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Corrigendum and Addendum: Singular Cauchy Problems and Asymptotic Behaviour for a Class of n-th Order Differential Equations

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We continue the numbering of sections and formulas in [1].

5. Corrigendum

In [1], after theorem 4.3, the author made the naive claim that "all the properties stated in theorems B and C are also true for the nonlinear case taking into account conditions (4.2), (4.3) instead of (3.2), (3.14)". This is indeed false and we shall now point out the striking differences between the three cases: sublinear, linear and superlinear perturbations of the operator L.

In the next section we shall return to the linear case and give another extension of th. A which, unlike theorems B and C, applies to any operator L.

For brevity we only list results concerning the first case examined in [1]: i.e. $1/p_i \notin L^1 \forall i$. The second case is treated similarly. Throughout the present paper we shall study relationships between the following statements:

S_k, $k \in \{0, 1, \cdots, n-1\}$: *"Equation* $(**)$ *has a solution* u_k *satisfying property* 7), *or equivalently* 8), *of th B".*

Theorem 5.1 below is a generalization of that part of th. B concerning operators with property (2.6) while th. 5.2 deals with operators satisfying (2.7).

Theorem 5.1. *Consider equation* $(**)$ *with* f *satisfying assumptions* (4.1), L *satisfying* (2.6) *and* $1/p_i \notin L^1 \forall i$. *We then have three separate cases:*

I) *If* $0<\alpha<1$ *then* $S_0 \Rightarrow S_{n-1}$.

II) *If* $\alpha=1$ *then for each fixed* $k \in \{0, 1, \cdots, n-1\}$ *we have* $S_k \Leftrightarrow S_{n-k-1}$.

III) *If* $\alpha>1$ *then* $S_{n-1} \Rightarrow S_0$.

Proof. Condition (2.6) implies

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(5.1) $$a_1 M_{n-k-1}(t) \leq P_k(t) \leq a_2 M_{n-k-1}(t), \qquad k=0, 1, \cdots, n-1$$

where a_1, a_2 are suitable positive constants. Hence the two conditions (4.2) for $k=0$ and $k=n-1$ are respectively equivalent to

$$\int^{+\infty} |qM_0| < +\infty \quad \text{and} \quad \int^{+\infty} |q|\,(M_0)^\alpha < +\infty .$$

Since $\lim_{t\to+\infty} M_0(t) = +\infty$, cases I, II, III follow trivially from theorems 4.1, 4.2. In case II the assertions pertaining to the values $k \neq 0$, $n-1$ may be analogously inferred from theorems 4.1, 4.2 since (4.2) reduces to (3.2) and since two-sided estimates, analogous to (5.1), may be obtained between $P_k M_k$ and $P_{n-k-1} M_{n-k-1}$. q.e.d.

Theorem 5.2. *Assuming the same hypotheses as for theorem* 5.1 *and assuming that L satisfies* (2.7), *we again have three different situations*:

I) *If* $0 < \alpha < 1$ *then the following are equivalent properties*:

I)$_1$: S_k *holds for some* $k \in \{0, 1, \cdots, n-1\}$.

I)$_2$: S_h *holds for each* $h \in \{k, \cdots, n-1\}$.

I)$_3$: $\int^{+\infty} |q(t)| \left(\int_T^t 1/p\right)^{n-1+(\alpha-1)k} dt < +\infty$.

II) *If* $\alpha = 1$ *then property* I)$_1$ *is equivalent to each of the following*:

II)$_2$: S_k *holds for each* $k \in \{0, 1, \cdots, n-1\}$.

II)$_3$: $\int^{+\infty} |q(t)| \left(\int_T^t 1/p\right)^{n-1} dt < +\infty$.

III) *If* $\alpha > 1$ *then each of the properties* I$_1$ *and* I)$_3$ *is equivalent to the following*: *"S_h holds for each $h \in \{0, \cdots, k\}$".*

Note that th. 5.2 is a considerable improvement on Waltman's results in [2] for the case $Lu \equiv u^{(n)}$, in which case I)$_3$ is equivalent to

$$\int^{+\infty} |q(t)|\, t^{n-1+(\alpha-1)k} dt < +\infty .$$

Proof. From formulas (2.1), (2.5) and assumption (2.7) it follows that there exist two suitable positive constants a_1, a_2 such that for t sufficiently large and for each $k=0, \cdots, n-1$ we have

$$a_1 P_k(t) \leq \left(\int_T^t 1/p\right)^k \leq a_2 P_k(t); \qquad a_1 M_k(t) \leq P_{n-k-1}(t) \leq a_2 M_k(t).$$

Hence condition (4.2) is equivalent to I)$_3$. Our conclusions simply follow from I)$_3$ and theorems 4.1, 4.2. q.e.d.

From the foregoing theorems, criteria for asymptotic equivalence may be inferred which improve corollary 3.A both in the linear and nonlinear cases for those special classes of operators satisfying (2.6) or (2.7). The following theorem is a particularly significant corollary of th. 5.2. At the end of the next section, an improvement of corollary 3.A in the linear case valid for any operator will be found.

Theorem 5.3. *Assuming the hypotheses of theorem* 5.2 *the two equations* $(**)$ *and* $Lw=0$ *are asymptotically equivalent iff any one of the following properties holds*:

1) $(**)$ *has a solution asymptotically equivalent* (*as* $t\to+\infty$) *to a nontrivial solution of* $Lw=0$ *with the least order of growth* (*if* $0<\alpha<1$); *with any order of growth* (*if* $\alpha=1$); *with the greatest order of growth* (*if* $\alpha>1$).

2) $$\int^{+\infty}|q(t)|\left(\int_T^t 1/p\right)^{n-1}dt<+\infty, \qquad \text{if } \alpha\leq 1$$

or $$\int^{+\infty}|q(t)|\left(\int_T^t 1/p\right)^{\alpha(n-1)}dt<+\infty, \qquad \text{if } \alpha>1.$$

Proof. It must first be pointed out that asymptotic equivalence between the two equations is meant in the sense of property 9) of th. B if $(**)$ is linear, and in the sense of property (P), [1, p. 210], when it is not. Theorem 5.3 now follows from th. 5.2 and the proof sketched in corollary 3.A. q.e.d.

We end this section by indicating three misprints in [1]: a) p. 194, 1. ↓ 6 and 1. ↓ 8: $V(t)$ reads $v(t)$; b) p. 199, 1. ↓ 13: $|q(t)|$ reads $|q(\tau)|$; c) p. 201, 1. ↑ 7: $d\tau/P_{n-1}(\tau)$ reads $d\tau/p_{n-1}(\tau)$.

6. Another extension of theorem A in the linear case

Theorems B and C present two different extensions of th. A concerning n-th order operators satisfying (2.6) or (2.7). However a careful reading of corollaries 3.A, 3.B shows that there is an extension worth-noting which, though weaker than those just quoted, is valid for any n-th order operator.

Theorem 6.1. *If* $n\geq 3$ *then for equation* $(*)$, *or* $(**)$ *with* $\alpha=1$, *the following are equivalent properties*:

1) S_k *holds for every* $k\in\{1,\cdots,n-2\}$,

2) S_k *holds for every* $k\in\{0,\cdots,n-1\}$.

Proof. By looking at formulas (2.1), (2.5) it is quite elementary to verify that for each $t\geq T$ we have the inequalities (whatever the p_i's might be):

$$P_0(t)M_0(t)\leq P_1(t)M_1(t); \qquad P_{n-1}(t)M_{n-1}(t)\leq P_{n-2}(t)M_{n-2}(t).$$

From theorems 3.1, 3.2 it follows that $S_1\Rightarrow S_0$ and $S_{n-2}\Rightarrow S_{n-1}$, and in particular our theorem. q.e.d.

By combining the results of sectons 5 and 6 other corollaries might be obtained which are of special interest in the case of third and fourth order equations.

References

[1] Granata, A., Singular Cauchy problems and asymptotic behaviour for a class of n-th order differential equations, Funkcial. Ekvac., **20** (1977), 193–212.

[2] Waltman, P., On the asymptotic behavior of solutions of an n-th order equation, Monatsh. Math., **69** (1965), 427–430.

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