Funkcialaj Ekvacioj, **22** (1978), 1-49

Local Theory of Fuchsian Systems with Certain Discrete Monodromy Groups III

By

Masaaki YOSHIDA and Shuzo HATTORI

(Kobe University and Miki High School, Japan)

§ 0. Introduction and preliminary

This is a continuation of the papers [17] and [18] with the same title. Let D be the domain in $\boldsymbol{C}^2$:

$$D=\{(z, u) \in \boldsymbol{C}^2 \mid \operatorname{Im} z-|u|^2>0\}.$$

The domain D can be regarded as a domain of the complex projective plane $\boldsymbol{P}^2(\boldsymbol{C})$ by the natural embedding of $\boldsymbol{C}^2$ into $\boldsymbol{P}^2(\boldsymbol{C})$. If $v=\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is a homogeneous coordinate of $\boldsymbol{P}^2(\boldsymbol{C})$ related to (z, u) by $z=v_1/v_3$, $u=v_2/v_3$ and if

$$H=\begin{bmatrix} 0 & 0 & i \\ 0 & -2 & 0 \\ -i & 0 & 0 \end{bmatrix},$$

then the domain is expressible as

$$D=\{v \in \boldsymbol{P}^2(\boldsymbol{C}) \mid v^* H v>0\},$$

where v^* is the transpose of the complex conjugate of v. We denote by $\bar{D}$ and ∂D the closure and the boundary of D in $\boldsymbol{P}^2(\boldsymbol{C})$ respectively. Then $\bar{D}$ meets the line at infinity $v_3=0$ at the unique point $P=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ on ∂D. A complex projective line passing through P is given, in terms of (z, u), by

$$u=u_0$$

where u_0 is a constant.

The complex analytic automorphism group Aut (D) of D is identified with the guotient group of the subgroup of $GL(3, \boldsymbol{C})$:

$$\{X \in GL(3, \boldsymbol{C}) \mid X^* H X=k H \text { for some } k>0\}$$

by the multiplicative group $C^{\times}$ of C. For the sake of simplicity we express an element of Aut (D) by a suitable matrix belonging to the corresponding rest class. Under this convention, an element X of Aut (D) keeps the point P fixed in a geodesic sense (for the definition, see [17]) if and only if X is of the form

$$[\mu, a, r]=\begin{bmatrix} 1 & 2i\mu\bar{a} & r+i|a|^2 \\ 0 & \mu & a \\ 0 & 0 & 1 \end{bmatrix}$$

where $\mu. b \in C, |\mu|=1$ and $r \in R$ ([17]; Proposition 3.1). We denote by G the subgroup of Aut (D) consisting of elements X which leave P fixed.

In our preceding paper [17], we treated a special kind of discrete subgroup of G which is generated by two unitary reflections (Definition 1.8) of order 4. In this paper, we list up all the discrete subgroups $\{\Gamma\}$ of G of locally finite volume (Theorem 1.6) and find all the groups $\{\Gamma\}' \subset \{\Gamma\}$ which are generated by unitary reflections (Theorem 1.9). Next, for each, $\Gamma \in \{\Gamma\}$, we construct the nonsingular model of the quotientspace added by the point P: $D/\Gamma \cup \{P\}$ (Theorem 2.1, 3.1, 4.1) and pick up all the subgroups belonging to $\{\Gamma\}$ such that the space $D/\Gamma \cup \{P\}$ is nonsingular (Corollary 2.4, 3.4, 4.4). Combining Corollary 2.4, 3.4, 4.4 and Theorem 1.9, we have the following theorem.

Theorem 0.1.[1)] *Let Γ be a discrete subgroup of G of locally finite volume. Then $D/\Gamma \cup \{P\}$ is nonsingular if and only if Γ is generated by unitary reflections.*

Then, for each $\Gamma \in \{\Gamma\}'$, we choose a system of parameters x, y around P of the space $D/\Gamma \cup \{P\}$ so that the mapping $D \to D/\Gamma \cup \{P\}$ is expressible globally by the use of well known functions (Proposition 2.7, 4.8). Finally, we construct a completely integrable system of differential equations (E_Γ), for each $\Gamma \in \{\Gamma\}'$, of the form

$$\begin{cases} \dfrac{\partial^2\xi}{\partial x^2}=p_{11}^1(x,y)\dfrac{\partial\xi}{\partial x}+p_{11}^2(x,y)\dfrac{\partial\xi}{\partial y}+p_{11}^0(x,y)\xi \\ \dfrac{\partial^2\xi}{\partial x\partial y}=-p_{22}^2(x,y)\dfrac{\partial\xi}{\partial x}-p_{11}^1(x,y)\dfrac{\partial\xi}{\partial y}+p_{12}^0(x,y)\xi \\ \dfrac{\partial^2\xi}{\partial y^2}=p_{22}^1(x,y)\dfrac{\partial\xi}{\partial x}+p_{22}^2(x,y)\dfrac{\partial\xi}{\partial y}+p_{22}^0(x,y)\xi \end{cases}$$

with regular singularity at the origin such that the group of linear fractional transformations derived from the local monodromy of (E_Γ) (which is called the projective monodromy group of (E_Γ)) coincides with Γ (Theorem 2.8, 3.6, 4.9). We also prove that the system (E_Γ) is Fuchsian, i.e., it is defined in $P^2(C)$ and regular singular at every point. The monodromy representation of the system (E_Γ) is given

1) Compare this theorem with [18]; Corollary 1.2.

as follows. Let Z_Γ be the singular locus of the system (E_Γ) in the finite plane $\boldsymbol{C}^2$ in $\boldsymbol{P}^2(\boldsymbol{C})$ and l be a line in $\boldsymbol{C}^2$ such that l intersects normally with Z_Γ and that the intersection number of l and Z_Γ is minimal. The number of $\{q_j\}:=l\cap Z_\Gamma$, which is the minimal degree of the defining equation of Z_Γ, is equal to the number of generating unitary reflections $\{h_j\}$ of Γ given in Theorem 1.9. Then, by a suitable permutation of q_j if necessary, we can find the roops α_j around q_j in l (which form a system of generators of the fundamental group $\pi_1(\boldsymbol{C}^2-Z_\Gamma)$ of $\boldsymbol{C}^2-Z_\Gamma$) so that the monodromy representation $\pi_1(\boldsymbol{C}^2-Z_\Gamma)\to\Gamma$ may be given by $\alpha_j\to h_j$.

In concluding this section, we shall briefly review the procedure of calculating the coefficients of (E_Γ). Since the system (E_Γ) is assumed to be completely integrable, we can express all the coefficients of (E_Γ) by the four coefficients $p_{11}^1, p_{11}^2, p_{22}^1$ and p_{22}^2 as follows:

$$p_{11}^0=-\frac{\partial p_{11}^1}{\partial x}-\frac{\partial p_{11}^2}{\partial y}+2(p_{11}^1)^2-2p_{22}^2p_{11}^2,$$

$$p_{12}^0=\frac{\partial p_{22}^2}{\partial x}+\frac{\partial p_{11}^1}{\partial y}+p_{22}^1p_{11}^2-p_{11}^1p_{22}^2,$$

$$p_{22}^0=-\frac{\partial p_{22}^2}{\partial y}-\frac{\partial p_{22}^1}{\partial x}+2(p_{22}^2)^2-2p_{11}^1p_{22}^1.$$

The four coefficients of the system (E_Γ) are calculated, as in [17], by the Schwarzian derivatives $S_{ij}^\alpha(z, u; x, y)$ of (z, u) with respect to the variables (x, y):

$$p_{11}^1(x, y)=S_{11}^1(z, u; x, y):=\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial z}+\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial u}+2\frac{1}{\xi_0}\frac{\partial \xi_0}{\partial x},$$

$$p_{22}^2(x, y)=S_{22}^2(z, u; x, y):=\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial z}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial u}+2\frac{1}{\xi_0}\frac{\partial \xi_0}{\partial y},$$

$$p_{22}^1(x, y)=S_{22}^1(z, u; x, y):=\frac{\partial^2 z}{\partial y^2}\frac{\partial x}{\partial z}+\frac{\partial^2 u}{\partial y^2}\frac{\partial x}{\partial u},$$

$$p_{11}^2(x, y)=S_{11}^2(z, u; x, y):=\frac{\partial^2 z}{\partial x^2}\frac{\partial y}{\partial z}+\frac{\partial^2 u}{\partial x^2}\frac{\partial y}{\partial u}$$

where $\xi_0=\{\det(\partial(z, u)/\partial(x, y))\}^{-1/3}$ and (x, y) is the local coordinate of the space $D/\Gamma\cup\{P\}$ around P obtained above.

As for the notation and terminology, we follow [17] and [18].

Table of contents

§ 1 Discrete subgroups of G of locally finite volume

Definition (c.f. [17], § 1). A subgroup Γ of G is said to be of locally finite volume (at P) if and only if $\Gamma D(N)/\Gamma$ has finite volume (with respect to the Aut (D)-invariant measure of D) for sufficiently large $N>0$, where

$$D(N)=\{(z, u) \in C^2 \mid \operatorname{Im} z-|u|^2>N\} \subset D.$$

In this section, we shall study discrete subgroups of locally finite volume of $G=\{[\mu, a, r]\}$. Notice that $C=\{[1, 0, r]\}$ is the center of G and

$$[\mu_1, a_1, r_1][\mu_2, a_2, r_2]=[\mu_1\mu_2, a_1+\mu_1 a_2, r_1+r_2-2 \operatorname{Im} \mu_1 \bar{a}_1 a_2].$$

Let G_1 be the normal subgroup of G consisting of all the elements of the form

$$[a, r]:=[1, a, r].$$

Then we have the following proposition due to Hemperly [4].

Proposition 1.1.

(i) *If Γ is a discrete subgroup of G of locally finite volume, then $\Gamma_1=\Gamma \cap G_1$ is also of locally finite volume and the index $[\Gamma : \Gamma_1]$ is finite.*

(ii) *For each discrete subgroup Γ_1 of G_1 of locally finite volume, there exist a lattice L $(=\Gamma_1/\Gamma_1 \cap C)$ of a complex plane, a positive number*

$$q:=\min_{[0,r]\in \Gamma_1} |r|$$

and a mapping

$$r(\cdot): L \rightarrow R/qZ$$

such that $[a, r]$ is an element of Γ_1 if and only if $a \in L$ and

$$r \equiv r(a) \mod q.$$

Since the lattice L, the positive number q and the mapping $r(\cdot)$ characterize the group Γ_1, we shall call the triple $\{L, q, r(\cdot)\}$ the characteristic of the group Γ_1.

First we shall study subgroups of G_1. Let Γ_1 be the subgroup of G_1 with characteristic $\{L, q, r(\cdot)\}$. Put

$$w=e^{(2\pi i/q)z},$$
$$D_u=\{w \in C \mid 0<|w|<e^{-2\pi|u|^2/q}\} \times \{u\},$$
$$\bar{D}_u=\{w \in C \mid 0 \leq |w|<e^{-2\pi|u|^2/q}\} \times \{u\}$$

and for $\eta \in L$,

$$g(u, \eta)=\exp \frac{2\pi i}{q}(2i\bar{\eta}u+r(\eta)+i|\eta|^2).$$

For two points $(w, u) \in \bar{D}_u$ and $(w', u+\eta) \in \bar{D}_{u+\eta}$, we define the equivalence $(w, u) \sim (w', u+\eta)$ by

$$\eta \in L, \qquad w'=g(u, \eta)w$$

and denote by $[w, u]$ the rest class of (w, u) with respect to the equivalence relation $\sim$. The following proposition is also due to Hemperly [4].

Proposition 1.2. *The quotient space D/Γ_1 is biholomorphically equivalent to $(\bigcup_{u\in C} D_u)/\sim$ and the nonsingular model $\overline{D/\Gamma_1}$ of $D/\Gamma_1 \cup \{P\}$ is identified with $(\bigcup_{u\in C} \overline{D_u})/\sim = (\bigcup_{u\in C} D_u)/\sim \cup E$, where E is the elliptic curve $\{[0, u]\} \cong C/L$.*

Corollary 1.3 ([4]; Lemma 2.1). *Let $\xi(\Gamma_1)$ be the line bundle over E with the transition functions $\{g(u, \eta) | \eta \in L\}$. Then $\overline{D/\Gamma_1}$ is identified with a tublar neighborhood of the zero section ($\cong E$) of the line bundle $\xi(\Gamma_1)$.*

Let $L=Z\eta_1+Z\eta_2$ ($\mathrm{Im}\, \bar{\eta}_1\eta_2>0$). The equality

$$[\eta_1, 0][\eta_2, 0][\eta_1, 0]^{-1}[\eta_2, 0]^{-1}=[0, -4\, \mathrm{Im}\, \bar{\eta}_1\eta_2]$$

implies that

$$p:=\frac{4\, \mathrm{Im}\, \bar{\eta}_1\eta_2}{q}$$

is a positive integer. Note that p depends only on L and q and is independent of the choice of Z-basis, η_1, η_2 of L.

Proposition 1.4. *Let Γ_1 be a subgroup of G_1 with characteristic $\{L, q, r(\cdot)\}$ and $g(u)$ be a nonzero section of the line bundle $\xi^{-1}(\Gamma_1)$. Then*

(i) *the degree of $g(u)$ is equal to p,*

(i)′ *the self-intersection number E^2 of E in $\overline{D/\Gamma_1}$ is equal to $-p$,*

(ii) *if $\{a_j\}$, $\{b_j\}$ be the zeros and poles of $g(u)$ respectively in the fundamental paralleogram of $L=Z\eta_1+Z\eta_2$,*

$$\sum_j a_j - \sum_j b_j \equiv \frac{p}{2}(\eta_1+\eta_2)+\frac{r(\eta_2)}{q}\eta_1-\frac{r(\eta_1)}{q}\eta_2 \quad \mathrm{mod}\, L.$$

Lemma 1.5. *Let $\vartheta(u)$ be a theta function with respect to the lattice $L=Z\eta_1+Z\eta_2$ which satisfies*

$$\vartheta(u+\eta)=a(\eta)e^{\pi h(\eta)(u+\eta/2)}\vartheta(u) \qquad \textit{for every } \eta \in L.$$

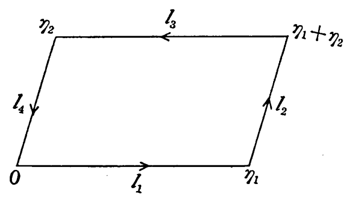
Then

(i) $d=\#\{a_j\}-\#\{b_j\}=\dfrac{1}{2i}(h(\eta_1)\eta_2-h(\eta_2)\eta_1)$,

(ii) $\sum a_j-\sum b_j\equiv\dfrac{d}{2}(\eta_1+\eta_2)+\dfrac{1}{2\pi i}(\eta_1\log a(\eta_2)-\eta_2\log a(\eta_1)) \mod L$,

where $\{a_j\}$ *and* $\{b_j\}$ *are the zeros and poles of* $\vartheta(u)$ *respectively in the fundamental paralleogram of* L.

Proof. (i) is known (e.g. see [12]). To prove (ii), we have only to calculate the following:



$$I=\frac{1}{2\pi i}\sum_{j=1}^{4}\int_{l_j}u\cdot\frac{\vartheta'(u)}{\vartheta(u)}du.$$

Since the relation

$$\frac{\vartheta'(u+\eta)}{\vartheta(u+\eta)}=\pi h(\eta)+\frac{\vartheta'(u)}{\vartheta(u)}$$

implies

$$\int_{l_3}u\frac{\vartheta'(u)}{\vartheta(u)}du=-\int_{l_1}(u+\eta_2)\frac{\vartheta'(u+\eta_2)}{\vartheta(u+\eta_2)}du$$
$$=-\pi h(\eta_2)\int_{l_1}(u+\eta_2)du-\eta_2\int_{l_1}\frac{\vartheta'(u)}{\vartheta(u)}du-\int_{l_1}u\frac{\vartheta'(u)}{\vartheta(u)}du,$$

if we notice that

$$\int_{l_1}\frac{\vartheta'(u)}{\vartheta(u)}du\equiv\log\frac{\vartheta(\eta_1)}{\vartheta(0)}\equiv\pi h(\eta_1)\frac{\eta_1}{2}+\log a(\eta_1) \mod 2\pi i,$$

I is easily calculated. **Q.E.D.**

Proof of Proposition 1.4. Let $g(u)$ be a nonzero section of $\xi^{-1}(\Gamma_1)$. Then by the definition of $\xi(\Gamma_1)$, $g(u)$ satisfies

$$g(u+\eta)=g(u,\eta)^{-1}g(u) \qquad \text{for every } \eta\in L.$$

Since

$$g(u, \eta)^{-1} = e^{-(2\pi i/q)r(\eta)} e^{\pi(4\bar{\eta}/q)(u+\eta/2)},$$

if we put $a(\eta) = e^{-(2\pi i/q)r(\eta)}$, $h(\eta) = \frac{4\bar{\eta}}{q}$ and $\vartheta(u) = g(u)$, we can apply Lemma 1.5. Hence

$$d = \frac{1}{2i}\left(\frac{4\bar{\eta}_1}{q}\eta_2 - \frac{4\bar{\eta}_2}{q}\eta_1\right) = \frac{4 \operatorname{Im} \bar{\eta}_1\eta_2}{q} = p$$

which proves (i). (i) yields $(i)'$, because, $\xi(\Gamma_1)$ is the normal bundle of E in $\overline{D/\Gamma_1}$ (Corollary 1.3) and E^2 is equal to the degree of a nonzero section of $\xi(\Gamma_1)$. The assertion (ii) is an immediate consequence of Lemma 1.5, (ii). **Q.E.D.**

The mapping $r(\cdot)$ is completely determined by the values $r(\eta_1)$ and $r(\eta_2)$, because, the equality

$$[a, r(a)][a', r(a')] = [a+a', r(a)+r(a') - 2 \operatorname{Im} \bar{a}a']$$

implies

$$r(a+a') \equiv r(a) + r(a') + 2 \operatorname{Im} \bar{a}a' \mod q$$

and so

$$r(n\eta_1 + m\eta_2) \equiv nr(\eta_1) + mr(\eta_2) + 2nm \operatorname{Im} \bar{\eta}_1\eta_2 \mod q.$$

For the element $I(\alpha) \in \operatorname{Aut}(D)$ of the form

$$I(\alpha) = \begin{bmatrix} |\alpha|^2 & & \\ & \alpha & \\ & & 1 \end{bmatrix}, \qquad \alpha \in \boldsymbol{C}^{\times},$$

we have

$$I(\alpha)[a, r]I^{-1}(\alpha) = [\alpha a, |\alpha|^2 r].$$

This proves that if Γ_1 has the characteristic $\{L, q, r(\cdot)\}$, then the characteristic of $I(\alpha)\Gamma_1(\alpha)^{-1}$ is $\{\alpha L, |\alpha|^2 q, |\alpha|^2 r(\cdot)\}$, and the number p leaves invariant under this operation. Thus every discrete subgroup Γ_1 of G_1 of locally finite volume is conjugate (in $\operatorname{Aut}(D)$) to the group with characteristic $\{L = \boldsymbol{Z} + \tau\boldsymbol{Z}, q, r(\cdot)\}$ for some $\tau(\operatorname{Im}\tau > 0)$, which we shall denote by $\Gamma_1(\tau; p, r_1, r_\tau)$ where $r_1 \equiv r(1)$, $r_\tau \equiv r(\tau) \mod q$ and $p = 4 \operatorname{Im} \tau/q$.

Next, we shall study subgroups of G.

Theorem 1.6. *Every discrete subgroup of G of locally finite volume which is not contained in G_1 is conjugate in* Aut (D) *to one of the following groups*:

Type II $\Gamma_{II}(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)=\sum_{\nu=1}^{2} \Gamma_1(\tau; p, r_1, r_\tau)[-1, 0, r]^\nu$

where $r_1=\frac{\varepsilon_1 q}{2}$, $r_\tau=\frac{\varepsilon_2 q}{2}$, $q=\frac{\varepsilon q}{2}$, $q=\frac{4 \operatorname{Im} \tau}{p}$ *and* $\varepsilon_1, \varepsilon_2, \varepsilon \equiv 0, 1 \mod 2$.

Type IV $\Gamma_{IV}(p, \varepsilon'; \varepsilon)=\sum_{\nu=1}^{4} \Gamma_1(i; p, r', r')[i, 0, r]^\nu$

where $r'=\frac{\varepsilon' q}{2}$, $r=\frac{\varepsilon q}{4}$, $q=\frac{4}{p}$ *and* $\varepsilon' \equiv 0, 1 \mod 2$, $\varepsilon \equiv 0, 1, 2, 3 \mod 4$.

Type III $\Gamma_{III}(p; \varepsilon)=\sum_{\nu=1}^{3} \Gamma_1(\zeta; p, 1/\sqrt{3}, -1/\sqrt{3})[\zeta^2, 0, r]^\nu$

where $\zeta=e^{2\pi i/6}$, $r=\frac{\varepsilon q}{3}$, $q=\frac{2\sqrt{3}}{p}$ *and* $\varepsilon \equiv 0, 1, 2 \mod 3$.

Type VI $\Gamma_{VI}(p; \varepsilon)=\sum_{\nu=1}^{6} \Gamma_1(\zeta; p, \sqrt{3}, \sqrt{3})[\zeta, 0, r]^\nu$

where $r=\frac{\varepsilon q}{6}$, $q=\frac{2\sqrt{3}}{p}$ *and* $\varepsilon \equiv 0, 1, \cdots, 5 \mod 6$.

Proof. Let Γ be a discrete subgroup of G of locally finite volume and put $\Gamma_1=G_1 \cap \Gamma$. If $\Gamma_1 \neq \Gamma$, there exists an element $[\mu, b, r] \in \Gamma-\Gamma_1$ such that Γ has the following coset decomposition:

$$\sum_{\nu=1}^{n} \Gamma_1[\mu, b, r]^\nu, \qquad \mu^n=1$$

for some $n>1$. Indeed, we have only to choose an element of Γ whose order in Γ/Γ_1 is maximal. Since we have

$$\begin{aligned}&[x, 0][\mu, b, r][x, 0]^{-1}\\ &\quad=[\mu, b+(1-\mu)x, r-2 \operatorname{Im}(\bar{x}b-\mu\bar{b}x-\mu|x|^2)],\end{aligned}$$

if we take conjugate of Γ by the element $[x, 0]=\left[\frac{b}{\mu-1}, 0\right]$, we can assume that $b=0$. Moreover, taking conjugate by a suitable element $I(\alpha)$, we can also assume that the characteristic of Γ_1 is $\{L=\boldsymbol{Z}+\boldsymbol{Z}\tau, q, r(\cdot)\}$. On the other hand, the equality

$$[\mu, 0, r][a, r'][\mu, 0, r]^{-1}[a, r']^{-1}=[(\mu-1)a, 2|a|^2 \operatorname{Im} \bar{\mu}], \qquad [a, r'] \in \Gamma$$

yields $\mu L \subset L$. This shows that the parameters n, μ, τ in the expression of Γ:

$$\sum_{\nu=1}^{n} \Gamma_1(\tau; p, r_1, r_\tau)[\mu, 0, r]^\nu$$

have four possibilities:

(i) $n=2$, $\mu=-1$ for arbitrary τ,
(ii) $n=4$, $\mu=\tau=i$,
(iii) $n=3$, $\mu=\zeta^2$, $\tau=\zeta$,

and

(iv) $n=6$, $\mu=\tau=\zeta$.

Since the above expression represents the group Γ with the normal subgroup Γ_1, we must have

$$[\mu, 0, r][a, r(a)][\mu, 0, r]^{-1}=[\mu a, r(a)] \in \Gamma_1 \qquad \text{for each } [a, r(a)] \in \Gamma_1,$$

and

$$[\mu, 0, r]^n=[0, nr] \in \Gamma_1.$$

Hence we have $r(\mu a)\equiv r(a)$ and $nr\equiv 0 \mod q$. The first equality implies, for each cases, the followings:

(i) $2r_1, 2r_\tau \equiv 0 \mod q$.
(ii) $r_1 \equiv r_i$ and $2r_1 \equiv 0 \mod q$.
(iv) $r_1 \equiv r_\zeta \equiv \sqrt{3} \mod q$. In fact, we have $r(1)\equiv r(\zeta)\equiv r(\zeta^2)$ and

$$\begin{aligned} r(\zeta^2) &= r(\zeta-1) \\ &\equiv r(\zeta)-r(1)+2\,\mathrm{Im}\,\zeta. \end{aligned}$$

(iii) $r_1 \equiv -r_\zeta \equiv 1/\sqrt{3} \mod q$, because we have $r(1)\equiv r(\zeta^2)\equiv r(\zeta^4)\equiv -r(\zeta)$. These complete the proof. **Q.E.D.**

In the groups which we have list upped in Theorem 1.6, there are several pairs which are conjugate to each other.

Proposition 1.7. (i) *The group $\Gamma_{\text{II}}(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)$ is conjugate to $\Gamma_{\text{II}}(-1/\tau; p, \varepsilon_2, \varepsilon_1; \varepsilon)$, $\Gamma_{\text{II}}(\tau; p, \varepsilon_1+p, \varepsilon_2; \varepsilon+\varepsilon_2)$ and $\Gamma_{\text{II}}(\tau; p, \varepsilon_1, \varepsilon_2+p; \varepsilon+\varepsilon_1)$.*

(ii) *The group $\Gamma_{\text{IV}}(p, \varepsilon'; \varepsilon)$ is conjugate to $\Gamma_{\text{IV}}(p, \varepsilon'+p; \varepsilon+2\varepsilon'+3p)$. Here, the additions in* (i) *are* mod 2, *and the additions for ε' and ε in* (ii) *are* mod 2 *and* mod 4, *respectively.*

Proof. (i) By definition, we have

$$\Gamma_{\mathrm{II}}(\tau;p,\varepsilon_1,\varepsilon_2;\varepsilon)$$
$$=\sum_{\nu=1}^{2}\left\{\left[n+m\tau,\ \frac{2\operatorname{Im}\tau}{p}(n\varepsilon_1+m\varepsilon_2+pnm+2l)\right];\ n,m,l\in Z\right\}\left[-1,0,\frac{2\operatorname{Im}}{p}\varepsilon\right]^{\nu}.$$

The equalities

$$I(-1/\tau)\left[n+m\tau,\frac{2\operatorname{Im}\tau}{p}(n\varepsilon_1+m\varepsilon_2+pnm+2l)\right]I(-1/\tau)^{-1}$$
$$=\left[-\frac{1}{\tau}n-m,\ \frac{2\operatorname{Im}\tau}{|\tau|^2p}(n\varepsilon_1+m\varepsilon_2+pnm+2l)\right]$$
$$=\left[(-m)+\left(-\frac{1}{\tau}\right)n,\ \frac{2\operatorname{Im}(-1/\tau)}{p}((-m)\varepsilon_2+n\varepsilon_1+p(-m)n\right.$$
$$\left.+2(l+m\varepsilon+pmn))\right]$$

and

$$I(-1/\tau)\left[-1,0,\frac{2\operatorname{Im}\tau}{p}\varepsilon\right]I(-1/\tau)^{-1}=\left[-1,0,\frac{2\operatorname{Im}(-1/\tau)}{p}\varepsilon\right]$$

prove the conjugacy of $\Gamma_{\mathrm{II}}(\tau;p,\varepsilon_1,\varepsilon_2;\varepsilon)$ and $\Gamma_{\mathrm{II}}(-1/\tau;p,\varepsilon_2,\varepsilon_1;\varepsilon)$. The formulae

$$[x,0][a,r][x,0]^{-1}=[a,r-4\operatorname{Im}\bar{x}a],$$
$$[x,0][\mu,0,r][x,0]^{-1}=[\mu,(1-\mu)x,r+2\operatorname{Im}\mu|x|^2]$$

lead to

$$[\tau/2,0]\left[n+m\tau,\ \frac{2\operatorname{Im}\tau}{p}(n\varepsilon_1+m\varepsilon_2+pnm+2l)\right][\tau/2,0]^{-1}$$
$$=\left[n+m\tau,\ \frac{2\operatorname{Im}\tau}{p}(n(\varepsilon_1+p)+m\varepsilon_2+pnm+2l)\right],$$

and

$$[\tau/2,0]\left[-1,0,\frac{2\operatorname{Im}\tau}{p}\varepsilon\right][\tau/2,0]^{-1}$$
$$=\left[-1,\tau,\frac{2\operatorname{Im}\tau}{p}\varepsilon\right]$$
$$=\left[\tau,\frac{2\operatorname{Im}\tau}{p}(\varepsilon_2+2l)\right]\left[-1,0,\frac{2\operatorname{Im}\tau}{2}(\varepsilon-\varepsilon_2+2l)\right]$$

which yield the conjugacy of $\Gamma_{\mathrm{II}}(\tau;p,\varepsilon_1,\varepsilon_2;\varepsilon)$ and $\Gamma_{\mathrm{II}}(\tau;p,\varepsilon_1+p,\varepsilon_2,\varepsilon+\varepsilon_2)$. If we put $x=-1/2$ instead of $x=\tau/2$ in the above formulae, we can analogously prove the conjugacy of $\Gamma_{\mathrm{II}}(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)$ and $\Gamma_{\mathrm{II}}(\tau; p, \varepsilon_1, \varepsilon_2+p; \varepsilon+\varepsilon_1)$.

(ii) Recall that

$$\Gamma_{\mathrm{IV}}(p, \varepsilon'; \varepsilon) = \sum_{\nu=1}^{4} \left\{ \left[n+mi, \frac{2}{p}((n+m)\varepsilon'+pnm+2l) \right]; n, m, l \in \boldsymbol{Z} \right\} \left[i, 0, \frac{\varepsilon}{p} \right]^{\nu}.$$

Making use of the above formulae by putting $x = \dfrac{i-1}{2}$, we have,

$$\begin{aligned} &\left[\frac{i-1}{2}, 0\right]\left[n+mi, \frac{2}{p}((n+m)\varepsilon'+pnm+2l)\right]\left[\frac{i-1}{2}, 0\right]^{-1} \\ &\quad = \left[n+mi, \frac{2}{p}((n+m)(\varepsilon'+p)+pnm+2l)\right] \end{aligned}$$

and

$$\begin{aligned} &\left[\frac{i-1}{2}, 0\right]\left[i, 0, \frac{\varepsilon}{p}\right]\left[\frac{i-1}{2}, 0\right]^{-1} \\ &\quad = \left[i, i, \frac{\varepsilon}{p}+1\right] \\ &\quad = \left[i, \frac{2}{p}(\varepsilon'+p+2l)\right]\left[i, 0, \frac{1}{p}(\varepsilon-p-2\varepsilon'-4l)\right] \end{aligned}$$

which show that $\Gamma_{\mathrm{IV}}(p, \varepsilon'; \varepsilon)$ and $\Gamma_{\mathrm{IV}}(p, \varepsilon'+p; \varepsilon-p-2\varepsilon')$ are conjugate to each other.

Q.E.D.

Definition 1.8. An element of G is called a unitary reflection with center $u=u_0$ if it has the form

$$h(\mu, u_0) = [\mu, u_0(1-\mu), 2|u_0|^2 \operatorname{Im} \mu],$$

where μ is a root of unity.

Notice that $h(\mu, u_0)$ keeps the line $\{u=u_0\}$ passing through P fixed, and that the order of $h(\mu, u_0)$ is equal to the order of μ in $\boldsymbol{C}^{\times}$. The following formula

$$h(\mu_1, u_1)h(\mu_2, u_2)h(\mu_1, u_1)^{-1} = h(\mu_2, u_1+\mu_1(u_2-u_1)) \tag{1.1}$$

is frequently used later. We are interested in the subgroups of G generated by unitary reflections.

Theorem 1.9. *Every discrete subgroup of G of locally finite volume which is generated by unitary reflections is conjugate in* Aut (D) *to one of the following groups*:

Type II $\Gamma_{\mathrm{II}}(\tau; 1, 0, 0; 0) = \langle h(-1, 0), h(-1, 1/2), h(-1, \tau/2)\rangle$

$$\Gamma_{\mathrm{II}}(\tau; 2, 0, 0; 0) = \left\langle h(-1, 0), h(-1, 1/2), h(-1, \tau/2), h\left(-1, \frac{1+\tau}{2}\right)\right\rangle$$

Type IV $\Gamma_{\mathrm{IV}}(1, 0; 0) = \langle h(i, 0), h(-1, 1/2)\rangle$

$$\Gamma_{\mathrm{IV}}(2, 0; 0) = \left\langle h(i, 0), h(-1, 1/2), h\left(-1, \frac{1+i}{2}\right)\right\rangle$$

$$\Gamma_{\mathrm{IV}}(2, 1; 0) = \left\langle h(i, 0), h\left(i, \frac{1+i}{2}\right)\right\rangle$$

$$\Gamma_{\mathrm{IV}}(4, 0; 0) = \left\langle h(i, 0), h\left(i, \frac{1+i}{2}\right), h(-1, 1/2)\right\rangle$$

Type III $\Gamma_{\mathrm{III}}(1; 0) = \left\langle h(\zeta^2, 0), h\left(\zeta^2, \frac{1+\zeta}{3}\right)\right\rangle$

$$\Gamma_{\mathrm{III}}(3; 0) = \left\langle h(\zeta^2, 0), h\left(\zeta^2, \frac{1+\zeta}{3}\right), h\left(\zeta^2, \frac{1+\zeta}{3}\right)\right\rangle$$

Type VI $\Gamma_{\mathrm{VI}}(1; 0) = \langle h(\zeta, 0), h(\zeta, 1)\rangle$

$$\Gamma_{\mathrm{VI}}(1; 2) = \left\langle h(-1, 0), h\left(\zeta^2, \frac{1+\zeta}{3}\right), h\left(\zeta^2, \frac{\zeta+\zeta^2}{3}\right)\right\rangle$$

$$\Gamma_{\mathrm{VI}}(1; 3) = \left\langle h\left(-1, \frac{1}{2}\right), h(-1, \zeta), h(\zeta^2, 0)\right\rangle$$

$$\Gamma_{\mathrm{VI}}(1; 5) = \left\langle h\left(-1, \frac{1}{2}\right), h\left(\zeta^2, \frac{1+\zeta}{3}\right)\right\rangle$$

$$\Gamma_{\mathrm{VI}}(2; 0) = \left\langle h(\zeta, 0), h\left(-1, \frac{1}{2}\right)\right\rangle$$

$$\Gamma_{\mathrm{VI}}(2; 4) = \left\langle h(-1, 0), h\left(-1, \frac{1}{2}\right), h\left(\zeta^2, \frac{1+\zeta}{3}\right)\right\rangle$$

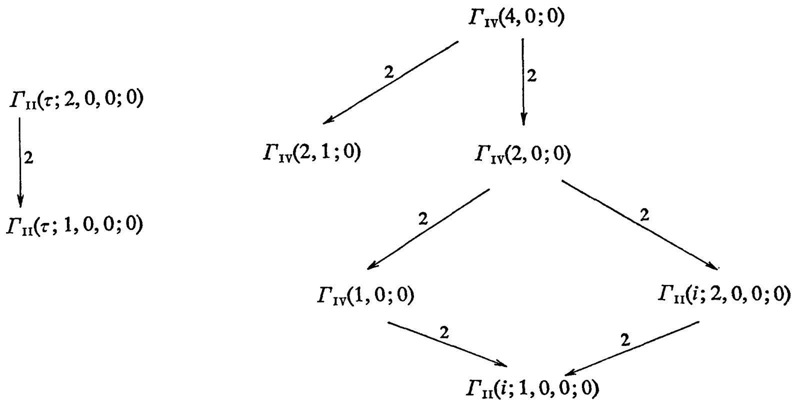
$$\Gamma_{\mathrm{VI}}(3; 0) = \left\langle h(\zeta, 0), h\left(\zeta^2, \frac{1+\zeta}{3}\right)\right\rangle$$

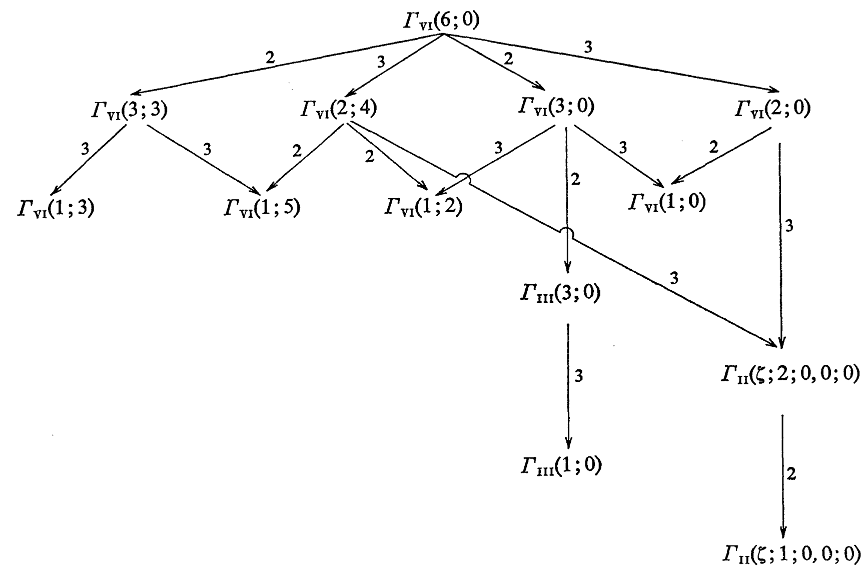
$$\Gamma_{\mathrm{VI}}(3; 3) = \left\langle h\left(-1, \frac{1}{2}\right), h(\zeta^2, 0), h\left(\zeta^2, \frac{1+\zeta}{3}\right)\right\rangle$$

$$\Gamma_{\mathrm{VI}}(6; 0) = \left\langle h(\zeta, 0), h\left(-1, \frac{1}{2}\right), h\left(\zeta^2, \frac{1+\zeta}{3}\right)\right\rangle.$$

Remark 1.10. It is the group $\Gamma_{\mathrm{IV}}(2, 1; 0)$ that we treated in [17].

Remark 1.11. If $\Gamma \xrightarrow{\nu} \Gamma'$ means that Γ' is a normal subgroup of Γ of index ν, then we have the following relations.





It was proved ([17], Proposition 3.3) that the group $\langle h_1, h_2\rangle$ generated by two unitary reflections h_1 and h_2 in G of respective orders n_1 and n_2 is descrete if and only if $n_j|4$ $(j=1, 2)$ or $n_j|6$ $(j=1, 2)$. This gives the following corollary of Theorem 1.9.

Theorem 1.12. *Every discrete subgroup of G which is generated by more than two unitary reflections is of locally finite volume except for the group generated bytwo unitary reflections of order two, which is unique up to conjugacy.*

Proof of Theorem 1.9. **Type II.** We find all the unitary reflections, which are of course of order 2, contained in the group $\Gamma_{II}(\tau; p, \varepsilon_1, \varepsilon_2, \varepsilon)$. Since

$$\Big[n+m\tau, \frac{2\operatorname{Im}\tau}{p}(n\varepsilon_1+m\varepsilon_2+pnm+2l)\Big]\Big[-1, 0, \frac{2\operatorname{Im}\tau}{p}\varepsilon\Big]$$
$$=\Big[-1, n+m\tau, \frac{2\operatorname{Im}\tau}{p}(n\varepsilon_1+m\varepsilon_2+pnm+\varepsilon+2l)\Big],$$

if we recall that

$$h(-1, u)=[-1, 2u, 0],$$

the set $R(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)$ which consists of all the unitary reflections in $\Gamma_{\mathrm{II}}(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)$ is given by

$$\left\{h\left(-1, \frac{n+m\tau}{2}\right) \middle| n, m \in \boldsymbol{Z}, n\varepsilon_1+m\varepsilon_2+pnm+\varepsilon\equiv 0 \bmod 2\right\}.$$

Next we study the group $\langle R(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)\rangle$ generated by the elements of $R(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)$. On account of the equality

$$h(-1, u_1)h(-1, u_2)=[2(u_1-u_2), 2 \operatorname{Im} \bar{u}_1 u_2]$$

and Proposition 1.7, we can conclude that

$$\begin{aligned}&\langle R(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)\rangle \cap G_1/\langle R(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)\rangle \cap C\\ &\qquad (=\{a \in \boldsymbol{C} \mid [a, r] \in R(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)\cap G_1 \text{ for some } r \in \boldsymbol{R}\})\\ &\qquad =L\ (=\boldsymbol{Z}+\tau\boldsymbol{Z})\end{aligned}$$

if and only if $\Gamma_{\mathrm{II}}(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)$ is conjugate to $\Gamma_{\mathrm{II}}(\tau; p, 0, 0; 0)$. By the relation (1, 1) and the expression of the elements of $R(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)$ obtained above, we have

$$\langle R(\tau; 2p'+1, 0, 0; 0)\rangle=\left\langle h(-1, 0), h\left(-1, \frac{1}{2}\right), h\left(-1, \frac{\tau}{2}\right)\right\rangle,$$

$$\langle R(\tau; 2p', 0, 0; 0)\rangle=\left\langle h(-1, 0), h\left(-1, \frac{1}{2}\right), h\left(-1, \frac{\tau}{2}\right), h\left(-1, \frac{1+\tau}{2}\right)\right\rangle.$$

Recall that

$$\begin{aligned}\Gamma_{\mathrm{II}}(\tau; 1, 0, 0; 0)&=\langle[1, 0][\tau, 0], [0, 4 \operatorname{Im} \tau], h(-1, 0)\rangle,\\ \Gamma_{\mathrm{II}}(\tau; 2, 0, 0; 0)&=\langle[1, 0], [\tau, 0], [0, 2 \operatorname{Im} \tau], h(-1, 0)\rangle\end{aligned}$$

then the equalities

$$\begin{aligned}&h\left(-1, \frac{1}{2}\right)h(-1, 0)=[1, 0],\ h\left(-1, \frac{\tau}{2}\right)h(-1, 0)=[\tau, 0],\\ &[\tau, 0][1, 0][\tau, 0]^{-1}[1, 0]^{-1}=[0, 4 \operatorname{Im} \tau],\\ &h\left(-1, \frac{1+\tau}{2}\right)h(-1, 0)[\tau, 0]^{-1}[1, 0]^{-1}=[0, 2 \operatorname{Im} \tau]\end{aligned}$$

prove the assertion.

Type IV. Notice that

$$h(i, u)=[i, u(1-i), 2|u|^2],$$

$$\left[n+mi, \frac{2}{p}(\varepsilon'(n+m)+pnm+2l)\right]\left[i, 0, \frac{\varepsilon}{p}\right]$$

$$=\left[i, n+mi, \frac{1}{p}(2\varepsilon'(n+m)+2pnm+\varepsilon+4l)\right]$$

and

$$\left[n+mi, \frac{2}{p}(\varepsilon'(n+m)+pnm+2l)\right]\left[i, 0 \frac{\varepsilon}{p}\right]^2$$

$$=\left[-1, n+mi, \frac{2}{p}(\varepsilon'(n+m)+pnm+\varepsilon+2l)\right]$$

then the sets $R_4(p, \varepsilon'; \varepsilon)$ and $R_2(p, \varepsilon'; \varepsilon)$ which consist of all the unitary reflections of $\Gamma_{\mathrm{IV}}(p, \varepsilon'; \varepsilon)$ of order 4 and order 2 are given by

$$\left\{h\left(\pm i, \frac{n-m+i(n+m)}{2}\right)\middle| n, m \in \boldsymbol{Z}, 2\varepsilon'(n+m)-p(n+m)^2+\varepsilon\equiv 0 \bmod 4\right\}$$

and

$$\left\{h\left(-1, \frac{n+mi}{2}\right)\middle| n, m \in \boldsymbol{Z}, \varepsilon'(n+m)+pnm+\varepsilon\equiv 0 \bmod 2\right\},$$

respectively. Put $R(p, \varepsilon'; \varepsilon)=\langle R_4(p, \varepsilon'; \varepsilon), R_2(p, \varepsilon'; \varepsilon)\rangle$. By making use of Proposition 1.7, we can show that

$$R(p, \varepsilon'; \varepsilon)\cap G_1/R(p, \varepsilon'; \varepsilon)\cap C=L$$

if and only if $\Gamma_{\mathrm{IV}}(p, \varepsilon'; \varepsilon)$ is conjugate to one of the following five groups: $\Gamma_{\mathrm{IV}}(4p'+1, 0; 0)$, $\Gamma_{\mathrm{IV}}(4p'+2, 0; 0)$, $\Gamma_{\mathrm{IV}}(4p'+2, 1; 0)$, $\Gamma_{\mathrm{IV}}(4p'+3, 0; 0)$, $\Gamma_{\mathrm{IV}}(4p', 0; 0)$, and that $R(4p'+1, 0; 0)=R(4p'+3, 0; 0)$. By the above results and (1, 1) we can conclude the followings:

$$R(4p'+1, 0; 0)=\left\langle h(i, 0), h\left(-1, \frac{1}{2}\right)\right\rangle$$

$$R(4p'+2, 0; 0)=\left\langle h(i, 0), h\left(-1, \frac{1}{2}\right), h\left(-1, \frac{1+i}{2}\right)\right\rangle,$$

$$R(4p'+2, 1; 0)=\left\langle h(i, 0), h\left(i, \frac{1+i}{2}\right)\right\rangle,$$

$$R(4p', 0; 0)=\left\langle h(i, 0), h\left(-1, \frac{1}{2}\right), h\left(i, \frac{1+i}{2}\right)\right\rangle.$$

The equality $\Gamma_{\mathrm{IV}}(2,1;0)=\left\langle h(i,0), h\left(i, \frac{1+i}{2}\right)\right\rangle$ is already proved in [17]. Recall

$$\Gamma_{\mathrm{IV}}(1,0;0)=\langle[1,0],[i,0],[0,4],h(i,0)\rangle,$$
$$\Gamma_{\mathrm{IV}}(2,0;0)=\langle[1,0],[i,0],[0,2],h(i,0)\rangle,$$
$$\Gamma_{\mathrm{IV}}(4,0;0)=\langle[1,0],[i,0],[0,1],h(i,0)\rangle.$$

Then the equalities

$$h\left(-1, \frac{1}{2}\right)h(i,0)^2=[1,0],$$
$$h(i,0)h\left(-1,\frac{1}{2}\right)h(i,0)=[i,0]$$
$$[1,0]^{-1}[i,0][1,0][i,0]^{-1}=[0,4],$$
$$[i,0]h\left(-1,\frac{1}{2}\right)h\left(-1,\frac{1+i}{2}\right)=[0,2],$$
$$h\left(i,\frac{1+i}{2}\right)h(i,0)h\left(-1,\frac{1}{2}\right)=[0,1]$$

lead to the conclusion.

Type III. Since we have

$$\Gamma_{\mathrm{III}}(p;\varepsilon)=\sum_{\nu=1}^{3}\left\{\left[n+m\zeta^2, (n+m-3nm)\frac{\sqrt{3}}{3}+\frac{2\sqrt{3}}{p}l\right]; n, m, l\in \boldsymbol{Z}\right\} \times\left[\zeta^2, 0, \frac{2\sqrt{3}}{3p}\varepsilon\right]^{\nu},$$

$$\left[n+m\zeta^2, (n+m-3nm)\frac{\sqrt{3}}{3}+\frac{2\sqrt{3}}{p}l\right]\left[\zeta^2, 0, \frac{2\sqrt{3}}{3p}\varepsilon\right] =\left[\zeta^2, n+m\zeta^2, \left(n+m-3nm+\frac{6l}{p}+\frac{2\varepsilon}{p}\right)\frac{\sqrt{3}}{3}\right]$$

and

$$h(\zeta^2,u)=[\zeta^2, u(1-\zeta^2), \sqrt{3}\,|u|^2],$$

the set $R(p;\varepsilon)$ which consists of every unitary reflection in $\Gamma_{\mathrm{III}}(p,\varepsilon)$ is given by

$$\left\{h\left(\zeta^{\pm 2}, \frac{2n-m}{3}+\frac{n+m}{3}\zeta^2\right); n, m\in \boldsymbol{Z}, \frac{(n+m)(n+m-1)}{2}p-\varepsilon\equiv 0 \bmod 3\right\}.$$

If $\langle R(p;\varepsilon)\rangle$ denotes the group generated by the elements of $R(p,\varepsilon)$, we can prove that

$$\langle R(p;\varepsilon)\rangle \cap G_1/\langle R(p;\varepsilon)\rangle \cap C = L$$

if and only if $p\equiv 0, 1 \bmod 3$ and $\varepsilon=0$. By (1, 1) we conclude that

$$\langle R(3p'+1;0)\rangle=\left\langle h(\zeta^2,0),\ h\left(\zeta^2,\frac{1+\zeta}{3}\right)\right\rangle,$$

$$\langle R(3p';0)\rangle=\left\langle h(\zeta^2,0),\ h\left(\zeta^2,\frac{1+\zeta}{3}\right),\ h\left(\zeta^2,\frac{\zeta+\zeta^2}{3}\right)\right\rangle.$$

On the other hand, we have

$$\Gamma_{\mathrm{III}}(1;0)=\left\langle\left[1,\frac{\sqrt{3}}{3}\right],\left[\zeta^2,\frac{\sqrt{3}}{3}\right],[0,2\sqrt{3}],h(\zeta^2,0)\right\rangle,$$

$$\Gamma_{\mathrm{III}}(3;0)=\left\langle\left[1,\frac{\sqrt{3}}{3}\right],\left[\zeta^2,\frac{\sqrt{3}}{3}\right],\left[0,\frac{2}{3}\sqrt{3}\right],h(\zeta^2,0)\right\rangle.$$

Thus the following equalities

$$h\left(\zeta^2,\frac{1+\zeta}{3}\right)h^{-1}(\zeta^2,0)=\left[1,\frac{\sqrt{3}}{3}\right],$$

$$\left[1,\frac{\sqrt{3}}{3}\right]^{-1}h^{-1}\left(\zeta^2,\frac{1+\zeta}{3}\right)h(\zeta^2,0)=\left[\zeta^2,\frac{\sqrt{3}}{3}\right],$$

$$\left[1,\frac{\sqrt{3}}{3}\right]\left[\zeta^2,\frac{\sqrt{3}}{3}\right]^{-1}\left[1,\frac{\sqrt{3}}{3}\right]^{-1}\left[\zeta^2,\frac{\sqrt{3}}{3}\right]=[0,2\sqrt{3}],$$

$$h(\zeta^2,0)h\left(\zeta^2,\frac{1+\zeta}{3}\right)h\left(\zeta^2,\frac{\zeta+\zeta^2}{3}\right)=\left[0,\frac{2}{3}\sqrt{3}\right]$$

yield the desired equalities.

Type VI. Since we have

$$\Gamma_{\mathrm{VI}}(p;\varepsilon)=\sum_{\nu=1}^{6}\left\{\left[n+m\zeta^2,(n+m-nm)\sqrt{3}+\frac{2\sqrt{3}}{p}l\right];n,m,l\in\boldsymbol{Z}\right\}\left[\zeta,0,\frac{\sqrt{3}}{3p}\varepsilon\right]^{\nu}$$
$$=\left\{\left[\zeta^{\nu},n+m\zeta^2,\left(n+m-nm+\frac{2l}{p}+\frac{\nu\varepsilon}{3p}\right)\sqrt{3}\right];n,m,l\in\boldsymbol{Z},\nu=1,\cdots,6\right\}$$

and

$$h(\zeta,u)=[\zeta,u(1-\zeta),\sqrt{3}\,|u|^2],$$

the sets $R_6(p;\varepsilon)$, $R_3(p;\varepsilon)$ and $R_2(p;\varepsilon)$ which consist of all the unitary reflections of respective order 6, 3 and 2 in $\Gamma_{\mathrm{VI}}(p;\varepsilon)$ are given by

$$\{h(\zeta^{\pm 1}, n+m\zeta^2); \varepsilon=0\},$$

$$\left\{h\left(\zeta^{\pm 2}, \frac{n+m}{3}+\frac{2m-n}{3}\zeta^2\right); \frac{(n+m)(n+m+3)}{2}p+2\varepsilon\equiv 0 \bmod 3\right\}$$

and

$$\left\{h\left(-1, \frac{n+m\zeta^2}{2}\right); p(n+m-nm)+k\equiv 0 \bmod 2\right\}$$

respectively. This shows that

$$R(p; \varepsilon)\cap G_1/R(p; \varepsilon)\cap C=L$$

if and only if $(p; \varepsilon)=(6p'+1; 0)$, $(6p'+2; 0)$, $(6p'+3; 0)$, $(6p'; 0)$, $(6p'+1; 5)$, $(6p'+1; 2)$, $(6p'+2; 4)$, $(6p'+3; 3)$, $(6p'+1; 3)$, $(6p'+5; 0)$, $(6p'+4; 0)$, $(6p'+5; 1)$, $(6p'+5; 4)$, $(6p'+4; 2)$, $(6p'+5; 3)$ and $(6p'; 3)$, where $R(p; \varepsilon)=\langle R_6(p; \varepsilon), R_3(p; \varepsilon), R_2(p; \varepsilon)\rangle$. Moreover we have $R(6p'+5; 0)=R(6p'+1; 0)$, $R(6p'+4; 0)=R(6p'+2; 0)$, $R(6p'+5; 1)=R(6p'+1; 5)$, $R(6p'+5; 4)=R(6p'+1; 2)$, $R(6p'+4; 2)=R(6p'+2; 4)$, $R(6p'+5; 3)=R(6p'+1; 3)$ and $R(6p'; 3)=\Gamma_{\mathrm{III}}(3, 0)$. The equation (1, 1) leads to

$$R(6p'+1; 0)=\langle h(\zeta, 0), h(\zeta, 1)\rangle$$

$$R(6p'+2; 0)=\left\langle h(\zeta, 0), h\left(-1, \frac{1}{2}\right)\right\rangle$$

$$R(6p'+3; 0)=\left\langle h(\zeta, 0), h\left(\zeta^2, \frac{1+\zeta}{3}\right)\right\rangle$$

$$R(6p'; 0)=\left\langle h(\zeta, 0), h\left(\zeta^2, \frac{1+\zeta}{3}\right), h\left(-1, \frac{1}{2}\right)\right\rangle$$

$$R(6p'+1; 5)=\left\langle h\left(\zeta^2, \frac{1+\zeta}{3}\right), h\left(-1, \frac{1}{2}\right)\right\rangle$$

$$R(6p'+1; 2)=\left\langle h\left(\zeta^2, \frac{1+\zeta}{3}\right), h\left(\zeta^2, \frac{\zeta+\zeta^2}{3}\right), h(-1, 0)\right\rangle$$

$$R(6p'+2; 4)=\left\langle h\left(\zeta^2, \frac{1+\zeta}{3}\right), h(-1, 0), h\left(-1, \frac{1}{2}\right)\right\rangle$$

$$R(6p'+3; 3)=\left\langle h(\zeta^2, 0), h\left(\zeta^2, \frac{1+\zeta}{3}\right), h\left(-1, \frac{1}{2}\right)\right\rangle$$

$$R(6p'+1; 3)=\left\langle h(\zeta^2, 0), h\left(-1, \frac{1}{2}\right), h(-1, \zeta)\right\rangle.$$

On the other hand, we have

$$\Gamma_{\text{VI}}(1;0)=\langle[1,\sqrt{3}],[\zeta^2,\sqrt{3}],[0,2\sqrt{3}],h(\zeta,0)\rangle$$
$$\Gamma_{\text{VI}}(2;0)=\langle[1,0],[\zeta^2,0],[0,\sqrt{3}],h(\zeta,0)\rangle$$
$$\Gamma_{\text{VI}}(3;0)=\left\langle[1,\sqrt{3}],[\zeta^2,\sqrt{3}],\left[0,\frac{2}{3}\sqrt{3}\right],h(\zeta,0)\right\rangle$$
$$\Gamma_{\text{VI}}(6;0)=\left\langle[1,0],[\zeta^2,0],\left[0,\frac{\sqrt{3}}{3}\right],h(\zeta,0)\right\rangle$$
$$\Gamma_{\text{VI}}(1;5)=\left\langle[1,\sqrt{3}],[\zeta^2,\sqrt{3}],[0,2\sqrt{3}],\left[\zeta,0,\frac{5}{3}\sqrt{3}\right]\right\rangle$$
$$\Gamma_{\text{VI}}(1;2)=\left\langle[1,\sqrt{3}],[\zeta^2,\sqrt{3}],[0,2\sqrt{3}],\left[\zeta,0,\frac{2}{3}\sqrt{3}\right]\right\rangle$$
$$\Gamma_{\text{VI}}(2;4)=\left\langle[1,\sqrt{3}],[\zeta^2,\sqrt{3}],[0,\sqrt{3}],\left[\zeta,0,\frac{2}{3}\sqrt{3}\right]\right\rangle$$
$$\Gamma_{\text{VI}}(3;3)=\left\langle[1,\sqrt{3}],[\zeta^2,\sqrt{3}],\left[0,\frac{2}{3}\sqrt{3}\right],\left[\zeta,0,\frac{\sqrt{3}}{3}\right]\right\rangle$$
$$\Gamma_{\text{VI}}(1;3)=\langle[1,\sqrt{3}],[\zeta^2,\sqrt{3}],[0,2\sqrt{3}],[\zeta,0,\sqrt{3}]\rangle.$$

The equalities

$$h^{-1}(\zeta,1)h(\zeta,0)h(\zeta,1)h(\zeta,0)^{-1}=[1,\sqrt{3}],$$
$$(h(\zeta,1)h(\zeta,0))^2h(\zeta,0)h^{-1}(\zeta,1)=[\zeta^2,\sqrt{3}],$$
$$(h(\zeta,0)h(\zeta,1))^3=[0,2\sqrt{3}]$$

imply $\Gamma_{\text{VI}}(1;0)\subset R(1;0)$. If we notice that $h(\zeta,1)\in R(3;0)$, these equalities with

$$\left(h(\zeta,1)h\left(\zeta^2,\frac{1+\zeta}{3}\right)\right)^3=\left[0,\frac{2}{3}\sqrt{3}\right]$$

lead to $\Gamma_{\text{VI}}(3;0)\subset R(3;0)$. We have $\Gamma_{\text{VI}}(2,0)\subset R(2;0)$ and $\Gamma_{\text{VI}}(6;0)\subset R(6;0)$, since,

$$h\left(-1,\frac{1}{2}\right)h^3(\zeta,0)=[1,0],\ h^2(\zeta,0)h\left(-1,\frac{1}{2}\right)h(\zeta,0)=[\zeta^2,0],$$
$$\left(h(\zeta,0)h\left(-1,\frac{1}{2}\right)\right)^3=[0,\sqrt{3}],$$
$$h\left(\zeta^2,\frac{1+\zeta}{3}\right)h(\zeta,0)h\left(-1,\frac{1}{2}\right)=\left[0,\frac{\sqrt{3}}{3}\right].$$

On account of the equality

$$\left\langle h(\zeta^2,0),h\left(\zeta^3,\frac{1+\zeta}{3}\right)\right\rangle=\left\langle\left[1,\frac{\sqrt{3}}{3}\right],\left[\zeta^2,\frac{\sqrt{3}}{3}\right],[0,2\sqrt{3}],h(\zeta^2,0)\right\rangle$$

obtained in the proof of **type III**, taking conjugate by a suitable element, we can easily show that

$$[1, \sqrt{3}], [\zeta^2, \sqrt{3}], [0, 2\sqrt{3}] \in \left\langle h\left(\zeta^2, \frac{1+\zeta}{3}\right), h\left(\zeta^2, \frac{\zeta+\zeta^2}{3}\right)\right\rangle.$$

If we notice that

$$\left\langle h\left(\zeta^2, \frac{1+\zeta}{3}\right), h\left(\zeta^2, \frac{\zeta+\zeta^2}{3}\right)\right\rangle \subset R(1;5) \cap R(1;2) \cap R(2;4) \cap R(3;3),$$

then the equalities

$$h\left(\zeta^2, \frac{1+\zeta}{3}\right)h\left(\zeta^2, \frac{\zeta+\zeta^2}{3}\right)h\left(-1, \frac{1}{2}\right)[1, \sqrt{3}] = \left[\zeta, 0, \frac{5}{3}\sqrt{3}\right],$$

$$h\left(\zeta^2, \frac{1+\zeta}{3}\right)h\left(\zeta^2, \frac{\zeta+\zeta^2}{3}\right)h(-1, 0) = \left[\zeta, 0, \frac{2}{3}\sqrt{3}\right],$$

$$h(-1, 0)h\left(-1, \frac{1}{2}\right)[1, \sqrt{3}] = [0, \sqrt{3}],$$

$$h(\zeta^2, 0)h^{-1}\left(\zeta^2, \frac{1+\zeta}{3}\right)[1, \sqrt{3}] = \left[0, \frac{2}{3}\sqrt{3}\right],$$

$$h\left(\zeta^2, \frac{1+\zeta}{3}\right)h\left(\zeta^2, \frac{\zeta+\zeta^2}{3}\right)h\left(-1, \frac{1}{2}\right)[1, \sqrt{3}]^{-1}\left[0, \frac{2}{3}\sqrt{3}\right] = \left[\zeta, 0, \frac{\sqrt{3}}{3}\right],$$

yield $\Gamma_{\text{VI}}(1;5) \subset R(1;5)$, $\Gamma_{\text{VI}}(1;2) \subset R(1;2)$, $\Gamma_{\text{VI}}(2;4) \subset R(2;4)$ and $\Gamma_{\text{VI}}(3;3) \subset R(3;3)$. Finally, we have $\Gamma_{\text{VI}}(1;3) \subset R(1;3)$, because we have

$$h\left(-1, \frac{\zeta}{2}\right)h(\zeta^2, 0)h\left(-1, \frac{1}{2}\right)h^{-1}(\zeta^2, 0) = [1, \sqrt{3}],$$

$$[1, \sqrt{3}]h^{-1}(\zeta^2, 0)h\left(-1, \frac{\zeta}{2}\right)h(\zeta^2, 0)h\left(-1, \frac{1}{2}\right)h\left(-1, \frac{\zeta}{2}\right)h\left(-1, \frac{1}{2}\right) = [0, 2\sqrt{3}],$$

$$[0, 2\sqrt{3}]h\left(-1, \frac{\zeta}{2}\right)h\left(-1, \frac{1}{2}\right) = [\zeta^2, \sqrt{3}],$$

$$h\left(-1, \frac{1}{2}\right)[1, \sqrt{3}]h^2(\zeta^2, 0) = [\zeta, 0, \sqrt{3}].$$

This completes the proof of Theorem 1.9.

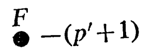
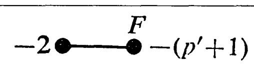
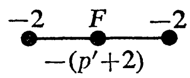
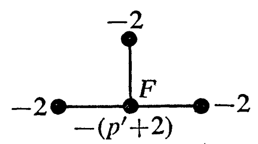
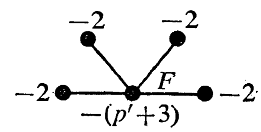
§ 2. Subgroup of Type II

In this section, we shall investigate the subgroup of **Type II**:

$$\Gamma_{\text{II}}(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon) = \sum_{\nu=1}^{2} \Gamma_1(\tau; p, r_1, r_\tau) A^\nu$$

where $A = [-1, 0, r]$. We shall first study the nonsingular model of the quotient space $D/\Gamma_{\text{II}}(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon) \cup \{P\}$. Let $\pi: M \to V$ be a resolution of a 2-dimentional analytic space V with a unique singular point $P \in V$. If $N = \pi^{-1}(P)$ consists of non-singularly embedded rational curves which intersect transversely and such that no three intersect at a point, it is customary to represent N by its dual weighted graph as follows. Let $\{N_j\}$ be the irreducible components of N. These N_j are the vertices of the graph. An edge connecting two vertices N_j and N_k corresponds to a point of intersection of N_j and N_k. Their intersection number is always $+1$, by the assumption. To each vertex N_j of the graph, we associate the self-intersection number N_j^2. We shall represent the resolution $\pi: M \to V$ by the weighted graph thus obtained, which will be called the graph of nonsingular model M of V.

Theorem 2.1. *For each $\Gamma = \Gamma_{\text{II}}(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)$, the quotient space $D/\Gamma \cup \{P\}$ has a singular point at most at P and has the nonsingular model S, of which graph is given by the following table.*

$\Gamma = \Gamma_{\text{II}}(\tau; p, \varepsilon_1, \varepsilon_2; \varepsilon)$	Graph of S
$\Gamma_{\text{II}}(\tau; 2p'+2, 0, 0; 0)$	
$\Gamma_{\text{II}}(\tau; 2p'+1, 0, 0; 0)$	
$\Gamma_{\text{II}}(\tau; 2p'+2, 1, 0; 0)$ $\Gamma_{\text{II}}(\tau; 2p'+2, 1, 1; 0)$	
$\Gamma_{\text{II}}(\tau; 2p'+1, 1, 1; 0)$	
$\Gamma_{\text{II}}(\tau; 2p'+2, 0, 0; 1)$	

Here F denotes the rational curve $F = E/\langle A \rangle$ and p' stands for nonnegative integer. Since every subgroup of **Type II** *is conjugate to one of the above groups by Proposition* 1.7, *we omitted the graphs of the remaining groups.*

Proof. Since $\Gamma_1 = \Gamma_1(\tau; p, r_1, r_2)$ is a normal subgroup of Γ, $A = [-1, 0, r]$ operates naturally on $\overline{D/\Gamma_1}$ as follows:

$$A: [w, u] \mapsto [e^{2\pi i(r/q)}w, -u].$$

In order to construct S, we shall resolve the singularities of $(\overline{D/\Gamma_1})/\langle A\rangle$, where $\langle A\rangle$ denotes the cyclic group generated by the operator A. The fixed points of A on the elliptic curve $E=\{[0, u] \in \overline{D/\Gamma_1}\}$ are $\{[0, \omega_j]\}_{j=0,1,2,3}$ where $\omega_0=0$, $\omega_1=1/2$, $\omega_2=(1+\tau)/2$ and $\omega_3=\tau/2$.

Lemma 2.2. There exist a neighborhood U_j of $[0, \omega_j]$ in $\overline{D/\Gamma_1}$ and a local parameter (s, t) around $[0, \omega_j]$ such that A operates on U_j as follows:

$$A: (s, t) \mapsto \left(s \exp 2\pi i\left\{\frac{r}{q}+\frac{r(2\omega_j)}{q}\right\}, -t\right).$$

Proof. Let $t=u-\omega_j$. Since we have

$$A[w, t+\omega_j]=[e^{2\pi i r/q}w, -t-\omega_j]$$
$$=[e^{2\pi i r/q}g(-t-\omega_j, 2\omega_j)w, -t+\omega_j],$$

and

$$g(-t-\omega_j, 2\omega_j)=\exp\frac{2\pi i}{q}\{2i(2\bar{\omega}_j)(-t-\omega_j)+r(2\omega_j)+i\,|2\omega_j|^2\}$$
$$=\exp\frac{8\pi\bar{\omega}_j t}{q}\cdot\exp 2\pi i\frac{r(2\omega_j)}{q},$$

we have only to put $s=w\exp\dfrac{4\pi\bar{\omega}_j t}{q}$. **Q.E.D.**

Corollary 2.3.

(i) *If* $r+r(2\omega_j)\equiv 0 \bmod q$, *then* $U_j/\langle A\rangle$ *is a nonsingular surface parametrized by* (s, t^2).

(ii) *If* $r+r(2\omega_j)\equiv q/2 \bmod q$, *then* $U_j/\langle A\rangle$, *with the unique singular point* $[0, \omega_j]$, *has a resolution which has the graph* $\underset{-2}{\bullet}$ *consisting of a single curve* C. *The nonsingular model of* $U_j/\langle A\rangle$ *is parametrized by* $(s/t, t^2)$ *and* $(t/s, s^2)$, *and the curve* C *is represented by* $t^2=0$.

Proof. Easy. See for instance [8] and [17]. **Q.E.D.**

Recall that

$$\frac{r}{q}=\frac{\varepsilon}{2},\ \frac{r(2\omega_0)}{q}=0,\ \frac{r(2\omega_1)}{q}=\frac{\varepsilon_1}{2},\ \frac{r(2\omega_3)}{q}=\frac{\varepsilon_2}{2}$$

and

$$\frac{r(2\omega_2)}{q}=\frac{r(1+\tau)}{q}$$
$$\equiv\frac{1}{q}(r(1)+r(\tau)+2\operatorname{Im}\tau) \bmod 1$$
$$=\frac{1}{2}(\varepsilon_1+\varepsilon_2+p) \bmod 1$$

and resolve the singularities of $(\overline{D/\Gamma_1})/\langle A\rangle$. Then, if we notice that $F=E/\langle A\rangle$ is nonsingularly embedded, we obtain the nonsingular model S of $D/\Gamma\cup\{P\}$ stated in the theorem. It remains to calculate the self-intersection number of F. Let K be the canonical line bundle of S. By the adjunction formular (see e.g. [7]), we have

$$F^2=-K\cdot F-2,$$

where $K\cdot F$ is the intersection number of K and F. If $g(u)$ is a section of the line bundle $\xi^{-1}(\Gamma_1)$, then $\omega=g(u)g(-u)\ (dw\wedge du)^{\otimes 2}$ is Γ-invariant, i.e., ω is a meromorphic section of $K^{\otimes 2}$. Since $2K\cdot F=K^{\otimes 2}\cdot F$ is equal to the degree of the divisor of ω on F, we shall find out zeros and poles of ω on F. $g(u)g(-u)$ has p zeros on F by Proposition 1.4. By Corollary 2.3, we conclude the followings:

(i) If $r+r(2\omega_j)\equiv 0 \bmod q$, then $(dw\wedge du)^{\otimes 2}$ has a simple pole at $[0,\omega_j]\in F$. Indeed, we have

$$(ds\wedge dt)^{\otimes 2}=\frac{1}{4y}(dx\wedge dy)^{\otimes 2},$$

where $(x, y)=(s, t^2)$ is the local parameter at $[0, \omega_j]$ in S.

(ii) If $r+r(2\omega_j)\equiv q/2 \bmod q$, then $(dw\wedge du)^{\otimes 2}$ is holomorphic and nowhere vanishing in the neighborhood of $[0,\omega_j]\in F$. In fact, for $(x, y)=(s/t, t^2)$, we have

$$(ds\wedge dt)^{\otimes 2}=\frac{1}{4}(dx\wedge dy)^{\otimes 2}.$$

The completion of the proof is now immediate. **Q.E.D.**

A nonsingularly embedded rational curve of self-intersection number -1 can be blow down to a point. Thus we have

Corollary 2.4. *For a group Γ of* **type II**, *$D/\Gamma\cup\{P\}$ is nonsingular if and only if Γ is conjugate to $\Gamma_{\text{II}}(\tau; 1, 0, 0; 0)$ or $\Gamma_{\text{II}}(\tau; 2, 0, 0; 0)$ for some τ.*

Remark 2.5. These two groups are generated by unitary reflections and have the following relation.

$$\Gamma_{\text{II}}(\tau; 2, 0, 0; 0)=\langle\Gamma_{\text{II}}(\tau; 1, 0, 0; 0), [0, 2\operatorname{Im}\tau]\rangle.$$

Next, for $\Gamma=\Gamma_{\mathrm{II}}(\tau;1,0,0;0)$, we shall express the natural map $D\mapsto D/\Gamma\cup\{P\}$ in terms of coordinates.

Lemma 2.6. *The even entire function* $g(u)=\vartheta_3(\tau;u)e^{(\pi/2\,\mathrm{Im}\,\tau)u^2}$ *is a section of the line bundle* $\xi^{-1}(\Gamma_1(\tau;1,0,0))$ *with zeros only at* $u\equiv\omega_2 \bmod L$, *where* $\vartheta_3(\tau;u)$ *is the even theta function defined by*

$$\vartheta_3(\tau;u):=\sum_{n=-\infty}^{\infty}\exp\pi i(\tau n^2+2nu).$$

Proof. By the definition of ξ, $g(u)$ is a section of ξ^{-1} if and only if

$$g(u+1)=\exp\left(\frac{2\pi}{4\,\mathrm{Im}\,\tau}(2u+1)\right)g(u),$$
$$g(u+\tau)=\exp\left(\frac{2\pi}{4\,\mathrm{Im}\,\tau}\bar{\tau}(2u+\tau)\right)g(u).$$

On the other hand, it is known that ϑ_3 has zeros only at $u\equiv\omega_2 \bmod L$ and satisfies

$$\vartheta_3(\tau;u+1)=\vartheta_3(\tau;u),$$
$$\vartheta_3(\tau;u+\tau)=e^{-\pi i(2u+\tau)}\vartheta_3(\tau;u).$$

Now the verification of $g(u)$ being a section of ξ^{-1} is a straightforward calculation.

Q.E.D.

Proposition 2.7. *There exists a system of local parameters* x, y *of* $S_0=D/\Gamma\cup\{P\}$ *around* P *such that the natural map* $D\to S_0$ *is expressed by*

$$x=g(u)\exp\left(\frac{\pi i}{2\,\mathrm{Im}\,\tau}z\right),$$
$$y=\left(g(u)\exp\left(\frac{\pi i}{2\,\mathrm{Im}\,\tau}z\right)\right)^2 f(u),$$

where $f(u)=(e_1-e_2)/(\wp(u)-e_2)$ *and* $\wp(u)=\wp(\tau;u)$ *is Weierstrass'* $\wp$*-function and* $e_j=\wp(\omega_j)$.

Proof. Since the proof is almost the same to that of [17]; Theorem 2, we shall skech the outline of the proof. Let S be the nonsingular model of $D/\Gamma\cup\{P\}$ constructed in Theorem 2.1 including the exceptional curve F and C with $FC=1$, $F^2=-1$, $C^2=-2$. We define the map ϕ: $\overline{D/\Gamma_1}\to \boldsymbol{P}^1\times \boldsymbol{C}$ by

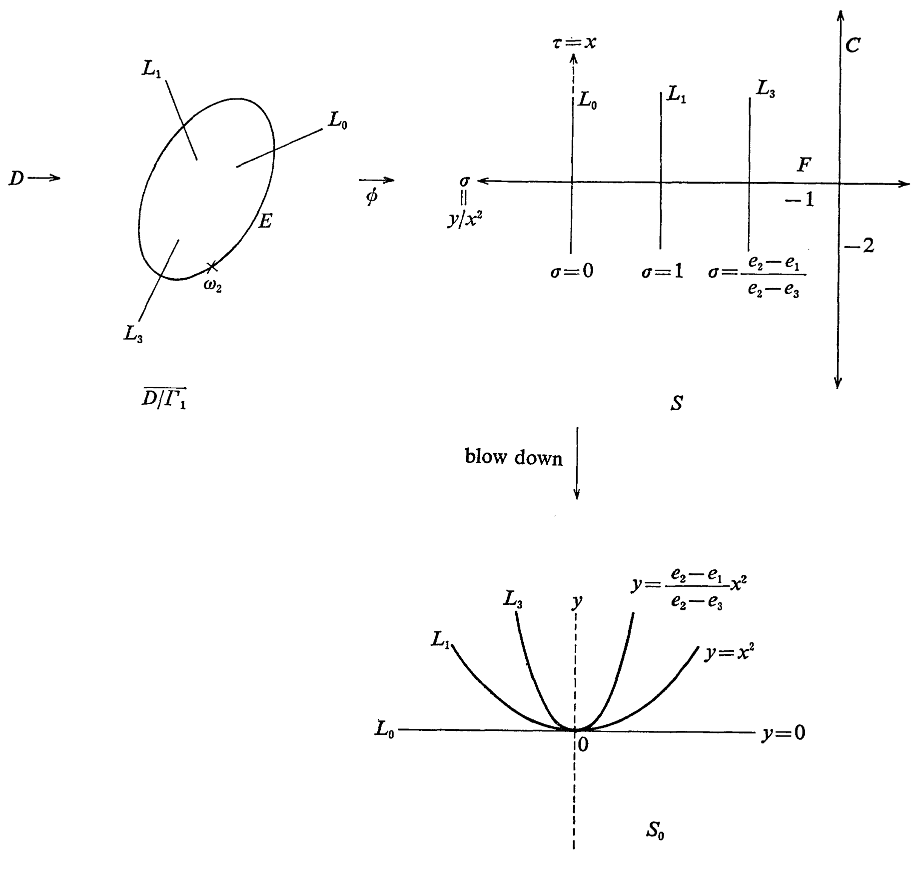
$$[w,u]\mapsto(\sigma,\tau)=(f(u),g(u)w).$$

On account of Lemma 2.6, we see that ϕ is well defined and A-invariant. Again by Lemma 2.6, in a sufficiently small neighborhood of $[0,\omega_2]\in\overline{D/\Gamma_1}$, ϕ is biholomorphically equivalent to the map

$$(s, t) \mapsto \left(\frac{1}{\sigma}, \tau\right) = (t^2, ts).$$

Thus by blowing up the range twice at $(\sigma, \tau) = (\infty, 0)$ as in [17]; Lemma 7.2, we obtain S. This shows that ϕ can be regarded as a rational map of $\overline{D/\Gamma_1}$ to S. The completion of the proof is now immediate. **Q.E.D.**

Let $L_0 = \{u=0\}$, $L_1 = \{u=\omega_1\}$, $L_3 = \{u=\omega_3\}$ be the centers of the generating unitary reflections of Γ. The same symbol L_j will denote the images of the line $\{u=\omega_j\}$ by the natural maps $D \to D/\Gamma_1$ and $D \to D/\Gamma$. Then the natural map $D \to S_0$ can be visualized by the following figure.



In figures hereafter, a line segment with two hooks denotes a nonsingular rational curve, and a numeral beside a curve denotes the self-intersection number of the curve.

Finally, for two groups $\Gamma_{\mathrm{II}}(\tau; 1, 0, 0; 0)$ and $\Gamma_{\mathrm{II}}(\tau; 2, 0, 0; 0)$, we construct the Fuchsian systems of differential equations.

Theorem 2.8. *The two systems* $(E_{\Gamma_{\mathrm{II}}(\tau;\,1,0,0;\,0)})$ *and* $(E_{\Gamma_{\mathrm{II}}(\tau;\,2,0,0;\,0)})$ *with the coefficients*

$$p_{11}^1=\left(\frac{1}{3}+k\right)\frac{x}{y-x^2}+\left(\frac{2}{3}+k\right)\frac{cx}{y+cx^2},$$
$$p_{22}^2=-\frac{1}{6y}+\left(\frac{1}{3}+\frac{k}{2}\right)\frac{1}{y-x^2}-\left(\frac{1}{6}+\frac{k}{2}\right)\frac{1}{y+cx^2},$$
$$p_{22}^1=\frac{1+k}{4}\frac{x}{y(y-x^2)}-\frac{k}{4}\frac{x}{y(y+cx^2)},$$
$$p_{11}^2=2k\frac{y}{y-x^2}+2(1+k)\frac{cy}{y+cx^2}$$

and

$$p_{11}^1=-\frac{1}{6x}+\left(\frac{1}{6}+\frac{k}{2}\right)\frac{1}{y-x}+\left(\frac{1}{3}+\frac{k}{2}\right)\frac{c}{y+cx},$$
$$p_{22}^2=-\frac{1}{6y}+\left(\frac{1}{3}+\frac{k}{2}\right)\frac{1}{y-x}-\left(\frac{1}{6}+\frac{k}{2}\right)\frac{1}{y+cx},$$
$$p_{22}^1=-\frac{1}{2}\left(1+k+\frac{k}{c}\right)\frac{1}{y}+\frac{1+k}{2}\frac{1}{y-x}+\frac{k}{2c}\frac{1}{y+cx},$$
$$p_{11}^2=\frac{1}{2}\left(1+k+\frac{k}{c}\right)\frac{c}{x}+\frac{k}{2}\frac{1}{y-x}-\frac{1+k}{2}\frac{c^2}{y+cx}$$

are completely integrable Fuchsian systerms of differential equations which have the projective monodromy groups $\Gamma_{\mathrm{II}}(\tau;1,0,0;0)$ *and* $\Gamma_{\mathrm{II}}(\tau;2,0,0;0)$ *respectively. Their singular points are only on* $\{y=0\}\cup\{y=x^2\}\cup\{y=-cx^2\}\cup H$ *and* $\{x=0\}\cup\{y=0\}\cup\{y=x\}\cup\{y=-cx\}\cup H$ *respectively, where* H *denotes the line at infinity in* $\boldsymbol{P}^2(\boldsymbol{C})$. *Here*

$$c=c(\tau)=\frac{e_1-e_2}{e_2-e_3},$$
$$k=k(\tau)=\frac{1}{e_1-e_3}\left(\frac{\pi}{\operatorname{Im}\tau}-e_1-2\eta_1\right),$$
$$e_j=e_j(\tau)=\wp(\tau;\omega_j),\qquad \eta_j=\eta_j(\tau)=\zeta(\tau;\omega_j)$$

and ζ *is the zeta function* (*the integral of* $\wp$*-function*).

Proof. We shall first calculate the four coefficients for $\Gamma=\Gamma_{\mathrm{II}}(\tau;1,0,0;0)$. By Proposition 2.7, one can easily check that [17]; Lemma 8.6 remains valid in this case.

Lemma 2.9 ([17], Lemma 8.6).

$$p_{11}^1=\frac{1}{x}-x\left(\frac{g'}{g}\right)'\left(\frac{\partial u}{\partial x}\right)^2+\frac{2}{3}\frac{f''}{f'}\frac{\partial u}{\partial x},$$

$$p_{22}^2=x\left(\frac{g'}{g}\right)'\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}-\frac{1}{3}\frac{f''}{f'}\frac{\partial u}{\partial y},$$

$$p_{22}^1=-x\left(\frac{g'}{g}\right)'\left(\frac{\partial u}{\partial y}\right)^2,$$

$$p_{11}^2=-\frac{2y}{x^2}-2y\left(\frac{g'}{g}\right)'\left(\frac{\partial u}{\partial x}\right)^2+x^2f'\frac{\partial^2 u}{\partial x^2},$$

where

$$g(u)=\vartheta_3(\tau;u)\exp\left(\frac{\pi u^2}{2\operatorname{Im}\tau}\right)$$

$$f(u)=(e_1-e_2)/(\wp-e_2).$$

Here a dash denotes the differentiation with respect to u.

Lemma 2.10.

(1) $\dfrac{\partial u}{\partial x}=2(e_1-e_2)\dfrac{x}{y}\dfrac{1}{\wp'}$,

(2) $\dfrac{\partial u}{\partial y}=-(e_1-e_2)\dfrac{x^2}{y^2}\dfrac{1}{\wp'}$,

(3) $f'=-\dfrac{1}{e_1-e_2}\dfrac{y^2}{x^4}\wp'$,

(4) $(\wp')^2=-4(e_1-e_2)^2(e_2-e_3)\dfrac{x^2}{y^3}(y-x^2)(y+cx^2)$,

(5) $\dfrac{\wp''}{(\wp')^2}=\dfrac{y}{2}\left(-\dfrac{1}{e_1-e_2}\dfrac{1}{y-x^2}+\dfrac{1}{e_1-e_2}\dfrac{1}{x}+\dfrac{1}{e_2-e_3}\dfrac{1}{y+cx^2}\right)$,

(6) $\dfrac{\partial^2 u}{\partial x^2}=2(e_1-e_2)\dfrac{x^2}{y}\left(\dfrac{1}{y-x^2}-\dfrac{c}{y+cx^2}\right)\dfrac{1}{\wp'}$,

(7) $\dfrac{f''}{f'}=\dfrac{1}{2}\dfrac{y}{e_1-e_2}\left(-\dfrac{1}{y-x^2}-3\dfrac{1}{x^2}+\dfrac{c}{y+cx^2}\right)\wp'$,

(8) $\left(\dfrac{g'}{g}\right)'=-\dfrac{1}{4(e_1-e_2)^2}\dfrac{y^2}{x^2}\left(\dfrac{1}{x^2}+\dfrac{1+k}{y-x^2}+\dfrac{kc}{y+cx^2}\right)(\wp')^2$.

Proof. By proposition 2.7, we have

$$\wp-e_2=(e_1-e_2)\frac{x^2}{y}.$$

Thus the equalities (1), (2) and (3) are obvious. The algebraic relation of $\wp$ and $\wp'$: $(\wp')^2=4(\wp-e_1)(\wp-e_2)(\wp-e_3)$ yields (4) and (5). (6) and (7) are easy consequences of (1), (3) and (5). To prove the last equality (8), apply the well known formulae

$$(\log \vartheta_3(u))''=-\wp(u+\omega_2)-\eta_1/\omega_1,$$

and

$$\wp(u+\omega_2)=\wp(u)-\frac{1}{2}\left(\frac{\wp''(u)}{\wp(u)-e_2}-\frac{(\wp'(u))^2}{(\wp(u)-e_2)^2}\right).$$

Then we have

$$\begin{aligned}(g'/g)'&=(\log \vartheta_3(u))''+\pi/\mathrm{Im}\,\tau\\&=(\wp')^2\left(-\frac{1}{2(\wp-e_2)^2}+\frac{1}{2(\wp-e_2)}\frac{\wp''}{(\wp')^2}+\frac{(-\wp+e_1)+(-e_1-2\eta_1+\pi/\mathrm{Im}\,\tau)}{(\wp')^2}\right).\end{aligned}$$

By (4) and (5), this is equal to

$$-\frac{y^2(\wp')^2}{4(e_1-e_2)^2x^2}\left(\frac{1}{x^2}+\frac{1}{y-x^2}+\frac{(e_1-e_3)ky}{(e_2-e_3)(y-x^2)(y+cx^2)}\right).$$

Thus if we notice that

$$\frac{(e_1-e_3)y}{(e_2-e_3)(y-x^2)(y+cx^2)}=\frac{1}{y-x^2}+\frac{c}{y+cx^2},$$

we have (8). **Q.E.D.**

By Lemma 2.9 and Lemma 2.11, we obtain the desired coefficients.

Next, we shall calculate the four coefficients for $\Gamma'=\Gamma_{\mathrm{II}}(\tau; 2, 0, 0; 0)$. Remark 2.5 implies the equality

$$D/\Gamma'=(D/\Gamma)/\langle[0, 2\,\mathrm{Im}\,\tau]\rangle.$$

Since the element $[0, 2\,\mathrm{Im}\,\tau]$ operates on D as $(z, u)\mapsto(z+2\,\mathrm{Im}\,\tau, u)$, using the expression of x, y in Proposition 2.8, we see that $[0, 2\,\mathrm{Im}\,\tau]$ operates on D/Γ as $(x, y)\mapsto(-x, y)$. Thus we can choose the local parameter (x', y') of $D/\Gamma'\cup\{P\}$ at P so that $(x', y')=(x^2, y)$. On the other hand, we have the following formula.

Lemma 2.11 ([18]; (2.2)). *If $x'=x^n$ and $y'=y^m$, then we have*

$$S_{11}^1(z, u; x', y')=-\frac{1}{3}\frac{n-1}{n}\frac{1}{x'}+S_{11}^1(z, u; x, y)\frac{1}{nx^{n-1}},$$

$$S_{22}^2(z, u; x', y')=-\frac{1}{3}\frac{m-1}{m}\frac{1}{y'}+S_{22}^2(z, u; x, y)\frac{1}{my^{m-1}},$$

$$S_{22}^{1}(z, u; x', y') = S_{22}^{1}(z, u; x, y)\frac{nx^{n-1}}{m^2 y^{2m-2}},$$

$$S_{11}^{2}(z, u; x', y') = S_{11}^{2}(z, u; x, y)\frac{my}{n^2 x^{2n-2}}.$$

If we put $n=2$ and $m=0$ in Lemma 2.11, then the coefficients of $(E_{\Gamma_{\mathrm{II}}(\tau;\, 2,\, 0,\, 0;\, 0)})$ are calculated by those of $(E_{\Gamma_{\mathrm{II}}(\tau;\, 1,\, 0,\, 0;\, 0)})$ obtained above.

It remains to prove that the two systems thus obtained are Fuchsian. For the system $(E_{\Gamma_{\mathrm{II}}(\tau;\, 2,\, 0,\, 0;\, 0)})$, at any point of $\boldsymbol{P}^2(\boldsymbol{C})$ other than the origin 0, we can easily check the hypotheses of [17]; Proposition 8.1, 8.2 and Corollary 8.3 and prove that it is regular singular. Moreover, we can prove that the system is regular singular at 0, thus Fuchsian, by making analogous consideration to the proof of [17]; Theorem 3. This and the argument above imply that $(E_{\Gamma_{\mathrm{II}}(\tau;\, 1,\, 0,\, 0;\, 0)})$ is also Fuchsian. **Q.E.D.**

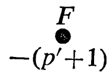
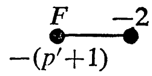
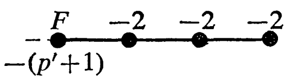
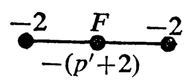
§ 3. Subgroup of Type IV

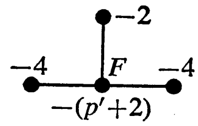
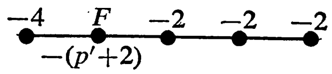
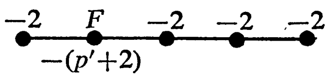
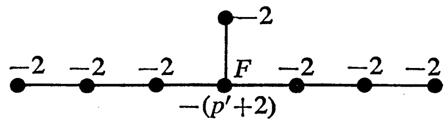
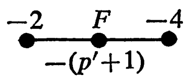
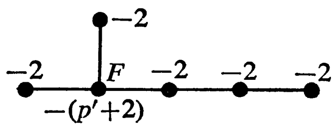
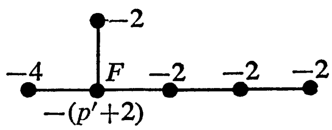
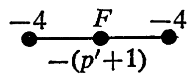
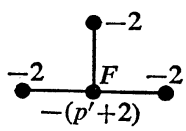
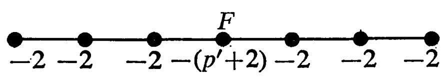
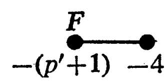
In this section, we shall study the subgroups of **Type IV**:

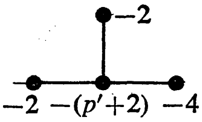
$$\Gamma_{\mathrm{IV}}(p, \varepsilon'; \varepsilon) = \sum_{\nu=1}^{4} \Gamma_1(i; p, r', r')A^{\nu}$$

where $A = [i, 0, r]$.

Theorem 3.1. *The following is the table of every subgroup Γ, up to conjugacy, of* **Type IV** *and its graph of nonsingular model S of the space $D/\Gamma \cup \{P\}$.*

$\Gamma = \Gamma_{\mathrm{IV}}(p, \varepsilon'; \varepsilon)$	Graph of S
$\Gamma_{\mathrm{IV}}(4p'+4, 0; 0)$	
$\Gamma_{\mathrm{IV}}(4p'+2, 0; 0)$ $\Gamma_{\mathrm{IV}}(4p'+2, 1; 0)$	
$\Gamma_{\mathrm{IV}}(4p'+1, 0; 0)$	
$\Gamma_{\mathrm{IV}}(4p'+4, 1; 0)$	

$\Gamma=\Gamma_{\mathrm{IV}}(p, \varepsilon'; \varepsilon)$	Graph of S
$\Gamma_{\mathrm{IV}}(4p'+4, 0; 1)$	
$\Gamma_{\mathrm{IV}}(4p'+4, 1; 1)$	
$\Gamma_{\mathrm{IV}}(4p'+4, 0; 2)$ $\Gamma_{\mathrm{IV}}(4p'+3, 0; 2)$ $\Gamma_{\mathrm{IV}}(4p'+3, 0; 3)$	
$\Gamma_{\mathrm{IV}}(4p'+4, 0; 3)$	
$\Gamma_{\mathrm{IV}}(4p'+1, 1; 0)$ $\Gamma_{\mathrm{IV}}(4p'+1, 1; 1)$	
$\Gamma_{\mathrm{IV}}(4p'+1, 1; 2)$	
$\Gamma_{\mathrm{IV}}(4p'+2, 0; 1)$	
$\Gamma_{\mathrm{IV}}(4p'+2, 1; 1)$	
$\Gamma_{\mathrm{IV}}(4p'+2, 1; 2)$	
$\Gamma_{\mathrm{IV}}(4p'+2, 1; 3)$	
$\Gamma_{\mathrm{IV}}(4p'+3, 0; 0)$	

$\Gamma=\Gamma_{\mathrm{IV}}(p, \varepsilon'; \varepsilon)$	Graph of S
$\Gamma_{\mathrm{IV}}(4p'+3, 0; 1)$	

Here F denotes the rational curve $E/\langle A\rangle$ *and* p' *stands for nonnegative integer.*

Proof. $A=[i, 0, r]$ operates on $\overline{D/\Gamma_1}$ as follows:

$$A: [w, u]\mapsto[e^{2\pi i r/q}w, iu].$$

The fixed points of $\langle A\rangle$ in $E=\{[0, u]\}\subset D/\Gamma_1$ are $[0, \omega_0]$, $[0, \omega_2]$ of order 4 and $[0, \omega_1]$, $[0, \omega_3]$ of order 2, where $\omega_0=0$, $\omega_1=1/2$, $\omega_2=(1+i)/2$ and $\omega_3=i/2$. Since A maps a neighborhood of $[0, \omega_1]$ in $\overline{D/\Gamma_1}$ biholomorphically to that of $[0, \omega_3]$, we shall study local behavior of $\langle A\rangle$ in the neighborhood of $[0, \omega_j]$ $(j=0, 1, 2)$.

Lemma 3.2. *There exist a neighborhood* U_j *of* $[0, \omega_j]$ *in* $\overline{D/\Gamma_1}$ *and a local parameter* (s, t) *around* $[0, \omega_j]$ *such that* A *operates on* U_j *as follows*:

$$A\ : (s, t)\mapsto(i^{\varepsilon}s, it) \qquad in\ U_0,$$
$$A^2: (s, t)\mapsto((-1)^{(\varepsilon+\varepsilon')}s, -t) \qquad in\ U_1,$$
$$A\ : (s, t)\mapsto(i^{(\varepsilon+2\varepsilon'-p)}s, it) \qquad in\ U_2.$$

Proof. If we notice that $r/q=\varepsilon/4$,

$$g(-\omega_1, 1)=\exp((2\pi i/q)(2i(-1/2)+r(1)+1))=\exp(2\pi i/2)\varepsilon'$$

and

$$g(i\omega_2, 1)=\exp((2\pi i/q)(2i\cdot i\omega_2+r(1)+i))=\exp(2\pi i/4)(2\varepsilon'-p),$$

then the proof is analogous to that of Lemma 2.2. **Q.E.D.**

Lemma 3.3. *Let* $\mathfrak{A}_\nu$ $(\nu=0, 1, 2, 3)$ *be the transformation group on* $X=\boldsymbol{C}^2$ *generated by an operation*

$$(s, t)\mapsto(i^{\nu}s, it),$$

ρ *be the natural map* $X\to V_\nu=X/\mathfrak{A}_\nu$ *and*

$$\pi: M_\nu\to V_\nu$$

be the minimal resolution. (V_ν *has a unique singularity at the origin* 0 *if* $\nu\neq0$ *and we regard* $M_\nu=V_\nu$ *if* $\nu=0$.) *For* $Y=\{(0, t)\}\subset X$, *we define* $\tilde{Y}$ *to be the closure of*

$\pi^{-1}\circ\rho(Y-\{0\})$ in M_ν, which is a nonsingular curve in M_ν, and d_ν the degree of zero of $\pi^*(ds\wedge dt)$ on $\tilde{Y}$. Then we have the following table of the graph of M_ν and the number d_ν.

ν	0	1	2	3
Graph of M_ν	/	• −4	• −2	•—•—• −2 −2 −2
d_ν	−3	−2	−1	0

Proof. Easy. **Q.E.D.**

Now we are ready to prove Theorem 3.1. Resolve the singularity of $(\overline{D/\Gamma_1})/\langle A\rangle$ by using Lemmas 3.2 and 3.3 then we have the nonsingular model S. The self-intersection number of $F=E/\langle A\rangle$ is obtained by calculating the degree of zeros on F of the form

$$\prod_{\nu=1}^{4} g(i^\nu u)(dw\wedge du)^{\otimes 4},$$

where $g(u)$ is a nonzero section of the line bundle $\xi^{-1}(\Gamma_1)$. Since the proof is similar to that of Theorem 2.1, we omit the details. **Q.E.D.**

Corollary 3.4. *For a group Γ of* **Type IV**, *the space $D/\Gamma\cup\{P\}$ is nonsingular if and only if Γ is conjugate to one of the following four groups*: $\Gamma_{\text{IV}}(4,0;0)$, $\Gamma_{\text{IV}}(2,0;0)$, $\Gamma_{\text{IV}}(2,1;0)$, $\Gamma_{\text{IV}}(1,0;0)$.

Remark 3.5. Four groups above are generated by unitary reflections and have the following relations:

$$\Gamma_{\text{IV}}(4,0;0)=\langle\Gamma_{\text{IV}}(2,1;0),[-1,1,0]\rangle,$$
$$\Gamma_{\text{IV}}(1,0;0)=\langle\Gamma_{\text{II}}(i;1,0,0;0),[i,0,0]\rangle,$$
$$\Gamma_{\text{IV}}(2,0;0)=\langle\Gamma_{\text{IV}}(1,0;0),[0,2]\rangle.$$

Since we have already constructed the system (E_Γ) with the projective monodromy group $\Gamma=\Gamma_{\text{IV}}(2,1;0)$ in [17], we shall construct three systems with the projective monodromy groups $\Gamma_{\text{IV}}(4,0;0)$, $\Gamma_{\text{IV}}(1,0;0)$ and $\Gamma_{\text{IV}}(2,0;0)$.

Theorem 3.6. *For each $\Gamma=\Gamma_{\text{IV}}(4,0;0)$, $\Gamma_{\text{IV}}(1,0;0)$, $\Gamma_{\text{IV}}(2,0;0)$, there exists a Fuchsian system (E_Γ) which has the projective monodromy group Γ. The singular locus Z_Γ in the finite plane and the four coefficients of (E_Γ) are given in the following table.*

Γ	Four coefficients of (E_Γ): p^1_{11}	p^2_{11}	Defining equation of Z_Γ
	p^1_{22}	p^2_{22}	
$\Gamma_{IV}(4,0;0)$	$-\dfrac{1}{6x}$	$\dfrac{5y}{4x(y-x)}$	$xy(y-x)=0$
	$\dfrac{x}{2y(y-x)}$	$\dfrac{x}{4y(y-x)}$	
$\Gamma_{IV}(1,0;0)$	$\dfrac{x^3}{3(x^4-y)}$	$\dfrac{x^2y}{x^4-y}$	$y(y-x^4)=0$
	$-\dfrac{x}{16y(x^4-y)}$	$\dfrac{2y-3x^4}{4(x^4-y)}$	
$\Gamma_{IV}(2,0;0)$	$\dfrac{y}{6x(x^2-y)}$	$\dfrac{y}{4(x^2-y)}$	$xy(y-x^2)=0$
	$-\dfrac{x}{8y(x^2-y)}$	$\dfrac{2y-3x^2}{12(x^2-y)}$	

Proof. (i) Construction of $(E_{\Gamma_{IV}(4,0,0)})$. It was proved ([17]; Theorem 2) that the local coordinate (x, y) of $D/\Gamma_{IV}(2,1;0)\cup\{P\}$ at P is given by

$$x=\vartheta(u)e^{\pi iz},$$
$$y=(\vartheta(u)e^{\pi iz})^2\frac{-e_1^2}{\wp^2(i;u)-e_1^2}$$

where $\vartheta(u)=\vartheta_0(i;u)\vartheta_0(i;iu)$. On the other hand, $[-1,1,0]$ operates on D as follows.

$$(z,u)\mapsto(z-2iu+i,\ -u+1).$$

If we use the relations $\vartheta(-u)=\vartheta(u)$ and $\vartheta(u+1)=-e^{\pi(2u+1)}\vartheta(u)$, we see that $[-1,1,0]$ operates on $D/\Gamma_{IV}(2,1;0)\cup\{P\}$ as

$$(x,y)\mapsto(-x,y).$$

Thus, by Remark 3.5 and Lemma 2.11, we can calculate the coefficients of $(E_{\Gamma_{IV}(4,0;0)})$ by those of $(E_{\Gamma_{IV}(2,1;0)})$ which are already known ([17]; Theorem 3).

(ii) Construction of $(E_{\Gamma_{IV}(1,0;0)})$.

Lemma 3.7. *$g(u)=\vartheta_3(i;u)e^{\pi u^2/2}$ is i-invariant, i.e., $g(iu)=g(u)$.*

Proof. Let $H^0(E, \Omega(\xi^{-1}))$ be the linear space of holomorphic sections of the line bundle $\xi^{-1}(\Gamma_1(i; 1, 0, 0))$. By Lemma 2.7, we have $g(u) \in H^0(E, \Omega(\xi^{-1}))$. Moreover, the equality $g(iu, i\eta)=g(u, \eta)$ implies $\tilde{g}(u):=g(iu) \in H^0(E, \Omega(\xi^1))$. On the other hand, we have $\dim H^0(E, \Omega(\xi^{-1}))=1$ by Proposition 1.4. Since $\tilde{g}(0)=g(0)$, these imply $\tilde{g}(u)=g(u)$. **Q.E.D.**

Let (x, y) be the local parameter of $S_0=D/\Gamma_{\text{II}}(i; 1, 0, 0; 0) \cup \{P\}$ obtained in Proposition 2.7. If we notice that $\wp(i; iu)=-\wp(i, u)$ and $e_2(i)=0$, then Proposition 2.7 and Lemma 3.7 imply that $[i, 0, 0] \in \Gamma_{\text{IV}}(1, 0; 0)$ operates on S_0 as

$$(x, y) \mapsto (x, -y).$$

Thus again by Remark 3.5 and Lemma 2.11, we obtain the coefficients of $(E_{\Gamma_{\text{IV}}(1, 0; 0)})$ from those of $(E_{\Gamma_{\text{II}}(i; 1, 0, 0; 0)})$ in Theorem 2.8. If we notice that $\eta_3(i)=-i\eta_1(i)$ then by Legendre's relation, we have $\eta_1(i)=\pi/2$. This yields the desired results.

(iii) Construction of $(E_{\Gamma_{\text{IV}}(2, 0; 0)})$. In (ii), we showed that

$$x'=g(u)e^{\pi iz/2}$$

$$y'=(g(u)e^{\pi iz/2})^4\frac{e_1^2}{\wp^2(i; u)}$$

is a system of local parameters of $D/\Gamma_{\text{IV}}(1, 0; 0) \cup \{P\}$ at P. Thus the element $[0, 2] \in \Gamma_{\text{IV}}(2, 0; 0)$ operates on $D/\Gamma_{\text{IV}}(1, 0; 0) \cup \{P\}$ as

$$(x', y') \mapsto (-x', y').$$

Therefore, Remark 3.5 and Lemma 2.11 lead to the conclusion.

Three systems thus obtained are Fuchsian since, we know that $(E_{\Gamma_{\text{IV}}(2, 1; 0)})$ and $(E_{\Gamma_{\text{II}}(i, 1, 0; 0; 0)})$ are Fuchsian. **Q.E.D.**

§ 4. Subgroups of Type III and Type VI

In this section, we shall study the subgroups of **Type III**:

$$\Gamma_{\text{III}}(p; \varepsilon)=\sum_{\nu=1}^{3} \Gamma_{\text{I, III}} A^{2\nu},$$

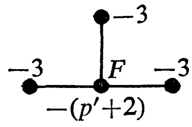
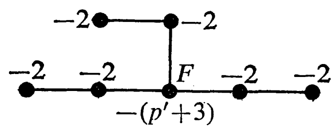
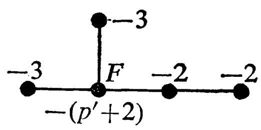
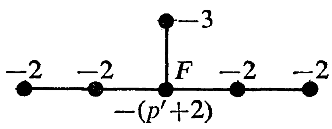
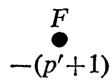
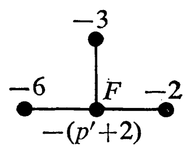
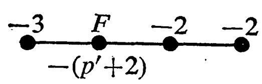
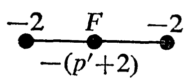
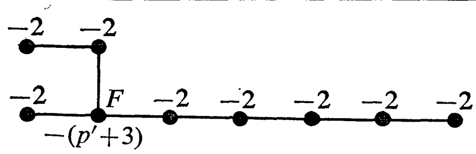
and **Type VI**:

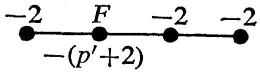
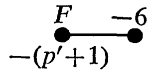
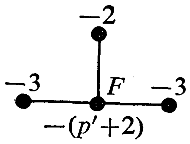
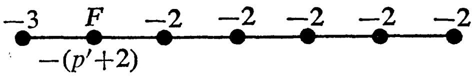
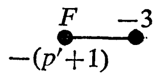
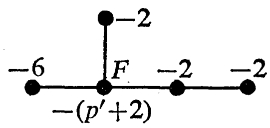
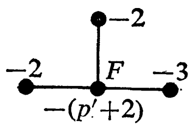
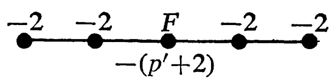
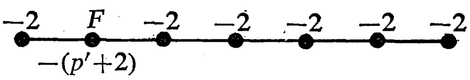
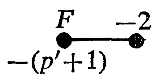
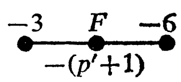
$$\Gamma_{\text{VI}}(p; \varepsilon)=\sum_{\nu=1}^{6} \Gamma_{\text{I,VI}} A^{\nu}$$

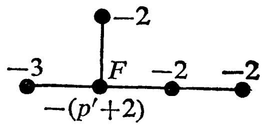
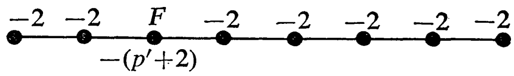
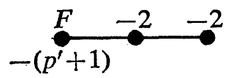
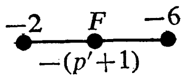
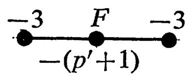
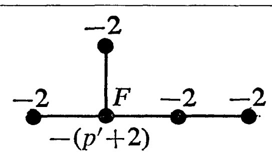
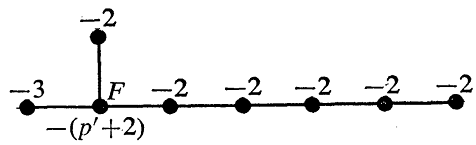
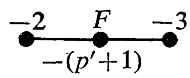
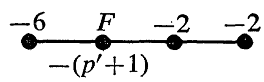
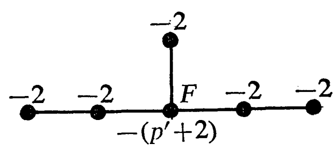
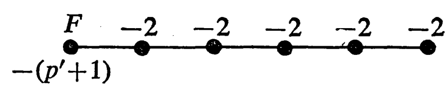
where $\Gamma_{\text{I, III}}=\Gamma_{\text{I, III}}(p)=\Gamma_1(\zeta; p, 1/\sqrt{3}, -1/\sqrt{3})$, $\Gamma_{\text{1, VI}}=\Gamma_{\text{1, VI}}(p)=\Gamma_1(\zeta; p, \sqrt{3},$

$-\sqrt{3}$), $A=[\zeta, 0, r]$, $r=\varepsilon/\sqrt{3}\,p$, and the index ε in $\Gamma_{\text{III}}(p, \varepsilon)$ (resp. $\Gamma_{\text{VI}}(p, \varepsilon)$) takes the values $\varepsilon=0, 1, 2$ (resp. $\varepsilon=0, 1, 2, 3, 4, 5$).

Theorem 4.1. *The graph of nonsingular model S of the space $D/\Gamma \cup \{P\}$, for each subgroup Γ of* **Type III** *and* **Type VI**, *is given by the following table.*

Γ	Graph of S
$\Gamma_{\text{III}}(3p'+3; 1)$	
$\Gamma_{\text{III}}(3p'+3; 2)$	
$\Gamma_{\text{III}}(3p'+2; 1)$	
$\Gamma_{\text{III}}(3p'+1; 2)$	
$\Gamma_{\text{III}}(3p'+3; 0)$ $\Gamma_{\text{VI}}(6p'+6; 0)$	
$\Gamma_{\text{VI}}(6p'+6; 1)$	
$\Gamma_{\text{VI}}(6p'+6; 2)$ $\Gamma_{\text{VI}}(6p'+6; 4)$	
$\Gamma_{\text{VI}}(6p'+6; 3)$	
$\Gamma_{\text{VI}}(6p'+6; 5)$	

<table>
<tr><th>Γ</th><th>Graph of S</th></tr>
<tr><td>$\Gamma_{\mathrm{VI}}(6p'+5;0)$
$\Gamma_{\mathrm{VI}}(6p'+5;3)$
$\Gamma_{\mathrm{VI}}(6p'+5;4)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{VI}}(6p'+5;1)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{VI}}(6p'+5;2)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{VI}}(6p'+5;5)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{III}}(3p'+2;0)$
$\Gamma_{\mathrm{VI}}(6p'+4;0)$
$\Gamma_{\mathrm{VI}}(6p'+4;2)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{VI}}(6p'+4;1)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{VI}}(6p'+4;3)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{III}}(3p'+2;2)$
$\Gamma_{\mathrm{VI}}(6p'+4;4)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{VI}}(6p'+4;5)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{VI}}(6p'+3;0)$
$\Gamma_{\mathrm{VI}}(6p'+3;3)$</td><td>
</td></tr>
<tr><td>$\Gamma_{\mathrm{VI}}(6p'+3;1)$</td><td>
</td></tr>
</table>

Γ	Graph of S
$\Gamma_{\mathrm{VI}}(6p'+3;2)$ $\Gamma_{\mathrm{VI}}(6p'+3;4)$	
$\Gamma_{\mathrm{VI}}(6p'+3;5)$	
$\Gamma_{\mathrm{III}}(3p'+1;0)$ $\Gamma_{\mathrm{VI}}(6p'+2;0)$ $\Gamma_{\mathrm{VI}}(6p'+2;4)$	
$\Gamma_{\mathrm{VI}}(6p'+2;1)$	
$\Gamma_{\mathrm{III}}(3p'+1;1)$ $\Gamma_{\mathrm{VI}}(6p'+2;2)$	
$\Gamma_{\mathrm{VI}}(6p'+2;3)$	
$\Gamma_{\mathrm{VI}}(6p'+2;5)$	
$\Gamma_{\mathrm{VI}}(6p'+1;0)$ $\Gamma_{\mathrm{VI}}(6p'+1;2)$ $\Gamma_{\mathrm{VI}}(6p'+1;3)$	
$\Gamma_{\mathrm{VI}}(6p'+1;1)$	
$\Gamma_{\mathrm{VI}}(6p'+1;4)$	
$\Gamma_{\mathrm{VI}}(6p'+1;5)$	

Here F denotes the rational curve $E/\langle A^2\rangle$ for the group of **Type III** *and $E/\langle A\rangle$ for the group of* **Type VI**, *and p' stands for nonnegative integer.*

Proof. A operates naturally on $\overline{D/\Gamma_{1,\mathrm{VI}}}$ as

$$A: [w, u] \mapsto [e^{2\pi i r/q} w, \zeta u].$$

As is easily seen, every fixed point of $\langle A\rangle$ in $E=\{[0, u]\}\subset\overline{D/\Gamma_{1,\mathrm{VI}}}$ is $\langle A\rangle$-conjugate to one of the three fixed points: $[0, 0]$ of order 6, $[0, 1/2]$ of order 2 and $[0, (1+\zeta)/3]$ of order 3. Similarly, A^2 operates on $\overline{D/\Gamma_{1,\mathrm{III}}}$ and the fixed points of $\langle A^2\rangle$ in $E\subset\overline{D/\Gamma_{1,\mathrm{III}}}$ are $[0, 0]$, $[0, (1+\zeta)/3]$ and $[0, (\zeta+\zeta^2)/3]$ of order 3. We shall study the local behavior of $\langle A\rangle$ and $\langle A^2\rangle$ in the neighborhood of each fixed point.

Lemma 4.2. *There exist neighborhoods U_ρ of $[0, \rho]$ ($\rho=0, 1/2, (1+\zeta)/3$) in $\overline{D/\Gamma_{1,\mathrm{VI}}}$ and V_σ of $[0, \sigma]$ ($\sigma=0, (1+\zeta)/3, (\zeta+\zeta^2)/3$) in $\overline{D/\Gamma_{1,\mathrm{III}}}$ and a local parameter (s, t) in each U_ρ and V_σ such that $\langle A\rangle$ and $\langle A^2\rangle$ operate on U_ρ and V_σ respectively as follows:*

$$A : (s, t) \mapsto (\zeta^\varepsilon s, \zeta t) \qquad \text{in } U_0,$$
$$A^3: (s, t) \mapsto ((-1)^{(p+\varepsilon)} s, -t) \qquad \text{in } U_{1/2},$$
$$A^2; (s, t) \mapsto ((\zeta^2)^{(p+\varepsilon)} s, \zeta^2 t) \qquad \text{in } U_{(1+\zeta)/3}$$

and

$$A^2: (s, t) \mapsto ((\zeta^2)^\varepsilon s, \zeta^2 t) \qquad \text{in } V_0 \text{ and } V_{(1+\zeta)/3},$$
$$A^2: (s, t) \mapsto ((\zeta^2)^{(\varepsilon-p)} s, \zeta^2 t) \qquad \text{in } V_{(\zeta+\zeta^2)/3}.$$

Proof. Analogous to those of Lemma 2.2 and Lemma 3.2. **Q.E.D.**

Lemma 4.3. *Let $\mathfrak{A}_{\nu\mu}$ ($\nu=0, 1, \cdots, 5, \mu=1, 2$) be the transformation group operating on $X=\boldsymbol{C}^2$ generated by*

$$(s, t) \mapsto (\zeta^\nu s, \zeta^\mu t)$$

ρ be the natural map $X\to V_{\nu\mu}=X/\mathfrak{A}_{\nu\mu}$ and

$$\pi: M_{\nu\mu}\to V_{\nu\mu}$$

be the minimal resolution. ($V_{\nu\mu}$ has a unique singularity at the origin 0 if $\nu\neq 0$ and we regard $M_{\nu\mu}=V_{\nu\mu}$ if $\nu=0$.) Put $Y=\{(0, t)\}\subset X$ and let $\tilde{Y}_{\nu\mu}$ be the closure of $\pi^{-1}\circ\rho(Y-\{0\})$ in $M_{\nu\mu}$, $d_{\nu\mu}$ the degree of zero of $\pi^*(ds\wedge dt)^{\otimes 6}$ on $\tilde{Y}_{\nu\mu}$ and $d'_{\nu 2}$ the degree of zero of $\pi^*(ds\wedge dt)^{\otimes 3}$ on $\tilde{Y}_{\nu 2}$. Then we have the following table of the graph of $M_{\nu\mu}$ and the numbers $d_{\nu\mu}$ and $d'_{\nu 2}$.

(ν, μ)	(0, 1)	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)
Graph of $M_{\nu\mu}$		● −6	● −3	● −2	●—● −2 −2	●—●—●—●—● −2 −2 −2 −2 −2
$d_{\nu\mu}$	−5	−4	−3	−2	−1	0

(ν, μ)	(0, 2)	(2, 2)	(4, 2)
Graph of $M_{\nu\mu}$		● −3	●—● −2 −2
$d_{\nu\mu}$	−4	−2	0
$d'_{\nu 2}$	−2	−1	0

Proof. Easy. **Q.E.D.**

By the lemmas above, we can prove the theorem. Since the proof is similar to that of Theorem 2.1, we omit it. **Q.E.D.**

Corollary 4.4. *For a group Γ of* **Type III** *and* **Type VI**, the space $D/\Gamma\{P\}$ *is non singular if and only if Γ is equal to the following eleven groups*: $\Gamma_{\mathrm{III}}(1;0)$, $\Gamma_{\mathrm{III}}(3;0)$, $\Gamma_{\mathrm{VI}}(1;0)$, $\Gamma_{\mathrm{VI}}(2;0)$, $\Gamma_{\mathrm{VI}}(3;0)$, $\Gamma_{\mathrm{VI}}(6;0)$, $\Gamma_{\mathrm{VI}}(1;3)$, $\Gamma_{\mathrm{VI}}(3;3)$, $\Gamma_{\mathrm{VI}}(1;5)$, $\Gamma_{\mathrm{VI}}(2;4)$, $\Gamma_{\mathrm{VI}}(1;2)$.

Remark 4.5. The eleven groups in the corollary are generated by unitary reflections and have the following relations.

$$\Gamma_{\mathrm{III}}(3;0)=\left\langle \Gamma_{\mathrm{III}}(1;0),\ \left[0, \frac{2}{3}\sqrt{3}\right]\right\rangle,$$

$$\Gamma_{\mathrm{VI}}(6;0)=\left\langle \Gamma_{\mathrm{VI}}(1;0),\ \left[0, \frac{\sqrt{3}}{3}\right]\right\rangle,$$

$$\Gamma_{\mathrm{VI}}(3;0)=\left\langle \Gamma_{\mathrm{VI}}(1;0),\ \left[0, \frac{2}{3}\sqrt{3}\right]\right\rangle,$$

$$\Gamma_{\mathrm{VI}}(2;0)=\langle \Gamma_{\mathrm{VI}}(1;0),\ [0, \sqrt{3}]\rangle,$$

$$\Gamma_{\mathrm{VI}}(3;3)=\left\langle \Gamma_{\mathrm{VI}}(1;3),\ \left[0, \frac{2}{3}\sqrt{3}\right]\right\rangle,$$

$$\Gamma_{\mathrm{VI}}(2;4)=\langle \Gamma_{\mathrm{VI}}(1;5),\ [0, \sqrt{3}]\rangle.$$

We shall express the natural map $D\to D/\Gamma\cup\{P\}$ in terms of coordinates (Proposition 4.8) for the groups $\Gamma=\Gamma_{\mathrm{III}}(1;0)$, $\Gamma_{\mathrm{VI}}(1;0)$, $\Gamma_{\mathrm{VI}}(1;2)$, $\Gamma_{\mathrm{VI}}(1;3)$ and

$\Gamma_{\mathrm{VI}}(1;5)$. Note that these are the groups situated at the bottom of the tree in Remark 1.11.

Lemma 4.6. *Let $\wp(u)=\wp(\zeta^2:u)$ be Weierstrass' $\wp$-function with respect to the lattice $L=\boldsymbol{Z}+\zeta^2\boldsymbol{Z}$. Then we have*

$$\wp(\zeta u)=-\zeta\wp(u),$$
$$\wp'(\zeta u)=-\wp'(u),$$
$$\operatorname{div}(\wp)=-2(0)+\left(\frac{1+\zeta}{3}\right)+\left(\frac{\zeta+\zeta^2}{3}\right),$$
$$\operatorname{div}(\wp')=-3(0)+\left(\frac{1}{2}\right)+\left(\frac{\zeta}{2}\right)+\left(\frac{\zeta^2}{2}\right),$$

where div (f) *denotes the divisor of f on $\boldsymbol{C}/L$.*

Proof. Well known except for the zeros of $\wp(u)$. Let q_1 and q_2 be the zeros of $\wp$ in the fundamental paralleogram of L. Since ζq_1 and ζq_2 are also the zeros of $\wp(u)$, we have

$$\zeta q_1\equiv q_2 \quad \text{and} \quad \zeta q_2\equiv q_1 \quad \operatorname{mod} L.$$

On the other hand, we have $q_1+q_2\equiv 0 \operatorname{mod} L$ (Abel's theorem). Now, one can easily solve these equalities to obtain the zeros of $\wp(u)$. **Q.E.D.**

Lemma 4.7. *Let ξ_{III} and ξ_{VI} denote the line bundles $\xi(\Gamma_{1,\mathrm{III}}(1))$ and $\xi(\Gamma_{1,\mathrm{VI}}(1))$ respectively. Then*

$$\vartheta_{\mathrm{III}}(u)=\exp\left(\frac{\pi}{\sqrt{3}}u^2-\frac{4}{3}\pi i u\right)\vartheta_1\left(u-\frac{\zeta+\zeta^2}{3}\right)$$

and

$$\vartheta_{\mathrm{VI}}(u)=\exp\left(\frac{\pi}{\sqrt{3}}u^2\right)\vartheta_1(u)$$

are holomorphic sections of ξ_{III}^{-1} and ξ_{VI}^{-1} respectively and satisfy

$$\vartheta_{\mathrm{III}}(\zeta^2 u)=\vartheta_{\mathrm{III}}(u), \qquad \operatorname{div}(\vartheta_{\mathrm{III}})=\left(\frac{\zeta+\zeta^2}{3}\right),$$
$$\vartheta_{\mathrm{VI}}(\zeta u)=\zeta\vartheta_{\mathrm{VI}}(u), \qquad \operatorname{div}(\vartheta_{\mathrm{VI}})=(0),$$

where $\vartheta_1(u)$ is the odd theta function defined by

$$\vartheta_1(u)=\vartheta_1(\zeta^2;u)=i\sum_{n=-\infty}^{\infty}(-1)^n\exp\left\{\pi i\zeta^2\left(\frac{2n-1}{2}\right)^2+\pi i(2n-1)u\right\}.$$

Remark. To find the above expression, we use Proposition 1.4, (ii).

Proof of Lemma 4.7. By the well known formula

$$\vartheta_1(u+1)=-\vartheta_1(u),$$
$$\vartheta_1(u+\zeta^2)=-e^{\pi i(2u+\zeta^2)}\vartheta_1(u),$$

one can easily check that ϑ_{III} and ϑ_{VI} are holomorphic sections of ξ_{III}^{-1} and ξ_{VI}^{-1} respectively. If we notice that ξ_{III} is ζ^2-invariant and ξ_{VI} is ζ-invariant, we see that $\vartheta_{III}(\zeta^2 u)$ and $\vartheta_{VI}(\zeta u)$ are sections of ξ_{III}^{-1} and ξ_{VI}^{-1} respectively. On the other hand, by Proposition 1.4, (i), we have

$$\dim H^0(E, \Omega(\xi_{III}^{-1}))=\dim H^0(E, \Omega(\xi_{VI}^{-1}))=1.$$

These imply

$$\vartheta_{III}(\zeta^2 u)=\lambda_1\vartheta_{III}(u)$$

and

$$\vartheta_{VI}(\zeta u)=\lambda_2\vartheta_{VI}(u)$$

for some $\lambda_1, \lambda_2 \in C^\times$. Since div $(\vartheta_1)=1\cdot(0)$, if we put $u=0$ in the first equality, we have $\lambda_1=1$. Differentiate the second equality and put $u=0$, then we have $\lambda_2=\zeta$.

Q.E.D.

Proposition 4.8. *For each* $\Gamma=\Gamma_{III}(1;0), \Gamma_{VI}(1;0), \Gamma_{VI}(1;2), \Gamma_{VI}(1;3), \Gamma_{VI}(1;5)$, *there exists a local parameter* (x, y) *of* $S_0=D/\Gamma\cup\{P\}$ *around* P, *such that the map* $D\to S_0$ *is given by*

$$(z, u)\mapsto(x, y)=(x(z, u), y(x, y)).$$

The expressions of $x=x(z, u)$ *and* $y=y(z, u)$ *are given by the following table.*

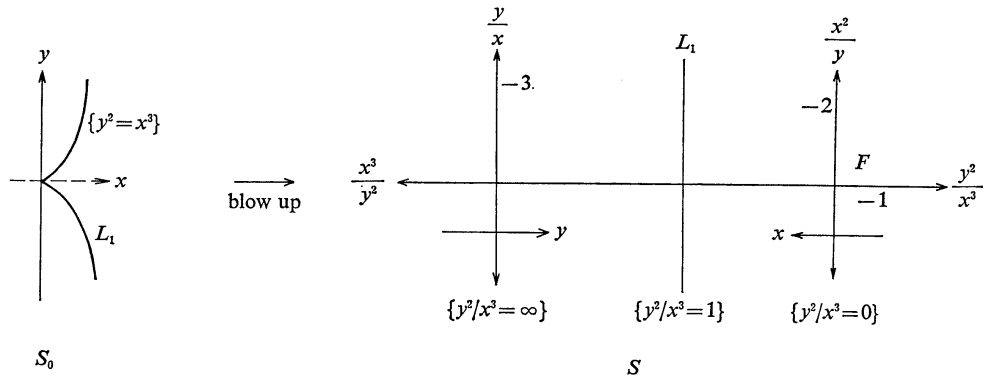
Γ	x	y	$f(u)$	$\theta(u)$
$\Gamma_{III}(1;0)$	$\theta(u)w$	$(\theta(u)w)^3 f(u)$	$\dfrac{2\wp'\left(\frac{1+\zeta}{3}\right)}{\wp'(u)+\wp'\left(\frac{1+\zeta}{3}\right)}$	$\vartheta_{III}(u)$
$\Gamma_{VI}(1;0)$	$(\theta(u)w)^2 f^{-1}(u)$	$(\theta(u)w)^3 f^{-1}(u)$	$1-\left\{\dfrac{e_1}{\wp(u)}\right\}^3$	$\vartheta_{VI}(u)\dfrac{\wp'(u)}{\wp(u)}$

Γ	x	y	$f(u)$	$\theta(u)$
$\Gamma_{\mathrm{VI}}(1;2)$	$(\theta(u)w)^2 f^{-1}(u)$	$(\theta(u)w)^3 f^{-1}(u)$	$1-\left\{\frac{\wp(u)}{e_1}\right\}^3$	$\vartheta_{\mathrm{VI}}(u)\wp'(u)$
$\Gamma_{\mathrm{VI}}(1;3)$	$(\theta(u)w)^2 f^{-1}(u)$	$(\theta(u)w)^3 f^{-2}(u)$	$\left\{\frac{\wp(u)}{e_1}\right\}^3$	$\vartheta_{\mathrm{VI}}(u)\{\wp(u)\}^2$
$\Gamma_{\mathrm{VI}}(1;5)$	$\theta(u)w$	$(\theta(u)w)^6 f(u)$	$\left\{\frac{\wp(u)}{e_1}\right\}^3$	$\vartheta_{\mathrm{VI}}(u)$

Here $\wp(u)=\wp(\zeta^2; u)$, $e_1=\wp(1/2)$ *and* $w=\exp\left(\frac{\pi i}{\sqrt{3}}z\right)$.

Remark. $f(u)$ *represents a covering projection* $E\to E/\langle A^2\rangle \cong \boldsymbol{P}^1$ *for the group* $\Gamma_{\mathrm{III}}(1;0)$ *and* $E\to E/\langle A\rangle \cong \boldsymbol{P}^1$ *for the other groups.*

Proof of Proposition 4.8. (i) Case $\Gamma=\Gamma_{\mathrm{VI}}(1;0)$. Let $L_1=\{u=0\}$ and $L_2=\{u=1\}\subset D$ be the centers of the generating unitary reflections. The same symbol L_j will denote the images of L_j by the natural maps $D\to D/\Gamma_1$ and $D\to D/\Gamma$. Under this convention, L_1 coincides with L_2 in $S_1:=D/\Gamma_1$, L_1 intersects transversely with F in the nonsingular model S (Theorem 4.1), and L_1 has a cusp of type (2, 3) in S_0. We shall express the map $S_1\to S$ in terms of coordinate. First we take the coordinate (x, y) of $S_0\subset \boldsymbol{C}^2$ such that $L_1=\{y^2=x^3\}$. Then if we blow up three times at (0, 0) $\in S_0$, the coordinate representation of S is obtained by the following figures.

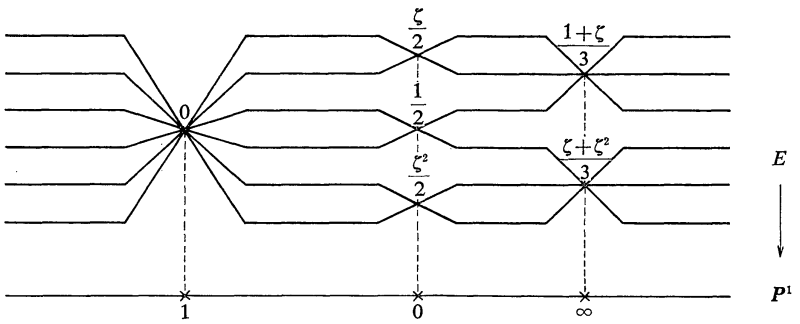


Next we define the map ϕ: $S_1\to \boldsymbol{P}^2(\boldsymbol{C})$ by

$$[w, u]\mapsto[z_1, z_2, z_3]=[\Theta(u)w, f(u), 1]$$

where $[z_1, z_2, z_3]$ is a homogeneous coordinate of $\boldsymbol{P}^2$, and Θ, f are the functions given in the second line of the table in the proposition. On account of Lemmas 4.6 and

4.8, ϕ is well defined. Note that $f(u)$ gives the covering projection $E \to E/\langle A \rangle \cong \boldsymbol{P}^1$ as follows:



Let R be the image of S_1 by ϕ. It will be shown that:

$$\psi: [z_1, z_2, z_3] \mapsto (y^2/x^3, x^2/y) = (z_2/z_3, z_1/z_2)$$

gives a birational map: $R \to S$, which gives the map $\psi \circ \phi$: $S_1 \to S$ and will complete the proof.

Since

$$\operatorname{div}(\theta(u)) = \left(\frac{1}{2}\right) + \left(\frac{\zeta}{2}\right) + \left(\frac{\zeta^2}{2}\right) - \left(\frac{1+\zeta}{3}\right) - \left(\frac{\zeta+\zeta^2}{3}\right),$$
$$\operatorname{div}(f(u)) = 2\left(\frac{1}{2}\right) + 2\left(\frac{\zeta}{2}\right) + 2\left(\frac{\zeta^2}{2}\right) - 3\left(\frac{1+\zeta}{3}\right) - 3\left(\frac{1+\zeta^2}{3}\right),$$

the map ϕ is smooth[2)] except on the lines $\{u=1/2\}$ and $\{u=(1+\zeta)/2\}$ in S_1, and so R is a union of the points $[0, 0, 1]$, $[0, 1, 0] \in \boldsymbol{P}^2$ and a open neighborhood of $\{[0, z_2, z_3] \in \boldsymbol{P}^2; z_2 \neq 0, z_3 \neq 0\}$. Moreover, by the two equalities above, we see that ϕ is locally biholomorphic to the map

$$\left[w, t+\frac{1}{2}\right] \to [tw, t^2, 1] \in \boldsymbol{P}^2$$

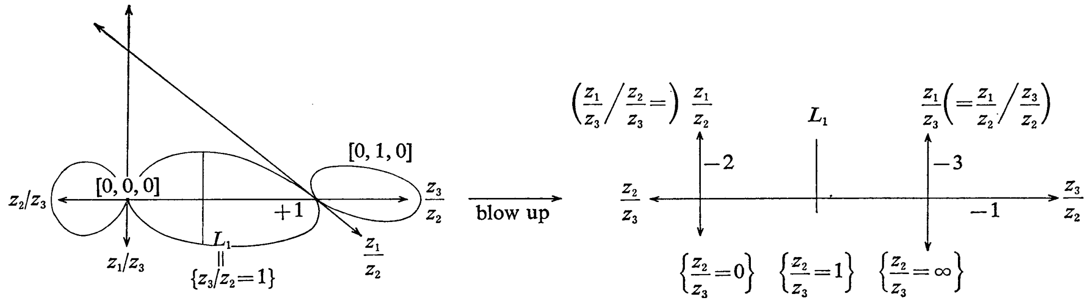
in the neighborhood of the line $\{u=1/2\} \subset S_1$, and

$$\left[w, t+\frac{1+\zeta}{3}\right] \to [t^2 w, 1, t^3] \in \boldsymbol{P}^2$$

in the neighborhood of the line $\{u=(1+\zeta)/2\} \subset S_1$. Thus by blowing up $R(\subset \boldsymbol{P}^2)$

2) In general, the map $\phi: M \to N$ is said to be smooth at $x \in M$ if the linear map $d\phi: T_x(M) \to T_{\phi(x)}(N)$ is surjective, where $T_x(M)$ is the tangent space of M at $x \in M$.

twice at [0, 0, 1] and three times at [0, 1, 0], we have the surface visualized by the following figure.[3]



$R\subset \boldsymbol{P}^2$

By the coordinate representation of S, this gives the map ψ.

(ii) For the remaining cases, the assertions are proved by the same principle as above: Let $L\subset D$ be the union of the centers of the generating unitary reflections of Γ. First we take coordinate (x, y) of $S_0\subset \boldsymbol{C}^2$ so that the image of L by the natural map $D\to S_0$ may be represented by a polynomial equation: $x(y^2-x^3)=0$, $y(y^2-x^3)=0$, $y(y-x^6)=0$ and $y(y-x^3)=0$ for each $\Gamma=\Gamma_{\mathrm{VI}}(1;2)$, $\Gamma_{\mathrm{VI}}(1;3)$, $\Gamma_{\mathrm{VI}}(1;5)$ and $\Gamma_{\mathrm{III}}(1;0)$ respectively. This induces a coordinate representation of the surface S obtained in Theorem 4.1. Next we define the map ϕ: $S_1\to X$ by

$$[w, u]\mapsto(z_1, z_2)=(\theta(u)w, f(u))$$

where $X=\boldsymbol{P}^2$ for $\Gamma=\Gamma_{\mathrm{VI}}(1;2)$, $\Gamma_{\mathrm{VI}}(1;3)$ and $X=\boldsymbol{C}\times\boldsymbol{P}^1$ for $\Gamma=\Gamma_{\mathrm{VI}}(1;5)$, $\Gamma_{\mathrm{III}}(1;0)$. Let R be the image of S_1 by ϕ and $\{l_j\}\subset S_1$ the set of lines on which ϕ is not smooth. Finally we blow up X suitable times at the points $\{\phi(l_j)\}\subset X$ so that R may be transformed to the surface of which the graph of the exceptional curves is the same as that of S. This gives the birational map: ψ: $R\to S$ and will complete the proof.

Q.E.D.

Finally we shall construct differential equations

Theorem 4.9. *For each Γ listed in Corollary 4.4, there exists a Fuchsian system (E_Γ) which has the projective monodromy group Γ. The singular locus Z_Γ in the finite plane and the four coefficients of (E_Γ) are given in the following table.*

[3] We blow up $\boldsymbol{P}^2$ successively to obtain the surface so that R may be transformed to a open subset of the surface thus obtained. To save space, we omit the details.

Γ	Four coefficients of (E_Γ) p^1_{11} / p^1_{22}	p^2_{11} / p^2_{22}	Defining Equation of Z_Γ
$\Gamma_{\mathrm{III}}(1;0)$	$\frac{x^2}{3(x^3-y)}$	$\frac{3xy}{x^3-y}$	$y(y-x^3)=0$
	$-\frac{x}{9y(x^3-y)}$	$-\frac{2x^3-y}{9y(x^3-y)}$	
$\Gamma_{\mathrm{III}}(3;0)$	$-\frac{x-2y}{9x(x-y)}$	$\frac{y}{3x(x-y)}$	$xy(y-x)=0$
	$-\frac{x}{3y(x-y)}$	$-\frac{2x-y}{9y(x-y)}$	
$\Gamma_{\mathrm{VI}}(1;0)$	$\frac{2x^2}{3(x^3-y^2)}$	$\frac{9xy}{4(x^3-y^2)}$	$y^2-x^3=0$
	$-\frac{4x}{9(x^3-y^2)}$	$-\frac{y}{9(x^3-y^2)}$	
$\Gamma_{\mathrm{VI}}(2;0)$	$\frac{2x^2}{3(x^3-y)}$	$\frac{9xy^2}{2(x^3-y)}$	$y(y-x^3)=0$
	$\frac{x}{9y(x^3-y)}$	$-\frac{3x^2-2y}{18y(x^3-y)}$	
$\Gamma_{\mathrm{VI}}(3;0)$	$\frac{2y^2}{9x(x-y^2)}$	$\frac{y}{4x(x-y^2)}$	$x(y^2-x)=0$
	$-\frac{4x}{3(x-y^2)}$	$-\frac{y}{9(x-y^2)}$	
$\Gamma_{\mathrm{VI}}(6;0)$	$\frac{2y}{9x(x-y)}$	$\frac{y}{2x(x-y)}$	$xy(y-x)=0$
	$-\frac{x}{3y(x-y)}$	$-\frac{3x-2y}{18y(x-y)}$	
$\Gamma_{\mathrm{VI}}(1;3)$	0	$\frac{3xy}{4(x^3-y)}$	$y(y^2-x^3)=0$
	$-\frac{4x}{9(x^3-y^2)}$	$-\frac{2x^3+y^2}{9y(x^3-y^2)}$	

Γ	Four coefficients of (E_Γ): p^1_{11}	p^2_{11}	Defining Equation of Z_Γ
	p^1_{22}	p^2_{22}	
$\Gamma_{\text{VI}}(3;3)$	$-\dfrac{2}{9x}$	$\dfrac{y}{12x(x-y^2)}$	$xy(y^2-x)=0$
	$-\dfrac{4x}{3(x-y^2)}$	$-\dfrac{2x+y^2}{9y(x-y^2)}$	
$\Gamma_{\text{VI}}(1;5)$	$\dfrac{x^5}{x^6-y}$	$\dfrac{12x^4y}{x^6-y}$	$y(y-x^6)=0$
	$\dfrac{12x^4y}{x^6-y}$	$-\dfrac{2}{9y}$	
$\Gamma_{\text{VI}}(2;4)$	$\dfrac{2x^3+y}{6x(x^3-y)}$	$\dfrac{3xy}{x^3-y}$	$xy(y-x^3)=0$
	$-\dfrac{x}{18(x^3-y)}$	$-\dfrac{2}{9y}$	
$\Gamma_{\text{IV}}(1;2)$	$\dfrac{4x^3+y^2}{6x(x^3-y^2)}$	$\dfrac{9xy}{4(x^3-y^2)}$	$x(y^2-x^3)=0$
	$-\dfrac{2x}{9(x^3-y^2)}$	$\dfrac{y}{9(x^3-y^2)}$	

Proof. First we treat the case $\Gamma=\Gamma_{\text{III}}(1;0)$. By Proposition 4.8, we have

$$x=\theta(u)w,$$
$$y=(\theta(u)w)^3 f(u).$$

Lemma 4.10.

$$p^1_{11}=-\frac{1}{3x}-\left(\frac{\theta'}{\theta}\right)'x\left(\frac{\partial u}{\partial x}\right)^2+\frac{2}{3}\left(\frac{f'}{f}\right)'\frac{f}{f'}\frac{\partial u}{\partial x},$$

$$p^2_{22}=-\frac{1}{3y}+\left(\frac{\theta'}{\theta}\right)'x\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}-\frac{1}{3}\left(\frac{f'}{f}\right)'\frac{f}{f'}\frac{\partial u}{\partial y},$$

$$p^1_{22}=-\left(\frac{\theta'}{\theta}\right)'x\left(\frac{\partial u}{\partial y}\right)^2,$$

$$p^2_{11}=-\frac{3y}{x^2}-3\left(\frac{\theta'}{\theta}\right)'y\left(\frac{\partial u}{\partial x}\right)^2+\frac{f'}{f}y\frac{\partial^2 u}{\partial x^2}.$$

Proof. Analogous to that of [17], Lemma 8.6. **Q.E.D.**

Lemma 4.11.

$$\frac{\partial u}{\partial x}=\frac{y}{\omega x(x^3-y)}\wp,$$
$$\frac{\partial u}{\partial y}=\frac{-1}{3\omega(x^3-y)}\wp$$
$$\frac{\partial^2 u}{\partial x^2}=-\frac{2xy}{\omega(x^3-y)^2}\wp,$$
$$\frac{f'}{f}=\frac{-3y}{\omega x^3}\wp^2,$$
$$\left(\frac{f'}{f}\right)'=-\frac{3(x^3+y)}{x^3}\wp,$$
$$\left(\frac{\theta'}{\theta}\right)'=\frac{y}{x^3}\wp,$$

where $\wp=\wp(\zeta;u)$ and $\omega=\wp'((1+\zeta)/3)$.

Proof. Since we have

$$\frac{y}{x^3}=f=\frac{2\omega}{\wp'+\omega}$$

and

$$(\wp')^2=4\wp^3-4e_1^3,$$

all the equations, except the last, are easily shown. The homogeneity of ζ-function: $\eta_3=\zeta^{-2}\eta_1$ and Legendre's relation: $\eta_1\omega_3-\eta_3-\omega_1=\pi i/2$ yield $\eta_1=\pi/\sqrt{3}$. Thus the equality $\theta(u)=\exp((\pi/\sqrt{3})u^2-(4/3)\pi iu)\vartheta_1(u-(\zeta+\zeta^2)/3)$ and the formula

$$\frac{d^2}{du^2}\log\vartheta_1(u)=-\wp(u)-\eta_1/\omega_1$$

yield

$$\left(\frac{\theta'}{\theta}\right)'=-\wp\left(u-\frac{\zeta+\zeta^2}{3}\right).$$

Hence, by the addition formula of $\wp$, we have the last equality. **Q.E.D.**

By Lemmas 4.10 and 4.11 with the equality

$$\wp_3=\frac{\omega^2x^3(x^3-y)}{y},$$

the coefficients are calculated.

For the groups $\Gamma=\Gamma_{\mathrm{VI}}(1;0)$, $\Gamma_{\mathrm{VI}}(1;3)$, $\Gamma_{\mathrm{VI}}(1;5)$, and $\Gamma_{\mathrm{VI}}(1;2)$, by the expression of x, y in Proposition 4.8, we can propose the lemmas analogous to Lemmas 4.10 and 4.11 and can calculate the coefficients. Thus we omit the proofs.

For the remaining groups, again by Proposition 4.9, we have

$$\left[0, \frac{2}{3}\sqrt{3}\right]:\quad (x, y)\mapsto(\zeta^2x, y) \qquad \text{on } S_0(\Gamma_{\mathrm{III}}(1;0))$$

$$[0, \sqrt{3}]:\quad (x, y)\mapsto(x, -y) \qquad \text{on } S_0(\Gamma_{\mathrm{VI}}(1;0))$$

$$\left[0, \frac{2}{3}\sqrt{3}\right]:\quad (x, y)\mapsto(-\zeta x, y) \qquad \text{on } S_0(\Gamma_{\mathrm{VI}}(1;0))$$

$$\left[0, \frac{\sqrt{3}}{3}\right]:\quad (x, y)\mapsto(-\zeta x, -y) \qquad \text{on } S_0(\Gamma_{\mathrm{VI}}(1;0))$$

$$\left[0, \frac{2}{3}\sqrt{3}\right]:\quad (x, y)\mapsto(-\zeta x, y) \qquad \text{on } S_0(\Gamma_{\mathrm{VI}}(1;3))$$

$$[0, \sqrt{3}]:\quad (x, y)\mapsto(-x, y) \qquad \text{on } S_0(\Gamma_{\mathrm{VI}}(1;5))$$

where $S_0(\Gamma)=D/\Gamma\cup\{P\}$ and (x, y) is the local coordinate of $S_0(\Gamma)$ obtained in Proposition 4.8. Therefore Lemma 2.11 and Remark 4.6 yield the desired results.

Since every group in question belongs to the same tree in Remark 1.11, and the Fuchsian system $(E_{\Gamma_{\mathrm{II}}(\zeta;1,0,0,0)})$ belongs to this tree, the systems thus obtained are Fuchsian. This completes the proof.

References

[1] Appell, P. et Goursat, E., *Théorie des fonctions algébriques d'une variable et des transcendentes qui s'y rattachant*, Tom 2, Gauthier-Villars et Fils, Paris 1930.

[2] Deligne, P., Equations différentielles à points réguliers singuliers, Lec. Notes in Math., **163**, Springer, 1970.

[3] Gunning, R. C., *Lectures on Riemann surfaces*, Princeton Univ. Press, Princeton, N. J., 1966.

[4] Hemperly, J. C., The parabolic contribution to the number of linearly independent automorphic forms on a certain bounded domain, Amer. J. Math., **94** (1972), 1078–1100.

[5] Hurwitz-Courant, *Allgemeine Funktionentheorie und elliptishe Funktionen*, Springer, Berlin, 1929.

[6] Kobayashi, S. and Nomizu, K., *Foundations of differential geometry*, Interscience publishers, 1969.

[7] Kodaira, K., On compact complex anaytic surfaces, I, Ann. of Math., **71**-1 (1960), 111–152.

[8] Laufer, H. B., *Normal two-dimentional singularities*, Princeton Univ. Press, Princeton,

N. J., 1971.

[9] Oda, T., On Schwarzian derivatives in several variables (in Japanese), Kokyuroku of R.I.M., Kyoto Univ., **226** (1974).

[10] Pyatetskii-Shapiro, I. I., *Automorphic functions and the geometry of Classical Domains*, Gordon and Breach, N. Y., 1969.

[11] Picard, E., Sur les fonctions de deux variables indépendentes analogues aux fonctions modulaires, Acta Math., **2** (1883), 114–126.

[12] Robert, A., Elliptic curve, Lec. Notes in Math., **326**, Springer, 1973.

[13] Shimura, G., On purely transcendental fields of automorphic functions of several variables, Osaka J. Math., **1** (1964), 1–14.

[14] Schwarz, H. A., Über diejenigen Fälle, in welchen die Gaussishe hypergeometrische algebraische Function ihres vierten Elements darstellt, J. de Crelle, **54** (1873), 292–335.

[15] Terada, T., Probléme de Riemann et fonctions automorphes provenant des fonctions hypergéométriques de plusieurs variables, J. Math. Kyoto Univ., **13**-3 (1973), 557–578.

[16] Yoshida, M., Canonical forms of some systems of linear partial differential equations, Proc. Japan Acad., **52**-9 (1976), 473–476.

[17] Yoshida, M., Local theory of Fuchsian systems with certain discrete monodromy groups I, Funkcial. Ekvac., **21**-2 (1978), 105–137.

[18] Yoshida, M., Local theory of Fuchsian systems with certain discrete monodromy groups II, Funkcial. Ekvac., **21**-3 (1978), 203–221.

[19] Yoshida M. and Takano, K., On a linear system of Phaffian equations with regular singular points, Funkcial. Ekvac., **19**-2 (1976), 175–189.

nuna adreso:
Yoshida, M.
Department of Mathematics
Faculty of Science
Kyushu University
Fukuoka, Japan
Hattori Sh.
Miki High School
Miki, Hyogo-ken, Japan

(Ricevita la 4-an de aprilo, 1977)