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On Isomorphism Between Certain Strong $\mathscr{L}^{(p,\lambda)}$ Spaces and the Lipschitz Spaces and its Applications

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§ 1. Introduction

In the study of partial differential equations of elliptic and parabolic type the theory of the $\mathscr{L}^{(p,\lambda)}$ spaces has proved to be very important (see for example [3] and [6]). The theory of the spaces was first studied by C. B. Morrey [6] and later was established by such various authors as S. Campanato, F. John—L. Nirenberg, G. N. Meyers and G. Stampacchia (see [12] and [14] for bibliography).

G. Stampacchia, on the other hand, introduced the theory of $\mathscr{L}^{(p,\lambda)}$ spaces of strong type in [13], which is more general and complicated than that of $\mathscr{L}^{(p,\lambda)}$ spaces, and some of the strong $\mathscr{L}^{(p,\lambda)}$ spaces were characterized in [8], [9], [10], [11], [12], [13] and others. In [8] and [11], we have proved the space $\mathscr{L}_r^{(p,\lambda)}$ (the definition is shown in 2) is imbedded into the space Lip $(n/r-\lambda/p, r)$ with their corresponding norms, where $1\leqq p<\infty$, $-p<\lambda<n$, $1\leqq r<\infty$ and $0<n/r-\lambda/p\leqq 1$. In this paper, we shall prove at first the converse imbedding theorem, that is, the latter space is imbedded into the former space with their corresponding norms and therefore isomorphism between two spaces holds. Secondly, with the aid of these results we obtain a Morrey-Sobolev type imbedding theorem concerning spaces of functions whose first derivatives u_x belong to some strong $\mathscr{L}^{(p,\lambda)}$ spaces which are isomorphic to the corresponding Lipschitz spaces. Finally, even if the strong $\mathscr{L}^{(p,\lambda)}$ spaces can never be isomorphic to any Lipschitz space, an analogous imbedding theorem will be proved under suitable conditions. These two theorems are generalizations and improvements of Theorems 1,2 in [10] and Proposition 1 in [11] and closely analogous to the Morrey-Sobolev type theorem due to Stampacchia in the $\mathscr{L}^{(p,\lambda)}$ spaces [13].

In 2, relevant definitions, additional remarks and main results are stated.

In 3, the isomorphism theorem is proved.

In 4, the proofs of the imbedding theorems are given. Main tools for the proof are theorems due to S. Campanato-G. N. Meyers in [1] and [5] respectively, F. John—L. Nirenberg in [4], S. M. Nikol'skii in [7], M. H. Taibleson in [15] and the author in [8] and [11] (with Y. Furushō).

§ 2. Preliminaries

We shall always consider subfamilies of real-valued integrable functions $u(x)=u(x_1, \cdots, x_n)$ defined on the n dimensional Euclidean space E^n "with supports contained in a fixed bounded cube".[1] Let Q_0 be a fixed bounded cube and we denote a generic subcube of Q_0 having its sides parallel to those of Q_0 by Q and its measure by $|Q|$. The mean-value of a function u on Q is denoted by u_Q: $u_Q=|Q|^{-1}\int_Q u(x)dx$.

Definition 1. *A function u is said to belong to the space $\mathscr{L}_r^{(p,\lambda)}=\mathscr{L}_r^{(p,\lambda)}(Q_0)$ (the $\mathscr{L}^{(p,\lambda)}$ space of strong type r), where $1\leqq p<\infty$, $-\infty<\lambda<\infty$ and $1\leqq r<\infty$, if for any system of subcubes Q_j of "finite number" $S=\{Q_j\colon \cup Q_j\subset Q_0\}$, no two of which have interior point in common, the relation*

$$(2.1) \qquad [u]_{\mathscr{L}^{(p,\lambda)}(Q_j)}=\sup_{Q\subset Q_j}\left\{|Q|^{\lambda/n-1}\int_Q |u(x)-u_Q|^p\, dx\right\}^{1/p}=K(Q_j)<\infty$$

holds and, furthermore, there exists a constant $L=L(u)$ such that

$$(2.2) \qquad \sup_{\{Q_j\}=S\in\bar{S}}\left[\sum_j |K(Q_j)|^r\right]^{1/r}=L$$

where $\bar{S}$ denotes the family of all systems of subcubes considered above. We denote L by $[u]_{\mathscr{L}_r^{(p,\lambda)}(Q_0)}$ and define a norm of the space $\mathscr{L}_r^{(p,\lambda)}(Q_0)$ by $[u]_{\mathscr{L}_r^{(p,\lambda)}(Q_0)}+\|u\|_{L^p(Q_0)}$. This norm renders the space $\mathscr{L}_r^{(p,\lambda)}(Q_0)$ with a structure of a Banach space.

Here, we make

Remark 2.1. (1) We may assume without loss of generality that each side of Q_0 is parallel to some axis, the side length is $2R_0$ and Q_0 has its center at the origin.

(2) Let $Q(2R_0)$ be the cube concentric with and parallel to Q_0, and the side length be twice that of Q_0 (hence, $Q(R_0)$ means Q_0). Now, we extend the domain of u to $Q(2R_0)-Q_0$: that is, if $R_0<x_k\leqq 2R_0$

$$(2.3) \qquad u(x_1, \cdots, x_k, \cdots, x_n)\equiv u(x_1, \cdots, 2R_0-x_k, \cdots, x_n).$$

Furthermore, we repeat the same procedure as above to $Q\ ((\sqrt{n}+2)R_0)$.

(3) Finally, we select an infinitely differentiable function $0\leqq\gamma(x)\leqq 1$ which is equal to unity on $Q((\sqrt{n}+1)R_0)$ and to 0 outside of $Q((\sqrt{n}+2)R_0)$, and define newly the function $u(x)$ as follows:

$$(2.4) \qquad \begin{aligned} u(x)&=\gamma(x)u(x) && \text{on } Q((\sqrt{n}+2)R_0)\\ &=0 && \text{outside of } Q((\sqrt{n}+2)R_0). \end{aligned}$$

[1] For the detail, see Remark 2.1.

Now, we can easily verify that if u belongs to the space $\mathscr{L}_r^{(p,\lambda)}(Q_0)$, then u belongs to the space $\mathscr{L}_r^{(p,\lambda)}(Q((\sqrt{n}+1)R_0))$.

Definition 2. *A function u is said to be Hölder continuous of strong type* $1 \leqq r < \infty$ *with exponent* $0<\alpha<1$ *on* Q_0, *if the following two conditions are satisfied*:

(1) *u is Hölder continuous with exponent* α *on* Q_0:

(2) *there exists a constant* $L=L(u)$ *such that, for any system of subcubes* Q_j *belonging to* $\bar{S}$ *as in Definition* 1, *one has*

$$\sup_{\{Q_j\}=S\in\bar{S}} \left[\sum_j |K(Q_j)|^r\right]^{1/r} = L \tag{2.5}$$

where $K(Q_j)$ *denotes the Hölder coefficient with exponent* α *of* $u|_{Q_j}$, *the restriction of* u *to the subcube* Q_j. *We denote* L *by* $[u]_{\mathscr{H}_r^\alpha(Q_0)}$ *and obtain a Banach space by taking* $\|u\|_{\mathscr{H}_r^\alpha(Q_0)}=[u]_{\mathscr{H}_r^\alpha(Q_0)}+\max_{x\in Q_0}|u(x)|$ *as the norm in* $\mathscr{H}_r^\alpha(Q_0)$.

Remark 2.2. By Campanato-Meyers' theorem in [1] and [5], it is well-known that for any $1\leqq p<\infty$ the space $\mathscr{L}_r^{(p,-p\alpha)}$ is isomorphic to the space $\mathscr{H}_r^\alpha(Q_0)$ with their corresponding norms (see Lemma 3.2).

Definition 3. *A function u is said to belong to the space* Lip (a,p) *on* Q_0, *where* $0<a<\infty$ *and* $1\leqq p\leqq\infty$, *that is, u is said to satisfy a Lipschitz condition of order a in* $L^p=L^p(Q_0)$, *if there exists a constant* $K=K(u)$ *such that*

$$\sup_{|h|\leqq\sqrt{n}|Q_0|^{1/n}} |h|^{-a+\bar{a}}\left[\int_{Q_0}|D^{\bar{a}}u(x+h)-D^{\bar{a}}u(x)|^p\,dx\right]^{1/p}=K \tag{2.6}$$

where $\bar{a}$ *is the greatest integer less than a. We denote K by* $[u]_{\mathrm{Lip}\,(a,p,Q_0)}$ *or* $[u]_{\mathrm{Lip}\,(a,p)}$ *for simplicity and define the norm* $\|u\|_{\mathrm{Lip}\,(a,p)}$ *by* $[u]_{\mathrm{Lip}\,(a,p)}+\|u\|_{L^p(Q_0)}$. *Endowed with this norm the space* Lip (a,p) *becomes a Banach space.*

Remark 2.3. It is obvious that if the function u, before extension to E^n as was mentioned in Remark 2.1 (2)–(3), belongs to the space Lip $(a,p,Q((\sqrt{n}+1)^{-1}R_0))$, $-p<\lambda<n$, then the newly defined function (2.4) also belongs to the space Lip (a,p) and even to the space Lip (a,p,E^n).

Now, our main results read as follows:

Theorem 1. *Let p, λ and r be arbitrary constants satisfying* $1\leqq p<\infty$, $1\leqq r<\infty$ *and* $0<n/r-\lambda/p<1$. *Then the space* $\mathscr{L}_r^{(p,\lambda)}$ *is isomorphic to the space* Lip $(n/r-\lambda/p, r)$ *and*

$$C^{-1}\|u\|_{\mathscr{L}_r^{(p,\lambda)}}\leqq\|u\|_{\mathrm{Lip}\,(n/r-\lambda/p,r)}\leqq C\|u\|_{\mathscr{L}_r^{(p,\lambda)}} \tag{2.7}$$

where $C(>1)$ *is a constant independent of* u.[2)]

Theorem 2. *Let u be a function such that its derivatives* u_x *belong to the space* $\mathscr{L}_r^{(p,\lambda)}$, *where* $1\leqq p<\infty$, $0<\lambda<n$, $1\leqq r<\infty$ *and* $0<n/r-\lambda/p<1$. *Then the following estimates hold for* u:

(1) *If* $p<\lambda$, *then u belongs to* $\mathscr{L}_{r_1}^{(\tilde{p},\lambda)}$ *and* [3)]

$$[u]_{\mathscr{L}_{r_1}^{(\tilde{p},\lambda)}}\leqq C\,\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}} \tag{2.8}$$

where r_1 *is an arbitrary constant such that* $(n/\lambda)p<r_1$.[4)]

(2) *If* $p=\lambda$, *then u belongs to* $\mathscr{L}_{r_1}^{(1,0)}$ *(the strong John-Nirenberg space: see* [4] *and Lemma* 3.2) *and*

$$[u]_{\mathscr{L}_{r_1}^{(1,0)}}\leqq C\,\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}} \tag{2.9}$$

where r_1 *is an arbitrary constant greater than* n.

(3) *If* $p>\lambda$, *then u belongs to* $\mathscr{H}_{r_1}^{1-\lambda/p}$ *and*

$$[u]_{\mathscr{H}_{r_1}^{1-\lambda/p}}\leqq C\,\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}} \tag{2.10}$$

where r_1 *is an arbitrary constant such that* $(n/\lambda)p<r_1$.

Theorem 3. *Among the conditions of Theorem* 2, *if we replace the condition* "$0<n/r-\lambda/p<1$" *by* "$(n/\lambda)p<r$, *that is* $n/r-\lambda/p<0$", *in which case the space* $\mathscr{L}_r^{(p,\lambda)}$ *cannot be isomorphic to any Lipschitz space, we obtain the following estimates for* u:

(1) *If* $p<\lambda$ *and* $(n/\lambda)p<r<(n/\lambda)\tilde{p}$, *then u belongs to* $\mathscr{L}_r^{(\tilde{p},\lambda)}$ *and*

$$[u]_{\mathscr{L}_r^{(\tilde{p},\lambda)}}\leqq C\,\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}. \tag{2.11}$$

(2) *If* $p=\lambda$, *then u belongs to* $\mathscr{L}_r^{(1,0)}$ *and*

$$[u]_{\mathscr{L}_r^{(1,0)}}\leqq C\,\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}. \tag{2.12}$$

(3) *If* $p>\lambda$, *then u belongs to* $\mathscr{H}_r^{1-\lambda/p}$ *and*

$$[u]_{\mathscr{H}_r^{1-\lambda/p}}\leqq C\,\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}. \tag{2.13}$$

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2) Throughout the remainder of this article we denote, for simplicity, positive constants possibly different but independent of functions under consideration by C or sometimes by $C(n)$, $C(n,p,\lambda,r)$ etc. only indicating arguments on which the constants may depend.

3) In the case of $1\leqq p<\lambda<n$, $\tilde{p}$ always means $(1/p-1/\lambda)^{-1}$.

4) We note that for arbitrary constants r and r_1 satisfying the inequality $1\leqq r<r_1<\infty$, the following relation holds with their corresponding norms:

$$\mathscr{L}_r^{(p,\lambda)}\subset\mathscr{L}_{r_1}^{(p,\lambda)}\subset\mathscr{L}_\infty^{(p,\lambda)}=\mathscr{L}^{(p,\lambda)}.$$

In addition, if $n/r-\lambda/p$ is equal to unity, then u_x belong to the Sobolev space $H^{1,r}$ and we can take r_1 equal to $(n/\lambda)p$.

§ 3. Proof of the isomorphism theorem

We have proved the following:

Lemma 3.1. ([8], [11]) *After extension to* E^n *as was made in Remark* 2.1 (2)–(3) *we have*

$$\mathscr{L}_r^{(p,\lambda)} \subset \mathrm{Lip}\,(n/r-\lambda/p, r) \tag{3.1}$$

with their corresponding norms, where $1 \leqq p < \infty$, $-p<\lambda<n$, $1 \leqq r < \infty$ *and* $0 < n/r - \lambda/p < 1$.

Therefore, it is sufficient for us to prove that the following converse imbedding relation holds.

Proposition 3.1. *Let* p, λ *and* r *be constants as in Lemma* 3.1. *Then we have*

$$\mathrm{Lip}\,(n/r-\lambda/p, r) \subset \mathscr{L}_r^{(p,\lambda)} \tag{3.2}$$

with their corresponding norms.

For this purpose, we need the following:

Lemma 3.2. *The space* $\mathscr{L}_r^{(p,\lambda)}$ *is isomorphic to the space* $\mathscr{L}_r^{(1,\lambda/p)}$ *and*

$$C(n, p, \lambda, r)[u]_{\mathscr{L}_r^{(p,\lambda)}} \leqq [u]_{\mathscr{L}_r^{(1,\lambda/p)}} \leqq [u]_{\mathscr{L}_r^{(p,\lambda)}} \tag{3.3}$$

where $1<p<\infty$, $-p<\lambda<n$ *and* $\lambda/p \leqq n/r$.

Remark 3.1. This lemma was proved in [1] and [5] independently ($-p<\lambda<0$), [4] ($\lambda=0$) and [11] ($0<\lambda<n$) respectively.

By this lemma Proposition 3.1 is equivalent to the following:

Proposition 3.2. *Under the same condition as in Proposition* 3.1, *we have*

$$\mathrm{Lip}\,(n/r-\lambda/p, r) \subset \mathscr{L}_r^{(r, r(\lambda/p))} \tag{3.4}$$

with their corresponding norms.

Proof. Let $\{u_m\}_{m=1,2,\dots} \subset \mathrm{Lip}\,(n/r-\lambda/p, r)$ be an arbitrary sequence of functions which converges strongly to 0 in the space $\mathrm{Lip}\,(n/r-\lambda/p, r)$ as m tends to infinity. Here, we note that it suffices to consider the seminorm part only combining Definitions 1,3 and (3.4). Now, we take an arbitrary system of disjoint subcubes $\{Q_j\colon \cup Q_j \subset Q_0\} \in \bar{S}$. Then, we have

$$
\begin{aligned}
&[u_m]_{\mathscr{L}^{(r,r(\lambda/p))}(Q_j)} \\
&\quad = \sup_{Q\subset Q_j}\left[|Q|^{r\lambda/np-1}\int_Q |u_m(x)-(u_m)_Q|^r\,dx\right]^{1/r} \\
&\quad \leqq \sup_{Q\subset Q_j}\left[|Q|^{r\lambda/np-r-1}\int_Q dx\left\{\int_Q |u_m(x)-u_m(y)|\,dy\right\}^r\right]^{1/r}
\end{aligned}
$$

by Minkowskii's inequality, this is

$$
\begin{aligned}
&\leqq \sup_{Q\subset Q_j}|Q|^{(1/r)(r\lambda/np-r-1)}\int_Q dy\left[\int_Q |u_m(x)-u_m(y)|^r\,dx\right]^{1/r} \\
&\leqq \sup_{Q\subset Q_j}|Q|^{\lambda/np-1-1/r}\int_{|h|\leqq\sqrt{n}\,|Q|^{1/n}} |h|^{(n/r-\lambda/p)-n/r+\lambda/p}dh \\
&\qquad\qquad\qquad \times\left[\int_Q |u_m(x+h)-u_m(x)|^r\,dx\right]^{1/r} \\
&\leqq C(n,p,\lambda,r)[u_m]_{\mathrm{Lip}\,(n/r-\lambda/p,r,Q_j)}.
\end{aligned}
$$

As we assume that the sequence $\{u_m\}$ converges to 0 in the space Lip $(n/r-\lambda/p, r)$ strongly, for every j there exists a positive integer m_j such that for any positive number ε, the following inequality holds:

$$
\begin{aligned}
[u_m]_{\mathscr{L}^{(r,r(\lambda/p))}(Q_j)} &\leqq C(n,p,\lambda,r)[u_m]_{\mathrm{Lip}\,(n/r-\lambda/p,r,Q_j)} \\
&\leqq 2^{-j/r}\varepsilon \qquad \forall m\geqq m_j.
\end{aligned}
\tag{3.5}
$$

Taking into account that the number of subcubes $Q_j \in S$ is "finite" (see Definition 1), we may set $m_0=\max_j m_j$ and the following inequality holds:

$$
[u_m]_{\mathscr{L}^{(r,r(\lambda/p))}(Q_j)}\leqq 2^{-j/r}\varepsilon \qquad \forall m\geqq m_0. \tag{3.6}
$$

Hence, we have

$$
\sum_j [u_m]^r_{\mathscr{L}^{(r,r(\lambda/p))}(Q_j)}\leqq \sum_j \frac{\varepsilon^r}{2^j}<\varepsilon^r.
$$

As the system $\{Q_j\}$ and the positive number ε are arbitrary, this means

$$
[u_m]_{\mathscr{L}_r^{(r,r(\lambda/p))}(Q_0)}<\varepsilon \tag{3.7}
$$

and the sequence $\{u_m\}$ must converge strongly to 0 in the space $\mathscr{L}_r^{(r,r(\lambda/p))}$ as m tends to infinity.

This completes the proof of this theorem.

§ 4. Proof of the Morrey-Sobolev type imbedding theorems

Before proceeding to the proof of Theorem 2, we state the following:

Lemma 4.1 ([7]). *We have*

$$\mathrm{Lip}\,(a, p) \subset \mathrm{Lip}\left(a - \left(\frac{1}{p} - \frac{1}{q}\right)n,\ q\right) \tag{4.1}$$

with their corresponding norms, where a, p and q are arbitrary constants satisfying $1 \leqq p < q \leqq \infty$ and $0 < a - (1/p - 1/q) \neq n$ integer.

Proof of Theorem 2. As u_x belong to the space $\mathcal{L}_r^{(p,\lambda)}$, we have by Theorem 1

$$\begin{aligned} C_1(n, p, \lambda, r)\, \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}} &\leqq \|u_x\|_{\mathrm{Lip}\,(n/r-\lambda/p,r)} \\ &\leqq C_2(n, p, \lambda, r)\, \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}} \end{aligned}$$

that is

$$C_1\, \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}} \leqq \|u\|_{\mathrm{Lip}\,(1+n/r-\lambda/p,r)} \leqq C_2\, \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}}$$

and applying Lemma 4.1, we obtain the following inequality:

$$[u]_{\mathrm{Lip}\,(a,(a-1+\lambda/p)^{-1}n)} \leqq C\, \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}} \tag{4.2}$$

where $\max\,(0, 1-\lambda/p) < a < 1$.

Here, we divide the proof into three cases in accordance with this theorem.

(1) $p < \lambda$: By taking $(a-1+\lambda/p)^{-1}n = r_1$, we have $n/r_1 = a-1+\lambda/p = a+\lambda/\tilde{p}$ and $\mathrm{Lip}\,(a, (a-1+\lambda/p)^{-1}n) = \mathrm{Lip}\,(n/r_1 - \lambda/\tilde{p}, r_1)$ which is isomorphic to the space $\mathcal{L}_{r_1}^{(\tilde{p},\lambda)}$ by making use of Theorem 1 again. In addition, as we can take a arbitrarily close to unity, r_1 also may be supposed to be an arbitrary constant greater than $(n/\lambda)p$.[5]

This completes the proof of this case.

(2) $p = \lambda$: The left hand side term of (4.2) reduces to $[u]_{\mathrm{Lip}\,(a,n/a)}$ and the conclusion is immediate by a similar argument as in (1).

(3) $p > \lambda$: By the same substitutions as in (1) we obtain

$$[u]_{\mathrm{Lip}\,(a,r_1)} = [u]_{\mathrm{Lip}\,(n/r_1-1+\lambda/p,r_1)}$$

and, as $-1+\lambda/p$ is negative, we can assert that by (4.2) and Theorem 1 the following inequality holds:

$$[u]_{\mathcal{H}_{r_1}^{1-\lambda/p}} \leqq C\, \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}}. \tag{4.3}$$

Hence, the proof of Theorem 2 is complete.

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[5] In the cases (2) and (3), the situations about r_1 is the same.

Now, to prove the last theorem we remark at first that in attaining the supremums $[u]_{\mathscr{L}_r^{(\tilde{p},\lambda)}}$ $(p<\lambda)$, $[u]_{\mathscr{L}_r^{(1,0)}}$ $(p=\lambda)$ and $[u]_{\mathscr{H}_r^{1-\lambda/p}}$ $(p>\lambda)$ respectively the corresponding spaces are isomorphic to the spaces $\mathrm{Lip}\,(n/r-\lambda/\tilde{p}, r)$, $\mathrm{Lip}\,(n/r, r)$ and $\mathrm{Lip}\,(n/r-1+\lambda/p, r)$ respectively and therefore we may suppose without loss of generality that $\{Q_j\}\in\bar{S}$ are always systems of congruent subcubes. Because, combining the following:

Lemma 4.2. ([8]) *Let p, λ and r be constants such that $1\leqq p<\infty$, $-p<\lambda<n$, $1\leqq r\leqq p$ and $0<n/r-\lambda/p\leqq 1$. Then we have*

$$\|u(x+h)-u(x)\|_{L^r(Q_j)}\leqq C(n)[u]_{\mathscr{L}^{(p,\lambda)}(Q'_j)}\,|h|^{n/r-\lambda/p} \tag{4.4}$$

where $|h|$ is the side length of Q_j and independent of j: furthermore, Q'_j is an arbitrary one of the subcubes which contain Q_j and have a common vertex and whose side length is $2|h|$. (This means that the following inequality holds:

$$[u]_{\mathrm{Lip}\,(n/r-\lambda/p,r)}\leqq C(n)[u]_{\mathscr{L}_r^{(p,\lambda)}}.): \tag{4.5}$$

and Proposition 3.2, our assertion is completely verified.

Finally, we prepare the following:

Lemma 4.3. ([15]) *Let a be a positive constant less than unity. Then we have*

$$\begin{aligned} C_1(n,a)[u]_{\mathrm{Lip}\,(a,p,Q_0)} &\leqq \sup_{|h|\leqq\sqrt{n}\,|Q_0|^{1/n}} |h|^{-a}\,\|u(x+h)-2u(x)+u(x-h)\|_{L^p(Q_0)} \\ &\leqq C_2(n,a)[u]_{\mathrm{Lip}\,(a,p,Q_0)}. \end{aligned} \tag{4.6}$$

Now, we are going to give the

Proof of Theorem 3. Set $|Q_j|=|h|^n$ and $v(x)=u(x)-u(x-h)$. Then, we have $u(x+h)-2u(x)+u(x-h)=v(x+h)-v(x)$ and, applying Theorem 1 to the case (1)

$$[u]_{\mathscr{L}_r^{(\tilde{p},\lambda)}}\leqq C(n,p,\lambda,r)[u]_{\mathrm{Lip}\,(n/r-\lambda/\tilde{p},r)}$$

making use of Lemma 4.3,

$$\leqq C(n,p,\lambda,r)[v]_{\mathrm{Lip}\,(n/r-\lambda/\tilde{p},r)}$$

applying Theorem 1 again

$$\leqq C(n,p,\lambda,r)[v]_{\mathscr{L}_r^{(\tilde{p},\lambda)}}$$

and by Lemma 3.2

$$\leqq C(n, p, \lambda, r)[v]_{\mathscr{L}_r^{(p,p(\lambda/\tilde{p}))}}$$

$$\leqq C(n, p, \lambda, r) \sup_{|Q_j|=|h|^n} \left[\sum_j \left(|Q_j|^{(\lambda-p)/n-1} \int_{Q_j} |v(x)-v_{Q_j}|^p \, dx\right)^{r/p}\right]^{1/r}$$

applying Poincarè's inequality we have

$$\leqq C(n, p, \lambda, r) \sup_{|Q_j|=|h|^n} \left[\sum_j \left(|Q_j|^{\lambda/n-1} \int_{Q_j} |v_x(x)|^p \, dx\right)^{r/p}\right]^{1/r}$$

$$= C(n, p, \lambda, r) \sup_{|Q_j|=|h|^n} \left[\sum_j \left(|Q_j|^{\lambda/n-1} \int_{Q_j} |u_x(x)-u_x(x-h)|^p \, dx\right)^{r/p}\right]^{1/r}$$

$$\leqq C(n, p, \lambda, r) \, \|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}.$$

For the proof of the last inequality, we refer the detail to Appendix 2 of [8]. Hence, the proof of the case (1) is complete.

By the similar calculations as in the case (1) we obtain the conclusion of the cases (2) and (3) respectively.

Therefore, our assertion is completely verified.

Remark 4.1. Stampacchia's imbedding theorem reads as follows:

Theorem. ([13]) *Let u be a function such that the first derivatives u_x belong to $\mathscr{L}^{(p,\lambda)} = \mathscr{L}_\infty^{(p,\lambda)}$, where $1 \leqq p < \infty$ and $0 \leqq \lambda \leqq n$. Then the following estimates hold for u.*

(1) *If $p < \lambda$, then u belongs to $\mathscr{M}^{(\tilde{p},\lambda)}$ (the definition is shown in* [12]) *and*

$$[u]_{\mathscr{M}^{(\tilde{p},\lambda)}} \leqq C \, \|u_x\|_{\mathscr{L}^{(p,\lambda)}}.$$

(2) *If $p = \lambda$, then u belongs to $\mathscr{L}^{(1,0)}$ and*

$$[u]_{\mathscr{L}^{(1,0)}} \leqq C \, \|u_x\|_{\mathscr{L}^{(p,\lambda)}}.$$

(3) *If $p > \lambda$, then u belongs to $C^{0,1-\lambda/p}$ and*

$$[u]_{C^{0,1-\lambda/p}} \leqq C \, \|u_x\|_{\mathscr{L}^{(p,\lambda)}}.$$

Therefore, we observe that as for the strong $\mathscr{L}^{(p,\lambda)}$ spaces, closely similar results to the Stampacchia's theorem are obtained except the case: *$p < \lambda$ and $(n/\lambda)\tilde{p} \leqq r$.*

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