ \| s \|'' = O(\delta''(\epsilon)) = o(1)^*, \quad s = (s_1, \ldots, s_n), \]

where \( \delta, \delta', \delta'' \) are order functions.

§ 3. Linear boundary value problems

In this section we consider a special case of the BVP \((2.3)\). In fact, we focus our attention on the linear parabolic BVP \((3.1)\) and obtain estimates for its solution. In turn, since the BVP is linear, these yield estimates for the difference between the solution and a formal approximation.

Consider the linear parabolic BVP for \( u = (u_1, \ldots, u_n) \) defined by

\[ \begin{cases} P_i u = \partial u_i / \partial t + \epsilon \partial_i P_{\epsilon} u + P_{\epsilon} u = f_i(t, x) & \text{in} \ G_p, \\ R_i u = u_i(t, x) = g_i(t, x) & \text{on} \ R_p - R_n, \\ \partial u_i / \partial n_a + \sum_{j=1}^{n} d_{ij} u_j = h_i(t, x) & \text{on} \ R_n, \quad i = 1, \ldots, n, \end{cases} \]

where

\[ P_{\epsilon} u = \sum_{j=1}^{n} a_{ij}(t, x) u_j + \sum_{k=1}^{m} b_{ik}(t, x) u_{i,k} - \sum_{j,k=1}^{m} c_{ijk}(t, x) u_{i,j,k}, \]

* The symbols \( O \) and \( o \) denote the Landau order symbols.
\[ P_{ii}u: = \sum_{j=1}^{n} \alpha_{ij}(t, x)u_j + \sum_{k=1}^{m} \beta_{ik}(t, x)u_{i,k}, \quad i = 1, \ldots, n. \]

Further assume that:
(i) the coefficients \( a_{ij}, b_{ik}, c_{ijk}, \alpha_{ij} \) and \( \beta_{ik} \) are defined and continuous in their arguments;
(ii) for fixed \( i \), the matrix with elements \( c_{ijk} \) is symmetric and positive semi-definite with respect to \( j \) and \( k \).

According to ‘theorem 2.1’, whenever there exists a test function for the BVP (3.1) then the implication (2.5) is true. We now give two sets of sufficient conditions which yield test functions for the BVP (3.1).

Set I. Consider the BVP (3.1) and assume that:

\[ (3.2) \quad e^{\gamma_{i}}a_{ij} + \alpha_{ij} \leq 0 \quad \text{and} \quad d_{ij} \leq 0, j \neq i, \quad i, j = 1, \ldots, n; \]

(the quasimonotonicity condition)

\[ \sum_{j=1}^{n} e^{\gamma_{i}}a_{ij} + \alpha_{ij} \geq a_0 \in \mathbb{R} \quad \text{(the set of all real numbers)} \quad \text{and} \]

\[ \sum_{j=1}^{n} d_{ij} \geq d_0 \in \mathbb{R}^+, \quad i = 1, \ldots, n. \]

Then a test function in the present case is given by \( v = (v_1, \ldots, v_n) \) where

\[ v_i(t) = e^{\gamma_{i}t}, \quad \eta \in \mathbb{R}^+, \quad t \in [0, T], \quad i = 1, \ldots, n. \]

Set II. Consider the BVP (3.1) and assume that the quasimonotonicity condition (3.2) is true. Further assume that:

(i) the coefficients \( a_{ij}, b_{ik}, c_{ijk}, d_{ij}, \alpha_{ij} \) and \( \beta_{ij} \) are bounded, \( \sum_{j=1}^{n} d_{ij} \geq 0, \quad i = 1, \ldots, n; \)

(ii) there exists a vector function \( \gamma = (\gamma_1, \ldots, \gamma_n), \gamma_i \in U \) with bounded first and second derivatives and independent of ‘\( t \)’ with the property

\[ \partial \gamma_i / \partial n_0 > 0 \quad \text{on} \quad R_n, \quad \gamma_0(x) = \gamma_i(x), \quad i = 1, \ldots, n. \]

Then a test function for this problem is given by

\[ v = (v_1, \ldots, v_n), \quad v_i(t, x) = m_i e^{\gamma_{i}t} + \gamma_0(x) > 0 \quad \text{on} \quad \bar{G}, \quad \eta, \quad m_i \in \mathbb{R}^+ \quad i = 1, \ldots, n. \]

Remarks. (i) There is a difference between the above sets of conditions. Set I rules out any consideration of second BVPs whereas set II allows us to consider second BVPs also.

(ii) The existence of a \( \gamma \) function defined in the set II has been discussed in certain cases by Walter [7, p. 249]. Also the significance of the condition \( \sum_{i=1}^{n} d_{ij} \geq d_0 > 0, i = 1, \ldots, n \) is mentioned in [7, p. 244].
The constants $m_i$, $i=1, \ldots, 14$, appearing in the rest of the paper are independent of the parameter $\epsilon$.

We now obtain estimates for solution of the BVP (3.1)

**Theorem 3.1.** Consider the BVP (3.1) and assume that the quasimonotonicity condition (3.2) is true. Further assume that there exists a vector function $w=(w_1, \ldots, w_n)$ such that

\[
\begin{align*}
&w_i = w_i(t, x) > 0 \quad \text{on} \ G, \ w_i \in U, \\
&P_i w \geq m_i \in R, \ R_i w \geq m \in R^+, \quad i = 1, \ldots, n.
\end{align*}
\]

Then we have

\[
|u_i(t, x)| \leq m_3 \{||f|| + ||g'|| + ||h''|| \} \quad \text{on} \ \overline{G}, \ m_3 \in R^+, \ i = 1, \ldots, n.
\]

**Proof.** The analysis carried out in the following is similar to that given in [6]. Consider the linear BVP (3.1). We set

\[
v = (v_1, \ldots, v_n), \ v_i = u_i/w_i \quad \text{on} \ \overline{G}, \ i = 1, \ldots, n,
\]

and find that $v$ must satisfy

\[
w_i \partial v_i/\partial t + \sum_{j=1}^{n} \delta_{ij} [\partial w_i/\partial t] v_j + \sum_{j=1}^{n} [\epsilon^t a_{ij} + \alpha_{ij}] v_j w_j + \sum_{j=1}^{n} \delta_{ij} \left( \sum_{k=1}^{m} \epsilon^t b_{ik} + \beta_{ik} \right) w_i k v_j - \sum_{j,k=1}^{m} \epsilon^t c_{ijk} v_i j w_k + \sum_{j,k=1}^{m} \delta_{ij} \left( \sum_{j,k=1}^{m} \epsilon^t c_{ijk} v_i j w_k \right) - \sum_{j,k=1}^{m} \epsilon^t c_{ijk} v_i j k w_k = f_i(t, x) \quad \text{in} \ G_p, \ i = 1, \ldots, n,
\]

\[
u_i w_i = g_i(t, x) \quad \text{on} \ R_p - R_n,
\]

where $i=1, \ldots, n$ and $\delta_{ij}$ is the Kronecker Symbol. The above problem can be written as a BVP for $v=(v_1, \ldots, v_n)$ in the following form similar to that of (3.1):

\[
\begin{align*}
P_i v : &= \partial v_i/\partial t + \sum_{j=1}^{n} A_{ij} v_j + \sum_{k=1}^{m} B_{ik} v_i k - \sum_{j,k=1}^{m} c_{ijk} \epsilon^t v_i j k \quad \text{in} \ G_p, \\
R_i v : &= f_i(t, x)/w_i(t, x) \quad \text{on} \ R_p - R_n, \\
v_i(t, x) &= \partial v_i/\partial n + \sum_{j=1}^{n} D_{ij} v_j = h_i(t, x)/w_i(t, x) \quad \text{on} \ R_n, \ i = 1, \ldots, n,
\end{align*}
\]
where

\[
A_{ij} = \left[ \delta_{ij}\partial w_i/\partial t + (\varepsilon^i a_{ij} + \alpha_{ij}) w_j + \delta_{ij} \sum_{k=1}^{m} (\varepsilon^i b_{ik} + \beta_{ik}) w_{t,k} \right.
\]
\[
- \delta_{ij} \sum_{f,k=1}^{m} \varepsilon^i c_{fjk} w_{t,f,k} \bigg] / w_t, \quad i, j = 1, \ldots, n,
\]
\[
B_{ik} = -(2/w_i) \sum_{j=1}^{m} \varepsilon^i c_{ijk} w_{t,jk} + (\varepsilon^i b_{ik} + \beta_{ik}), \quad k = 1, \ldots, m, \quad \text{and}
\]
\[
D_{ij} = [d_{ij} w_j + \delta_{ij} \partial w_i/\partial n_a]/w_t, \quad i, j = 1, \ldots, n.
\]

It is easy to verify the following inequalities:

\[
A_{ij} = [\varepsilon^i a_{ij} + \alpha_{ij}] w_j/w_i \leq 0, \quad j \neq i, i, j = 1, \ldots, n, \quad \text{by (3.2) and (3.4)},
\]
\[
D_{ij} = d_{ij} w_j/w_i \leq 0, \quad j \neq i, i, j = 1, \ldots, n, \quad \text{by (3.2) and (3.4)},
\]
\[
\sum_{j=1}^{n} A_{ij} = [P_i w]/w_i \geq m_i \in R, \quad i = 1, \ldots, n, \quad \text{by (3.4)},
\]
\[
\sum_{j=1}^{n} D_{ij} = [R_i w]/w_i \geq m_i/w_i \quad \text{on } R_n, \quad i = 1, \ldots, n, \quad \text{by (3.4)}.
\]

The above inequalities (3.8)–(3.11) show that, by the set I of sufficient conditions, the implication (2.5) is true for the BVP (3.7).

To obtain the required estimate, we consider the vector function \( y = (y_1, \ldots, y_n) \) defined by

\[
y_i(t, x) = m_i e^{\eta t}[\|f\| + \|g\| + \|h\|]' \quad \text{on } \mathcal{G}, \quad i = 1, \ldots, n,
\]

where \( m_i, \eta \in R^+ \) are yet to be determined.

We now show that, by a proper choice of \( m_i \) and \( \eta \), the vector function \( y \) defined by (3.12) is nothing but a superfunction with respect to a solution \( v \) of the BVP (3.7). For,

\[
P_i' y = \left[ \eta + \sum_{j=1}^{n} A_{ij} y_j \right] f_i/w_i \quad \text{in } G_p,
\]
\[
R_i' y = \left\{ \begin{array}{l}
y_i \geq g_i/w_i \quad \text{on } R_p-R_n, \\
\partial y_i/\partial n_a + \sum_{j=1}^{n} D_{ij} y_j \geq h_i/w_i \quad \text{on } R_n, \quad i = 1, \ldots, n,
\end{array} \right.
\]

by a proper choice of \( m_i, \eta \in R^+ \). Now (3.13) yields

\[
P_i' y \geq P_i' v \quad \text{and} \quad R_i' y \geq R_i' v, \quad i = 1, \ldots, n.
\]

which gives
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$v_i(t, x) \leq y_i(t, x)$ on $\bar{G}$, \hspace{1em} i=1, \ldots, n.

Since the BVP (3.7) is linear we have,

$|v_i(t, x)| \leq y_i(t, x)$ on $\bar{G}$, \hspace{1em} i=1, \ldots, n.

Hence

$|u_i(t, x)| \leq m_i y_i(t, x)$ on $\bar{G}$, \hspace{1em} i=1, \ldots, n,

where $u=(u_1, \ldots, u_n)$ is the solution of the BVP (3.1). Therefore,

$|u_i(t, x)| \leq m_i \|f\| + \|g\|' + \|h\|''$ on $\bar{G}$, \hspace{1em} i=1, \ldots, n, \hspace{1em} m_i \in \mathbb{R}^+.

Remark. In the above theorem, it has been assumed that the condition $R_i w > 0$, $i=1, \ldots, n$ holds good. This condition can be further relaxed to $R_i w \geq 0$.

Theorem 3.2. Assume that all the conditions of 'theorem 3.1' are satisfied. Further, let $Z$ be a formal approximation to the solution $u$ of the BVP (3.1) such that it satisfies (2.13)–(2.18). Then we have

$|u_i(t, x) - Z_i(t, x)| \leq O((\max (\delta, \delta', \delta''))$, on $\bar{G}$, \hspace{1em} i=1, \ldots, n.

Proof. To establish (3.15), we consider the linear parabolic BVP for $u-Z$ defined by

$(3.16) \hspace{1em} \begin{cases} P_i(u-Z)=q_i & \text{in } G_p, \\
R_i(u-Z)=r_i & \text{on } R_p - R_n, \\
R_i(u-Z)=s_i & \text{on } R_n, \hspace{1em} i=1, \ldots, n. \end{cases}$

We now apply 'theorem 3.1' to the BVP (3.16) which ensures the existence of a constant $m_s$ such that

$(3.17) \hspace{1em} |u_i(t, x) - Z_i(t, x)| \leq m_s (\|q\| + \|r\|' + \|s\|'')$.

The result (3.15) now follows from (3.17) and (2.16)–(2.18).

§ 4. Auxiliary linear boundary value problems

Throughout the 'section 3' the validity of the quasimonotonicity condition (3.2) has been assumed to obtain estimates for solution of the linear BVP (3.1). However, in general, this may not be the case. To handle the general case one can introduce an extended auxiliary problem (4.1) with respect to the BVP (3.1) as in [8, 9, 10] and use it to obtain results for the latter BVP.

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Then

\( \hat{u}_{1} = f_{i}(t, x) \),
and

\( \hat{u}_{i} = -(e^{i}a_{ij} + \alpha_{ij})^{+} \hat{u}_{j} - (e^{i}a_{ij} + \alpha_{ij})^{-} \hat{u}_{j+n} \) in \( G_{p} \),

\( \hat{u}_{i+n} = f_{i}(t, x) \) on \( R_{p} - R_{n} \),

\( \hat{u}_{i+n,j} = g_{i}(t, x) \)

(4.1)

where \( \hat{u} = (\hat{u}_{1}, \ldots, \hat{u}_{n}) \).

The following observations are made regarding the BVP (4.1):

(i) it satisfies the quasimonotonicity condition similar to that of (3.2), i.e., as defined in ‘definition 2.3’;

(ii) if \( (u_{1}, \ldots, u_{n}) \) is a solution of the BVP (3.1) then \( (-u_{1}, \ldots, -u_{n}, u_{1}, \ldots, u_{n}) \) is a solution of the BVP (4.1).

The following theorem, when the BVP (3.1) is not quasimonotone, gives estimates for the solution of the BVP (3.1) by making use of the extended auxiliary BVP (4.1).

**Theorem 4.1.** If the BVP (3.1) does not satisfy the quasi-monotonicity condition (3.2), we introduce the auxiliary BVP (4.1) and assume that there exists a vector function \( \hat{w} = (\hat{w}_{1}, \ldots, \hat{w}_{2n}) \) such that

\[
\begin{align*}
\hat{w}_{i}(t, x) & > 0 \quad \text{on } \partial G, \hat{w}_{i} \in U, \quad i = 1, \ldots, 2n, \\
\hat{P}_{i} \hat{w} & \geq m_{8}, \quad \hat{P}_{t+n} \hat{w} \geq m_{8} \in R, \\
\hat{R}_{i} \hat{w} & \geq m_{8}, \quad \hat{R}_{t+n} \hat{w} \geq m_{8} \in R^{+}, \quad i = 1, \ldots, n.
\end{align*}
\]

Then we have
(4.3) \( |u_i(t, x)| \leq m_i \|f\| + \|g\| + \|h\| \) on \( \bar{G} \), \( i=1, \ldots, n \), \( m_i \in R^+ \).

**Proof.** Consider the BVP (4.1). We set \( \hat{\theta}=(\hat{\theta}_1, \ldots, \hat{\theta}_n) \), \( \hat{\theta}_i=\hat{u}_i/\hat{w}_i \) on \( \bar{G} \), \( i=1, \ldots, 2n \), and find that, following a similar procedure as in the proof of 'theorem 3.1', \( \hat{\theta} \) must satisfy

\[
\begin{cases}
\hat{P}' \hat{\theta} = -f_i/\hat{w}_i & \text{in } G_p, \\
\hat{P}'_{i+n} \hat{\theta} = f_i/\hat{w}_{i+n} & \text{in } G_p, \\
\hat{R}' \hat{\theta} = \begin{cases}
g_i/\hat{w}_i & \text{on } R_p-R_n, \\
h_i/\hat{w}_i & \text{on } R_n,
\end{cases} \\
\hat{R}'_{i+n} \hat{\theta} = \begin{cases}
g_i/\hat{w}_{i+n} & \text{on } R_p-R_n, \\
h_i/\hat{w}_{i+n} & \text{on } R_n, \quad i=1, \ldots, n.
\end{cases}
\end{cases}
\]

Then, as in the case of 'theorem 3.1', we conclude that \( |\hat{u}_i(t, x)| \leq m_i \|f\| + \|g\| + \|h\| \) on \( \bar{G} \), \( i=1, \ldots, 2n \). Since \((-u_1, \ldots, -u_n, u_1, \ldots, u_n)\) is a solution of the BVP (4.1) we have the estimate (4.3).

**Theorem 4.2.** Assume that all the conditions of 'theorem 4.1' are satisfied. Further, let \( Z \) be a formal approximation to the solution \( u \) of the BVP (3.1) and satisfy (2.13)–(2.18). Then we have

\[
|u_i(t, x)-Z_i(t, x)| \leq O \left( \text{max} (\delta, \delta', \delta'') \right) \quad \text{on } \overline{G}, \quad i=1, \ldots, n.
\]

The proof follows in a straightforward manner if one carries out the steps as in the proof of 'theorem 3.2' and makes use of 'theorem 4.1'.

**Remark.** A set of sufficient conditions under which a function \( \hat{\psi}=(\hat{\psi}_1, \ldots, \hat{\psi}_n) \) of 'theorem 4.1' exists is the following:

\[
(\varepsilon^i a_{it} + \alpha_{it}) - \sum_{j=1, j \neq i}^n |\varepsilon^i a_{ij} + \alpha_{ij}| \geq m_{12} \in R; \\
d_{it} - \sum_{j=1, j \neq i}^n |d_{ij}| \geq m_{13} \in R^+, \quad i=1, \ldots, n.
\]

In fact, one can take \( \hat{\psi} \) as

\[
\hat{\psi}_i = m_{14} e^\varepsilon t, \quad i=1, \ldots, 2n, \quad t \in [0, T], \quad m_{14} \in R^+.
\]

§ 5. **Nonlinear boundary value problems**

We now consider the nonlinear BVP (2.3).

The constants \( n_i, i=1, \ldots, 6 \), appearing in this section are independent of the parameter \( \varepsilon \).

The implication (2.5) is true for the BVP (2.3) under any one of the following two sets of conditions.

**Set I.** Consider the BVP (2.3) and assume that:

(i) \( F_i(t, x, u, p, q) \) and \( H_i(t, x, u) \), for fixed \( i \), are quasimonotone decreasing in \( u \) according to the 'definition 2.3';
\(F_i(t, x, u + \alpha, p, q) - F_i(t, x, u, p, q) \geq n_i \alpha\),
\(n_i \in \mathbb{R}, \alpha = (\alpha_1, \cdots, \alpha_n), \alpha(t) = \alpha_i(t) > 0, \ i = 1, \cdots, n;\)
\(H_i(t, x, u + \alpha) - H_i(t, x, u) > 0, \ i = 1, \cdots, n.\)

The test function in the present case is given by
\[v = (v_1, \cdots, v_n), \quad v_i(t) = e^{\eta t}, \quad \eta \in \mathbb{R}^+, \quad t \in [0, T], \quad i = 1, \cdots, n.\]

Set II. Consider the BVP (2.3) and assume that \(F_i(t, x, u, p, q)\) and \(H_i(t, x, u)\), for fixed \(i\), are quasimonotone decreasing in \(u\). Further, suppose that \(F_i\) and \(H_i\), for fixed \(i\), satisfy Lipschitz conditions in \(G_p\) and \(R_n\) respectively with respect to every \(u = (u_1, \cdots, u_n), u_i \in U\) and its derivatives. Then, as in [10], one can construct a linear auxiliary Lipschitz problem for the BVP (2.3). Consequently, if the implication (2.5) is true for the Lipschitz problem then so is for the BVP (2.3) [10].

**Theorem 5.1.** Consider the nonlinear BVP (2.3) and assume that the implication (2.5) is true for it. Further assume that:

\[\pm F_i(t, x, Z \pm X, Z_{i,j}, e^{\eta t} Z_{i,j,k}) \mp F_i(t, x, Z, Z_{i,j}, e^{\eta t} Z_{i,j,k}) \geq n_i X_i, \quad i = 1, \cdots, n,\]

where \(n_i \in \mathbb{R}, n_i \in \mathbb{R}^+, Z\) is a formal approximation to the solution \(u\) of the BVP (2.3); \(X\) is a vector function whose components are \(X_i(t) = n_i e^{\eta t} \gamma(t), n_i\) and \(\eta\) belong to \(\mathbb{R}^+, \gamma(t) = \max (\delta, \delta', \delta'')\). Then we have

\[|u_i(t, x) - Z_i(t, x)| = O(\gamma(t)) = o(1) \quad \text{on } \overline{G}, \quad i = 1, \cdots, n.\]

**Proof.** Using 'Theorem 2.1', we shall show that the vector functions \(y, z\) defined by (5.4) below are respectively superfunction and subfunction with respect to the solution \(u\) of the BVP (2.3). Then the theorem follows immediately. Define

\[\begin{cases}
y = (y_1, \cdots, y_n), \quad y_i = Z_i + X_i, \\
z = (z_1, \cdots, z_n), \quad z_i = Z_i - X_i, \quad i = 1, \cdots, n,
\end{cases}\]

where \(X_i(t) = n_i e^{\eta t} \gamma(t), n_i\) and \(\eta\) belong to \(\mathbb{R}^+\) and are yet to be determined. Substituting \(v\) in the place of \(u\) in (2.3), we have

\[P_i y = \partial Z_i / \partial t + mn e^{\eta t} \gamma(t) + F_i(t, x, y, Z_{i,j}, e^{\eta t} Z_{i,j,k}) = \eta n e^{\eta t} \gamma(t) + F_i(t, x, y, Z_{i,j}, e^{\eta t} Z_{i,j,k}) - F_i(t, x, Z, Z_{i,j}, e^{\eta t} Z_{i,j,k}) + P_i Z \geq n e^{\eta t} \gamma(t) \geq n e^{\eta t} \gamma(t) + f_i - q_i, \quad \text{by (5.1) and (2.13)}, \]

\[\geq n e^{\eta t} \gamma(t) + f_i - K \gamma(t) + f_i, \quad \text{where } q_i \leq K \gamma(t) \quad \text{by (2.16), } \quad K \in \mathbb{R}^+, \]

\[\geq \gamma(t) n (\gamma + n_i) - K + f_i \geq f_i, \]

Theorem 5.1 holds.
by a proper choice of $\eta$ and $n_{a}$,

$$Z_{i} + n_{a}e^{\eta t} = g_{i} - r_{i} + n_{a}e^{\eta t} \gamma(\epsilon),$$
by (2.14),

$$\geq \gamma(\epsilon)[n_{a} - K] + g_{i} \geq g_{i},$$
on $R_{p} - R_{n}$.

$$R_{i}y = \partial Z_{i} / \partial n_{a} + H_{i}(t, x, y) - H_{i}(t, x, Z) + H_{i}(t, x, Z) \geq h_{i} + h_{i} \gamma(\epsilon)[n_{a} - K] \geq h_{i} \mathrm{s}(2.17), \quad \mathrm{s}(2.18),$$

and also a proper choice of $n_{a}$.

$$i.e.,$$

$$\begin{cases}
P_{i}y \geq f_{i} = P_{i}u & \text{in } G_{p},
R_{i}y \geq g_{i} = R_{i}u & \text{on } R_{p} - R_{n},
R_{i}y \geq h_{i} = R_{i}u & \text{on } R_{n}, \quad i = 1, \ldots, n,
\end{cases}$$

which, by ‘theorem 2.1’, yields

$$u_{i}(t, x) \leq Z_{i}(t, x) + X_{i}(t) \quad \text{on } \overline{G}, \quad i = 1, \ldots, n.$$  

Similarly one can show that

$$Z_{i}(t, x) - X_{i}(t) \leq u_{i}(t, x) \quad \text{on } \overline{G}, \quad i = 1, \ldots, n.$$  

Hence we have

$$|u_{i}(t, x) - Z_{i}(t, x)| \leq X_{i}(T) = n_{a} \gamma(\epsilon), \quad n_{a} \in \mathbb{R}^{+},$$
or

$$|u_{i}(t, x) - Z_{i}(t, x)| = O(\gamma(\epsilon)) = o(1) \quad \text{on } \overline{G}, \quad i = 1, \ldots, n.$$  

One can also state the following theorem the proof of which is similar to the above theorem.

**Theorem 5.2.** Consider the nonlinear BVP (2.3) and assume that the implication (2.5) is true for it. Further assume that:

$$F_{i}(t, x, Z \pm X, Z_{i,j} \pm X_{i,j}, \varepsilon^{i}[Z_{i,j,k} \pm X_{i,j,k}]) \geq F_{i}(t, x, Z, Z_{i,j}, \varepsilon^{i}Z_{i,j,k}) \geq n_{a} X_{i},$$

$$H_{i}(t, x, Z \pm X) \geq H_{i}(t, x, Z) \geq n_{a} X_{i},$$

where $n_{a} \in \mathbb{R}, n_{b} \in \mathbb{R}^{+}$ and $Z$ is a formal approximation to the solution $u$ of the BVP (2.3); $X$ is a vector function with components $X_{i}(t, x) = \gamma(\epsilon)[n_{a}e^{\eta t} + e_{i}(x)]$, where $e_{i}(x)$ is a bounded function with bounded first and second derivatives and independent of $t$ with the property $\partial e_{i} / \partial n_{a} > 0$ on $R_{n}, i = 1(1)n$; $n_{a} \in \mathbb{R}^{+}$ such that $n_{a}e^{\eta t} + e_{i}(x) > 0, i = 1, \ldots, n$. 
Then we have the estimate (5. 3) for the BVP (2.3).

Example [7, p 268].
If the system of differential equations in (2.3) is of the form

$$\partial u_i/\partial t = K_i \nabla^2 u_i + \partial \psi(t, x, u, \ldots, u_n), \quad i = 1(1)n,$$

and $K_i \geq 0$, $i = 1, \ldots, n$, are sufficiently small, then the nonlinear BVP (2.3) represents singular perturbation problems which arise in electrochemistry. In that case the above equations describe an $n$-component mixture with $u_i$ as the concentration of the $i^{th}$ component.

References


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