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Kneser's Type Properties for Caratheodory Differential Equations*)

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§ 0. Introduction

Consider the initial value problem

(E) $$\dot{x}=f(t, x)$$

(C) $$x(t_0)=x_0.$$

In 1923 Kneser [9] proved, in the case where f is continuous, that cross-sections of solution funnel emanating from an initial point are continuum (compact and connected). Among the interesting generalization which followed, we point out the Hukuhara's one [5]. He proved that the set of all solutions of the initial value problem (E)-(C) on any compact subinterval I of $[t_0, \alpha)$, where

$$[t_0, \alpha)=\cap\{\mathrm{Dom}\ x: x \text{ is a solution of (E)-(C)}\}\cap[t_0, +\infty),$$

is a continuum in the Banach space $C(I, \boldsymbol{C}^n)$.

In this paper we consider the case where the function f satisfies the Caratheodory conditions locally in its domain. The main purpose is to give some "informations" for the cross-section at the extreme point $t=\alpha$ in the case where $\alpha<+\infty$. These results are new even in the case when f is continuous.

Throughout this paper, we will denote by $|\cdot|$ one of the usual equivalent norms on the n-dimensional complex space $\boldsymbol{C}^n$ and by $B(A, \boldsymbol{C}^n)$ the vector space of all continuous and bounded $\boldsymbol{C}^n$-valued functions defined on a subset A of a metric space. This space with norm $\|\cdot\|$

$$\|h\|=\sup\{|h(z)|: z\in A\}$$

is a Banach space. In particular, if the set A is compact, then $B(A, \boldsymbol{C}^n)=C(A, \boldsymbol{C}^n)$. The conceptions of measure of a subset A of the real line take it out $\boldsymbol{R}$ of metrizability an dintegrability of a $\boldsymbol{C}^n$-valued function h defined on the set A should be

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understood in the sence of Lebesque. We will denote by $L(A, \boldsymbol{C}^n)$ the set of all integrable $\boldsymbol{C}^n$-valued functions defined on A, $L(A, \boldsymbol{R}) = L(A)$.

Let Ω be a subset of $\boldsymbol{R}\times\boldsymbol{C}^n$. The set of all Caratheodory functions on Ω (cf. [13, p. 183]) will be denoted by Car (Ω). The set of all $\boldsymbol{C}^n$-valued functions which satisfy the Caratheodory conditions locally in Ω, i.e. for any compact subset W of Ω it follows that $(f|\,W) \in$ Car (W), will be denoted by $\mathrm{Car}_{\mathrm{loc}}(\Omega)$.

We also use the notations

$$pr_1\Omega = \{t \in \boldsymbol{R}\colon \text{There exists } x \in \boldsymbol{C}^n \text{ with } (t, x) \in \Omega\}$$
$$pr_2\Omega = \{x \in \boldsymbol{C}^n\colon \text{There exists } t \in \boldsymbol{R} \text{ with } (t, x) \in \Omega\}.$$

If $h \in$ Car (Ω) and $m \in L(pr_1(\Omega))$ is a function such that

$$|f(t, x)| \leqq m(t) \qquad \text{for every } (t, x) \in \Omega$$

we will say that m characterizes $f \in$ Car (Ω). Moreover, for a solution x of (E)-(C) we will denote by $G(x; P_0)$ its graph, i.e.

$$G(x; P_0) = \{(t, x(t))\colon t \in \mathrm{Dom}\, x\}, \qquad P_0 = (t_0, x(t_0))$$

and by $G(x\,|\,A; P_0)$ the graph of its restriction on the subset A of Dom x.

§ 1. Existence, extension and dependence of solutions

The first theorem of this section is an adaptation of the well-known existence theorem of Caratheodory (cf. [13, p. 185]).

Theorem 1.1. *Consider the initial value problem* (E)-(C) *and let* $x_0 \in (pr_2\Omega)^0$ *and* U *be a neighborhood of the point* P_0 *such that the restriction* $(f|\,U\cap\Omega) \in$ Car $(U\cap\Omega)$. *If* $b>0$ *is such that for some* $\delta>0$

$$[t_0, t_0+\delta]\times\{x \in \boldsymbol{C}^n\colon |x-x_0| \leqq b\} \subseteqq U\cap\Omega$$

and $m \in L\,(pr_1(U\cap\Omega))$ *is a function characterizing the* $(f|\,U\cap\Omega) \in$ Car $(U\cap\Omega)$, *then there exists a solution* $x\colon [t_0, t_1]\to\boldsymbol{C}^n$ *of* (E)-(C) *with*

$$t_1 = \sup\left\{t \geqq t_0\colon \int_{t_0}^{t} m(s)ds \leqq b \text{ and } [t_0, t]\times\{x \in \boldsymbol{C}^n\colon |x-x_0| \leqq b\} \subseteqq U\cap\Omega\right\} > t_0.$$

If we replace the assumption $f \in \mathrm{Car}_{\mathrm{loc}}(\Omega)$ by $f \in$ Car (Ω) and set $U = \boldsymbol{C}^n$, then the above theorem leads to the usual formulation of Caratheodory theorem [13].

Remark 1.1. Under the assumptions of the above theorem, it is clear that all solutions of the initial value problem (E)-(C) are defined (at least) on the interval $[t_0, t_1]$, hence *the common interval of definition of all solutions of* (E)-(C) *is not trivial.* More generally: *If* W *is a compact subset of* Ω^0, *then there exists a* $\delta>0$ *such that all* *solutions* x *of the differential equation* (E) *with* $x(\tau)=\xi$ *and* $P=(\tau, \xi) \in W$ *are defined* (*at least*) *on the interval* $[\tau, \tau+\delta]$.

A solution x of the differential equation (E) is *continuable* if, and only if, there exists a solution of the equation (E) which is a proper extension of x. In the sequel we will denote by $\mathfrak{X}(P_0)$ the set of all noncontinuable solutions of the initial value problem (E)-(C). In general, however, the functions $x \in \mathfrak{X}(P_0)$ have not the same domain.

For example, the functions

$$x(t)=(1/\sqrt{1-t}, 0), \qquad 0 \leqq t<1,$$
$$y(t)=(1+(1/2)t+(3/8)t^2, (1/4)t^2), \qquad t \geqq 0$$

are clearly noncontinuable solutions of the initial value problem

$$\dot{x}_1=(1/2)x_1^3-[(15/4)+(10/4)t+(45/32)t^2+(27/64)t^3+(27/256)t^4]x_2, \qquad \dot{x}_2=\sqrt{x_2}$$
$$x_1(0)=1, \qquad x_2(0)=0.$$

On the other hand, by the previous Remark 1.1, it follows that the intersection of the domains of all $x \in \mathfrak{X}(P_0)$, i.e.

$$\text{Dom}\ \mathfrak{X}(P_0)=\bigcap\{\text{Dom}\ x : x \in \mathfrak{X}(P_0)\}$$

is a nontrivial interval.

In the following, our attention will be concentrated to the restrictions of the functions $x \in \mathfrak{X}(P_0)$ on some subinterval I of Dom $\mathfrak{X}(P_0)$. More precisely, we will be interesting in the set

$$\mathfrak{X}(I; P_0)=\{x \mid I : x \in \mathfrak{X}(P_0)\}.$$

Also we use the notations

$$\Omega^+=\{(t, x) \in \Omega : t>t_0\} \quad \text{and} \quad \Omega^-=\{(t, x) \in \Omega : t<t_0\}$$
$$\text{Dom}^+ x=[t_0, +\infty) \cap \text{Dom}\ x \ \text{ and } \ \text{Dom}^- x=(-\infty, t_0] \cap \text{Dom}\ x,\ x \in \mathfrak{X}(P_0) \ \text{ and}$$
$$\text{Dom}^+ \mathfrak{X}(P_0)=[t_0, +\infty) \cap \text{Dom}\ \mathfrak{X}(P_0) \ \text{ and } \ \text{Dom}^- \mathfrak{X}(P_0)=(-\infty, t_0] \cap \text{Dom}\ \mathfrak{X}(P_0).$$

We remark that the interval $\text{Dom}^+ x$ (resp. $\text{Dom}^- x$) is open at one of its endpoints if the set Ω^+ (resp. Ω^-) is open (and nonempty). Moreover, if Ω is open, then the interval Dom x is open too.

The next step is to prove that for the graph $G(x; P_0)$ of any solution $x \in \mathfrak{X}(P_0)$ we have

$$\text{Dist}\,(G(x; P_0), \partial\Omega)=0.$$

This is essentially contained in Prop. 1.1, Th. 1.2, and Cor. 1.1, the proofs of which follow the same lines of some known methods. However, these propositions generalize some results of Coppel [3, p. 15], Coddington and Levinson [2, p. 61], and Bebernes, Fulks and Meisters [1].

Also, the forms of these propositions will be such that to service the study of other problems of this paper.

Proposition 1.1. *Let W be a compact subset of Ω, $P_0 \in W^0$ and $x \in \mathfrak{X}(P_0)$. Then:*
α) *There exists a $t_1 \in \mathrm{Dom}^- x$ such that*

$$(t_1, x(t_1)) \in \partial W \quad and \quad G(x\,|\,[t_1, t_0]; P_0) \subseteqq W.$$

β) *There exists a $t_2 \in \mathrm{Dom}^+ x$ such that*

$$(t_2, x(t_2)) \in \partial W \quad and \quad G(x\,|\,[t_0, t_2]; P_0) \subseteqq W.$$

Theorem 1.2. *If the set Ω^+ (resp. Ω^-) is open and nonempty, then, for any solution $x \in \mathfrak{X}(P_0)$, every limit point of the graph $G(x\,|\,\mathrm{Dom}^+ x; P_0)$ (resp. $G(x\,|\,\mathrm{Dom}^- x; P_0)$) not lying to it, lies to the set*

$$\partial\Omega \cap [\{\sup \mathrm{Dom}\, x\} \times \boldsymbol{C}^n] \qquad (resp.\ \partial\Omega \cap [\{\inf \mathrm{Dom}\, x\} \times \boldsymbol{C}^n]).$$

Corollary 1.1. *Let $\Omega = I \times \boldsymbol{C}^n$, where I is an open interval of $\boldsymbol{R}$, and $x \in \mathfrak{X}(P_0)$. If $\mathrm{Dom}^+ x = [t_0, \alpha)$ and $\alpha \in I$, then*

$$\lim_{t \to \alpha} |x(t)| = +\infty.$$

Proof. We assume the contrary. Then, there exists a sequence $\{t_\nu\}$ in $[t_0, \alpha)$ with $\lim t_\nu = \alpha$ and such that the sequence $\{x(t_\nu)\}$ is bounded. Thus, for any limit point ξ of $\{x(t_\nu)\}$, we have $(\alpha, \xi) \in \Omega$. The point (α, ξ) is obviously a limit point of the graph $G(x\,|\,\mathrm{Dom}^+ x; P_0)$ and moreover does not belong to it. Therefore, by the previous theorem, we always have $(\alpha, \xi) \in \partial\Omega$, which contradicts to the fact that Ω is open.

The next theorem is a very useful result. In the case where the initial value problem (E)-(C) has a unique solution, this theorem reduces to the well-known theorem concerning, the continuous dependence of solutions on the initial values or "small perturbations" of the equation.

Theorem 1.3. *Let Ω be open and let $\{f_\nu\}$ be a sequence in the set $\mathrm{Car}_{\mathrm{loc}}(\Omega)$ such that*

$$\lim \int_I \sup_{x \in K} |f_\nu(t, x) - f(t, x)|\, dt = 0$$

for every compact restangle $I \times K \subseteqq \Omega$. Moreover, let $\{P_\nu\}$ be a sequence in Ω with

$\lim P_\nu = P_0$. *Suppose that, for any* $\nu \in \boldsymbol{N}$, *the function* x_ν *is a solution of the initial value problem*

$$x' = f_\nu(t, x), \quad x(t_\nu) = \xi_\nu, \quad P_\nu = (t_\nu, \xi_\nu).$$

Then, there exist a solution $x \in \mathfrak{X}(P_0)$ *and a subsequence* $\{x_{k_\nu}\}$ *of* $\{x_\nu\}$ *such that*

$$\lim x_{k_\nu} = x \text{ uniformly on compact subintervals of Dom } x,$$

that is, for any compact subinterval I of Dom x

i) $I \subseteqq \mathrm{Dom}\, x_{k_\nu}$ *for all large* $\nu \in \boldsymbol{N}$,

ii) $\lim x_{k_\nu} = x$ *uniformly on I.*

The proof of this theorem is omitted, since it can be carried out by similar arguments with those in [4, p. 14].

§ 2. Kneser's type properties of solutions

In this section we study topological and other properties of the set $\mathfrak{X}(P_0)$ of the solutions of the initial value problem (E)-(C). The fundamental Theorems 2.1 and 2.3 below have been proved (cf. [8], [11], [12]), for the general Volterra integral equation

$$x(t) = \varphi(t) + \int_0^t g(t, s, x(s))ds, \qquad t \in [0, c]$$

under general conditions on the function g like the Caratheodory's conditions. In the special case where

$$\varphi(t) = x_0 \quad \text{and} \quad g(t, s, x) = f(s, x), \ (t, s, x) \in [0, c]^2 \times \boldsymbol{C}^n$$

these conditions are reduced to

$$f \in \mathrm{Car}_{\mathrm{loc}}(\Omega), \qquad \text{where } \Omega = [0, c] \times \boldsymbol{C}^n.$$

Hence, Theorems 2.1 and 2.3 can be stated as corollaries of the main results of the papers mentioned above.

Theorem 2.1. *If the initial value problem* (E)-(C) *has a solution,* i.e. $\mathfrak{X}(P_0) \neq \emptyset$, *then for any* $r > 0$ *such that* $[t_0, t_0 + r] \subseteqq \mathrm{Dom}\, \mathfrak{X}(P_0)$, *the set* $\mathfrak{X}([t_0, t_0 + r]; P_0)$ *is a continuum in the space* $C([t_0, t_0 + r], \boldsymbol{C}^n)$.

Corollary 2.1. *If* $\mathfrak{X}(P_0) \neq \emptyset$ *and every solution* $x \in \mathfrak{X}(P_0)$ *is defined at some point* $t = \hat{t}$, *i.e.,* $\hat{t} \in \mathrm{Dom}\, \mathfrak{X}(P_0)$, *then the cross-section*

$$\mathfrak{X}(\hat{t}; P_0) = \{x(\hat{t}) : x \in \mathfrak{X}(P_0)\}$$

is a continuum in $\boldsymbol{C}^n$.

As an application of the previous theorem, we give a proof of the following density-convergence theorem.

Theorem 2.2. *If, for every $P \in \Omega$, the set $\mathfrak{X}(P_0)$ contains more than one solution then, for any solution $x \in \mathfrak{X}(P_0)$ and any compact interval $I \subseteqq \mathrm{Dom}\, x$, there exists a sequence $\{x_\nu\}$ in $\mathfrak{X}(P_0)$ with*

$$I \subseteqq \mathrm{Dom}\, x_\nu \text{ and } x_\nu \neq x \text{ on } I \text{ for every } \nu \in N$$

and such that

$$\lim x_\nu = x \text{ uniformly on } I.$$

Proof. Let $x \in \mathfrak{X}(P_0)$. We consider a propositional function P defined on $\mathrm{Dom}\, x$ by:

$P(\hat{t})$: There exists a sequence $\{x_\nu\}$ in the space $\mathfrak{X}([t_0, \hat{t}]; P_0)$ with $x_\nu \neq x$ on $[t_0, \hat{t}]$ for all $\nu \in N$ and $\lim x_\nu = x$ uniformly on $[t_0, \hat{t}]$.

If $t \in \mathrm{Dom}^+ \mathfrak{X}(P_0)$ then, by Theorem 2.1, the set $\mathfrak{X}([t_0, t]; P_0)$ is a continuum in $C([t_0, t], \boldsymbol{C}^n)$. On the other hand, every point of a connected set with more than one elements, like $\mathfrak{X}([t_0, t]; P_0)$, is a limit point of this set. Thus, for any $\hat{t} \in \mathrm{Dom}^+ \mathfrak{X}(P_0)$, $P(\hat{t})$ is true.

Let $t^* = \sup \{\hat{t} > t_0 : P(\hat{t})\}$. We suppose that $t^* \in \mathrm{Dom}\, x$, thus $G(x \mid [t_0, t^*]; P_0) \subseteqq \Omega^0$. Since the graph $G(x \mid [t_0, t^*]; P_0)$ is a compact subset of Ω^0, by Remark 1.1, there exists *a* $\delta > 0$ such that

$$[\hat{t}, \hat{t}+2\delta] \subseteqq \mathrm{Dom}^+ \mathfrak{X}(\hat{P}) \text{ for any } \hat{P} = (\hat{t}, x(\hat{t})) \in G(x \mid [t_0, t^*]; P_0).$$

We choose a $\hat{t} \in [t^* - \delta, t^*)$. Then there exists a sequence $\{\hat{x}_\nu\}$ in $\mathfrak{X}([\hat{t}, \hat{t}+2\delta]; \hat{P})$ such that

$$\hat{x}_\nu \neq x \text{ on } [\hat{t}, \hat{t}+2\delta] \text{ for every } \nu \in N \text{ and}$$
$$\lim \hat{x}_\nu = x \text{ uniformly on } [\hat{t}, \hat{t}+2\delta].$$

For every, $\nu \in N$, we consider the function x_ν defined by

$$x_\nu(t) = \begin{cases} x(t), & t_0 \leqq t \leqq \hat{t} \\ \hat{x}_\nu(t), & \hat{t} \leqq t \leqq \hat{t}+2\delta. \end{cases}$$

The sequence $\{x_\nu\}$ in the space $\mathfrak{X}([t_0, \hat{t}+2\delta]; P_0)$ satisfies:

$$x_\nu \neq x \text{ on the interval } [t_0, \hat{t}+2\delta] \text{ for every } \nu \in N,$$
$$\lim x_\nu = x \text{ uniformly on } [t_0, \hat{t}+2\delta].$$

Thus the proposition $P(\hat{t}+2\delta)$ is true. But, by the definition of t^*, this is a contradiction, since $t^*<t+2\delta$. Hence we have $t^*=\sup \operatorname{Dom} x$.

Theorem 2.3. *Let A be a continuum in Ω such that $\mathfrak{X}(P)\neq\emptyset$ for any $P\in A$. If all solutions $x\in\mathfrak{X}(A)$ are defined (at least) on some interval $[\alpha, \beta]$, then the set $\mathfrak{X}([\alpha, \beta], A)$ is a continuum in the space $C([\alpha, \beta], \boldsymbol{C}^n)$.*

Corollary 2.2. *Let A be a continuum in Ω with $pr_1A=\{t_0\}$. If for every point $(t_0, \xi)\in A$*

$$[t_0, t_0+\delta]\times\{x\in \boldsymbol{C}^n: |x-\xi|\leqq b\}\subseteqq\Omega,$$

where δ and b are positive numbers, then there exists a $r>0$ such that every solution $x\in\mathfrak{X}(A)$ is defined (at least) on the interval $[t_0, t_0+r]$. Moreover, for any such r, the set $\mathfrak{X}([t_0, t_0+r]; A)$ is a continuum in $C([t_0, t_0+r]; \boldsymbol{C}^n)$.

Proof. The assumptions of the existence theorem of Caratheodory are satisfied for any point of A and so, by the Remark 1.1, there exists a $r>0$ with

$$[t_0, t_0+r]\subseteqq\operatorname{Dom} x \qquad \text{for every } x\in\mathfrak{X}(A).$$

Furthermore, by the previous theorem, for any such r the set $\mathfrak{X}([t_0, t_0+r]; A)$ is a continuum in the space $C([t_0, t_0+r], \boldsymbol{C}^n)$.

In the following, we are going to study the case where

$$\Omega=[t_0, +\infty)\times\boldsymbol{C}^n.$$

In this case the intersection

$$\operatorname{Dom}\mathfrak{X}(P_0)=\cap\{\operatorname{Dom} x: x\in\mathfrak{X}(P_0)\}$$

is a right open interval, i.e. $\operatorname{Dom}\mathfrak{X}(P_0)=[t_0, \alpha)$ for some α. If $\alpha=+\infty$, then of course all solutions in $\mathfrak{X}(P_0)$ have the interval $[t_0, +\infty)$ as common domain of definition and the study of topological properties of $\mathfrak{X}(P_0)$ is included in the results which has been already given. So, in the following, we always suppose that $\alpha<+\infty$.

We remark, now, that there exists a solution $x\in\mathfrak{X}(P_0)$ such that

$$\operatorname{Dom} x=[t_0, \alpha) \quad \text{and} \quad \lim_{t\to\alpha}|x(t)|=+\infty.$$

Indeed, if we suppose that $\alpha\in\operatorname{Dom}\mathfrak{X}(P_0)$, then, by Corollary 2.1, $\mathfrak{X}(\alpha; P_0)$ is a continuum in $\boldsymbol{C}^n$. Thus, by the previous corollary, there exists a $r>0$ such that

$$[t_0, \alpha+r]\subseteqq\operatorname{Dom} x \qquad \text{for every } x\in\mathfrak{X}(P_0)\subseteqq\mathfrak{X}(\mathfrak{X}(\alpha; P_0))$$

which is obviously a contradiction. Therefore, by Corollary 1.1, for some solution $x\in\mathfrak{X}(P_0)$ with $\operatorname{Dom} x=[t_0, \alpha)$, we have

$$\lim_{t \to \alpha} |x(t)| = +\infty.$$

Our purpose, in the following is to study the topological structure of the two sets

$$\mathfrak{X}_F(P_0) = \{x \in \mathfrak{X}(P_0) : \alpha \in \operatorname{Dom} x\}$$
$$\mathfrak{X}_\infty(P_0) = \{x \in \mathfrak{X}(P_0) : \lim_{t \to \alpha} |x(t)| = +\infty\}$$

and especially of the first set $\mathfrak{X}_F(P_0)$. These sets constitute a partition of $\mathfrak{X}(P_0)$, i.e.

$$\mathfrak{X}_\infty(P_0) \cup \mathfrak{X}_F(P_0) = \mathfrak{X}(P_0) \quad \text{and} \quad \mathfrak{X}_\infty(P_0) \cap \mathfrak{X}_F(P_0) = \emptyset$$

and moreover $\mathfrak{X}_\infty(P_0)$ is always nonvoid.

The given results are new even in the particular case where the function f is continuous.

Definition 2.1. *A subset Φ of the space $C([t_0, \alpha), \boldsymbol{C}^n)$ is said to be absolutely equiconvergent to $+\infty$ at the point α if, and only if,*

$$(\forall h > 0)(\exists \varepsilon > 0)(\forall \varphi \in \Phi)(\forall t \in [\alpha - \varepsilon, \alpha]) \, |\varphi(t)| > h.$$

Theorem 2.4. *If $\Omega = [t_0, +\infty) \times \boldsymbol{C}^n$ and* $\operatorname{Dom} \mathfrak{X}(P_0) = [t_0, \alpha)$, *then:*

i) *The set $\mathfrak{X}_\infty(P_0)$ is absolutely equiconvergent to $+\infty$ at the point α.*

ii) *The set $\mathfrak{X}_\infty(P_0)$ is a compact subset of the space $C([t_0, \alpha), \boldsymbol{C}^n)$ and thus, for every $\hat{t} \in [t_0, \alpha)$, the set*

$$\mathfrak{X}_\infty(\hat{t}; P_0) \equiv \{x(\hat{t}) : x \in \mathfrak{X}_\infty(P_0)\}$$

is a compact subset in $\boldsymbol{C}^n$.

Proof. Suppose that the set $\mathfrak{X}_\infty(P_0)$ is not absolutely equiconvergent to $+\infty$ at the point α. Then there exist a sequence $\{t_\nu\}$ of points of $\boldsymbol{R}$, a sequence $\{x_\nu\}$ of solutions of $\mathfrak{X}_\infty(P_0)$ and a positive number $\hat{h}$ such that

$$\lim t_\nu = \alpha \quad \text{and} \quad |x_\nu(t_\nu)| \leqq \hat{h} \quad \text{for any } \nu \in \boldsymbol{N}.$$

By the Bolzano-Weierstrass theorem, we can assume, without loss of generality, that the sequence $\{x_\nu(t_\nu)\}$ is convergent. So, if we put

$$P = (\alpha, \lim x_\nu(t_\nu)) \quad \text{and} \quad P_\nu = (t_\nu, x_\nu(t_\nu)), \quad \nu = 1, 2, \cdots$$

then

$$\lim P_\nu = P \quad \text{and} \quad x_\nu \in \mathfrak{X}(P_0) \cap \mathfrak{X}(P_\nu), \quad \nu = 1, 2, \cdots.$$

Thus, by Theorem 1.3, there exist a solution $x \in \mathfrak{X}(P)$, a subsequence $\{x_{k_\nu}\}$ of $\{x_\nu\}$ and a positive number δ such that

$$\lim x_{k_\nu}=x \text{ uniformly on } [\alpha-\delta, \alpha].$$

Hence, we must have

$$\alpha \in \operatorname{Dom} x_{k_\nu} \qquad \text{for all large } \nu \in \boldsymbol{N}$$

which contradicts the fact that $x_{k_\nu} \in \mathfrak{X}_\infty(P_0)$ for every $\nu \in \boldsymbol{N}$.

ii) Let $\{x_\nu\}$ be a sequence in $\mathfrak{X}_\infty(P_0)$. By Theorem 1.3, there exist a solution $x \in \mathfrak{X}(P_0)$ and a subsequence $\{x_{k_\nu}\}$ of $\{x_\nu\}$ such that

$$\lim x_{k_\nu}=x \text{ uniformly on compact subsets of } \operatorname{Dom} x.$$

Obviously, $[t_0, \alpha)\subseteqq \operatorname{Dom} x$. If $\alpha \in \operatorname{Dom} x$, then we apply Theorem 1.3 to obtain $\alpha \in \operatorname{Dom} x_{k_\nu}$ for all large $\nu \in \boldsymbol{N}$. This is a contradiction, since $x_{k_\nu} \in \mathfrak{X}_\infty(P_0)$ for every $\alpha \in \boldsymbol{N}$. Thus $\alpha \notin \operatorname{Dom} x$, i.e. $x \in \mathfrak{X}_\infty(P_0)$.

We come back now to the set $\mathfrak{X}_F(P_0)$. We need the following lemma.

Lemma 2.1. *If H is a compact subset of $\mathfrak{X}_F(\alpha; P_0)$, then the set*

$$\mathfrak{X}([t_0, \alpha]; P_0)\cap \mathfrak{X}([t_0, \alpha]; H)$$

is a compact subset of the space $C([t_0, \alpha], \boldsymbol{C}^n)$.

Proof. Let $\{x_\nu\}$ be a sequence in $\mathfrak{X}([t_0, \alpha]; P_0)\cap \mathfrak{X}([t_0, \alpha]; H)$ and let $\{x_\nu(\alpha)\}$ be the corresponding sequence in H. Since H is compact, by restricting our consideration to a suitable subsequence, we may suppose that the sequence $\{x_\nu(\alpha)\}$ converges, that is

$$\lim x_\nu(\alpha)=\xi, \qquad \xi \in H.$$

By Theorem 1.3, there exist a subsequence $\{x_{k_\nu}\}$ of $\{x_\nu\}$, a solution $y \in \mathfrak{X}(H)$ and a positive number δ with $\delta<\alpha-t_0$ such that $y(\alpha)=\xi$ and

$$\lim x_{k_\nu}=y \quad \text{uniformly on} \quad [\alpha-\delta, \alpha].$$

On the other hand, every solution $x \in \mathfrak{X}(P_0)$ is defined on the interval $[t_0, \alpha-\delta]$ and moreover $\mathfrak{X}([t_0, \alpha-\delta]; P_0)$ is a continuum in the space $C([t_0, \alpha-\delta], \boldsymbol{C}^n)$. Hence, without loss of generality, we may suppose that there exists a solution $z \in \mathfrak{X}(P_0)$ such that

$$\lim x_{k_\nu}=z \quad \text{uniformly on} \quad [t_0, \alpha-\delta].$$

In particular, at the point $t=\alpha-\delta$ we have

$$z(\alpha-\delta)=\lim x_{k_\nu}(\alpha-\delta)=y(\alpha-\delta).$$

Therefore, the limit function x, which is defined by

$$x(t)=\begin{cases} z(t), & t_0\leqq t\leqq \alpha-\delta \\ y(t), & \alpha-\delta\leqq t\leqq \alpha, \end{cases}$$

is an element of the set $\mathfrak{X}([t_0,\alpha]; P_0)\cap \mathfrak{X}([t_0,\alpha]; H)$.

Theorem 2.5 below is the main result of this paper. For the sake of brevity, we introduce some notations. If $A\subseteq \Omega$ and I is an interval, we put

$$\mathfrak{X}(P_0, A)=\mathfrak{X}(P_0)\cap \mathfrak{X}(A),$$
$$\mathfrak{X}(I; P_0, A)=\{x\,|\,I\colon x\in \mathfrak{X}(P_0, A)\}.$$

In particular, if $I=\{\alpha\}$, we denote by

$$\mathfrak{X}(\alpha; P_0, A) \quad \text{the set} \quad \mathfrak{X}(\{\alpha\}; P_0, A)$$

while, if $A=\{P\}$, by

$\mathfrak{X}(P_0, P)$ and $\mathfrak{X}(I; P_0, P)$ the sets $\mathfrak{X}(P_0, \{P\})$ and $\mathfrak{X}(I; P_0, \{P\})$ respectively.

Also, we define the graph $G(\Phi)$ of a set of functions Φ by

$$G(\Phi)=\bigcup\{G(\varphi)\colon \varphi\in\Phi\}.$$

Theorem 2.5. *If the set $\mathfrak{X}_F(\alpha; P_0)$ is not empty, then it is not a bounded subset of C^n and moreover it has not isolated points, i.e.*

$$\mathfrak{X}_F(\alpha; P_0)=[\mathfrak{X}_F(\alpha; P_0)]^d.$$

Proof. We suppose that the set $\mathfrak{X}_F(\alpha; P_0)$ is bounded. Since it is obviously closed, it is also compact. Thus, by the previous lemma, the set $\mathfrak{X}_F([t_0,\alpha]; P_0)$ is compact and hence it is a bounded subset of the space $C([t_0,\alpha], C^n)$. Thus, there exists a positive number $\hat{h}$ such that

$$(1) \qquad (\forall x\in \mathfrak{X}_F([t_0,\alpha]; P_0))(\forall t\in[t_0,\alpha])\,|x(t)|<\hat{h}.$$

On the other hand, by Theorem 2.4 take it out, for every $t\in[t_0,\alpha)$, the set $\mathfrak{X}_\infty([t_0,t]; P_0)$ is also a compact subset of $C([t_0,t]; C^n)$ and moreover there exists an $\varepsilon>0$ such that

$$(\forall x\in \mathfrak{X}_\infty(P_0))(\forall t\in[\alpha-\varepsilon,\alpha))\,|x(t)|\geqq 2\hat{h}.$$

Thus, by (1), for every $\hat{t}\in[\alpha-\varepsilon,\alpha)$, we have

$$\mathfrak{X}_F(\hat{t}; P_0)\cap \mathfrak{X}_\infty(\hat{t}; P_0)=\emptyset \quad\text{and}\quad \mathfrak{X}_F(\hat{t}; P_0)\cup \mathfrak{X}_\infty(\hat{t}; P_0)=\mathfrak{X}(\hat{t}; P_0).$$

This is a contradiction, since the sets $\mathfrak{X}_F(\hat{t}; P_0)$ and $\mathfrak{X}_\infty(\hat{t}; P_0)$ are clearly compact and nonvoid and the set $\mathfrak{X}(\hat{t}; P_0)$ is a continuum.

To obtain $\mathfrak{X}_F(\alpha; P_0)=[\mathfrak{X}_F(\alpha; P_0)]^d$ we consider a point $\xi \in \mathfrak{X}_F(\alpha; P_0)$. It is enough to prove the existence of a sequence $\{\xi_\nu\}$ in $\mathfrak{X}_F(\alpha; P_0)$ such that

$$\lim \xi_\nu = \xi \quad \text{and} \quad \xi_\nu \neq \xi \quad \text{for every } \nu \in \boldsymbol{N}.$$

We suppose that this fails to be true. Then there exists a positive number ε such that

$$(2) \qquad |x(\alpha)-\xi| \geqq \varepsilon \qquad \text{for every } x \in \mathfrak{X}_F(P_0)\backslash\mathfrak{X}(P)$$

where $P=(\alpha, \xi)$. Lemma 2.1 ensures that the set $\mathfrak{X}([t_0, \alpha]; P_0, P)$ is a compact subset of the space $C([t_0, \alpha], \boldsymbol{C}^n)$. Thus, the graph $G(\mathfrak{X}([t_0, \alpha]; P_0, P))$ is also a compact subset of Ω. Therefore, by Remark 1.1, there exists a $\delta_1>0$ such that all solutions $x \in \mathfrak{X}(Q)$, $Q=(\tau, \eta)$ with $Q \in G(\mathfrak{X}([t_0, \alpha]; P_0, P))$ are defined on the interval $[\tau, \tau+\delta_1]$. In particular, if $Q \in G(\mathfrak{X}([\alpha-\delta_1, \alpha]; P_0, P))$, then it is clear that $\alpha \in \operatorname{Dom} \mathfrak{X}(Q)$.

Let now $\{x \in \boldsymbol{C}^n: |x| \leqq h\}$ be a neighborhood of the set $G(\mathfrak{X}([t_0, \alpha]; P_0, P))$ that is

$$(3) \qquad |x(t)|<h \qquad \text{for every } t \in [t_0, \alpha] \text{ and every } x \in \mathfrak{X}(P_0, P).$$

On the other hand, by virtue of Theorem 2.4, there exists a $\delta_2>0$ such that

$$|x(t)|>2h \qquad \text{for every } t \in [\alpha-\delta_2, \alpha) \text{ and every } x \in \mathfrak{X}_\infty(P_0).$$

Thus, if $\delta=\min\{\delta_1, \delta_2\}$, then, by (3), we have

$$(4) \qquad \mathfrak{X}_\infty(t; P_0) \cap \mathfrak{X}(t; P_0, P)=\emptyset \qquad \text{for every } t \in [\alpha-\delta, \alpha).$$

We consider now a point $\bar{t} \in [\alpha-\delta, \alpha)$. Then the set

$$\mathfrak{X}([t_0, \bar{t}]; P_0)\backslash\mathfrak{X}([t_0, \bar{t}]; P_0, P)$$

is not compact. In fact, in the opposite case this set and the compact set $\mathfrak{X}([t_0, \bar{t}]; P_0, P)$ constitute a partition of the continuum $\mathfrak{X}([t_0, \bar{t}]; P_0)$. This fact is of course a contradiction. So, the set $\mathfrak{X}([t_0, \bar{t}]; P_0)\backslash\mathfrak{X}([t_0, \bar{t}]; P_0, P)$ has a limit point in $\mathfrak{X}([t_0, \bar{t}]; P_0, P)$, i.e. there exist a sequence $\{x_\nu\}$ in $\mathfrak{X}([t_0, \bar{t}]; P_0)\backslash\mathfrak{X}([t_0, \bar{t}]; P_0, P)$ and a solution $x \in \mathfrak{X}([t_0, \bar{t}]; P_0, P)$ such that

$$\lim x_\nu = x \quad \text{uniformly on} \quad [t_0, \bar{t}].$$

Furthermore the set $\mathfrak{X}_\infty([t_0, \bar{t}]; P_0)$ contains at most a finite number of the terms of the sequence $\{x_\nu\}$. This is true because of the fact that the set $\mathfrak{X}_\infty([t_0, \bar{t}]; P_0)$ is compact and $x \notin \mathfrak{X}_\infty([t_0, \bar{t}]; P_0)$, by virtue of (4). Therefore we can assume, without loss of generality, that $x_\nu \in \mathfrak{X}_F([t_0, \bar{t}]; P_0)\backslash\mathfrak{X}([t_0, \bar{t}]; P_0, P)$ for every $\nu \in \boldsymbol{N}$, which means that

$$(5) \qquad x_\nu(\alpha) \neq \xi \qquad \text{for every } \nu \in \boldsymbol{N}.$$

Finally, if

$$(\bar{t}, x(\bar{t}))=\bar{P} \quad \text{and} \quad P_\nu=(\bar{t}, x_\nu(\bar{t})), \quad \nu=1, 2, \cdots$$

then we have $\lim P_\nu=\bar{P}$.

By virtue of Theorem 1.3, there exist a subsequence $\{x_{k_\nu}\}$ of $\{x_\nu\}$ and a solution $\hat{x} \in \mathfrak{X}(\bar{P})$ such that

$$\lim x_{k_\nu}=\hat{x} \quad \text{uniformly on} \quad [\bar{t}, \alpha].$$

Moreover, we have $\hat{x}(\alpha)=\xi$, since otherwise the continuum $\mathfrak{X}(\alpha; \bar{P})$ would connect the point ξ with the set $\mathfrak{X}_F(\alpha; P_0)\backslash\{\xi\}$. This obviously contradicts (2). Thus $\lim x_{k_\nu}(\alpha)=\xi$ which, in view of (5), also contradicts (2).

Example. We consider the initial value problem

$$(\mathrm{E}_1) \qquad \dot{x}_1=x_1^2x_2, \quad \dot{x}_2=\sqrt{x_2}, \quad \dot{x}_3=\sqrt{x_3}, \quad \dot{x}_4=x_4^2x_3$$

$$(\mathrm{C}_1) \qquad (x_1(0), x_2(0), x_3(0), x_4(0))=(1, 0, 0, 1)=P_0.$$

It is easy to verify that the set of solutions $\mathfrak{X}(P_0)$ consists of the two-parameter families

$$x_{\lambda\mu}(t)=\begin{cases}(1, 0, 0, 1), & 0\leqq t<\lambda\leqq\mu\\ (\varphi_\lambda(t), \psi_\lambda(t), 0, 1), & \lambda\leqq t<\mu\leqq\lambda+\sqrt[3]{12}\\ (\varphi_\lambda(t), \psi_\lambda(t), \psi_\mu(t), \varphi_\mu(t)), & \mu\leqq t<\lambda+\sqrt[3]{12}\end{cases}$$

$$y_{\mu\lambda}(t)=\begin{cases}(1, 0, 0, 1), & 0\leqq t<\mu\leqq\lambda\\ (1, 0, \psi_\mu(t), \varphi_\mu(t)), & \mu\leqq t<\lambda\leqq\mu+\sqrt[3]{12}\\ (\varphi_\lambda(t), \psi_\lambda(t), \psi_\mu(t), \varphi_\mu(t)), & \lambda\leqq t<\mu+\sqrt[3]{12}\end{cases}$$

where for any $\nu \in [0, +\infty]$

$$\varphi_\nu(t)=\frac{12}{12-(t-\nu)^3}, \; 0\leqq t<\nu+\sqrt[3]{12} \quad \text{and} \quad \psi_\nu(t)=\frac{(t-\nu)^2}{4}, \; t \in \boldsymbol{R}.$$

If we set

$$\mathfrak{X}_{\lambda\mu}(P_0)=\{x_{\lambda\mu}\colon 0\leqq\lambda\leqq\mu\leqq\lambda+\sqrt[3]{12}\} \quad \text{and} \quad \Psi_{\mu\lambda}(P_0)=\{y_{\mu\lambda}\colon 0\leqq\mu\leqq\lambda\leqq\mu+\sqrt[3]{12}\},$$

then we obviously have

$$\mathfrak{X}_\infty(P_0)=\mathfrak{X}_{o\mu}(P_0)\cup\Psi_{o\lambda}(P_0)$$

and

$$\mathfrak{X}_F(P_0)=[\mathfrak{X}_{\lambda\mu}(P_0)-\mathfrak{X}_{o\mu}(P_0)]\cup[\Psi_{\mu\lambda}(P_0)-\Psi_{o\lambda}(P_0)].$$

Consequently, the set $\mathfrak{X}_F(\alpha; P_0)$ is the union of two disjoint and connected subsets in $\boldsymbol{R}^4$ and the set $\mathfrak{X}_\infty(P_0)$ is a compact subset of $C([0, \sqrt[3]{12}), \boldsymbol{R})$.

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